Astérisque

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Astérisque, tome 157-158 (1988), p. 59-74

<http://www.numdam.org/item?id=AST_1988__157-158__59_0>

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Société Mathématique de France
Astérisque 157-158 (1988)

Some asymptotic laws for crossings and excursions †

by Krzysztof BURDZY, Jim W. PITMAN and Marc YOR

1. Introduction.

Consider the number of times $N_{AC}(t)$ that the trajectory of a Markov process X has crossed between two subsets A and C of its state space by time t. We want to consider the distribution of $N_{AC}(t)$ for large values of t. For instance, suppose X is planar Brownian motion, with A and C two closed subsets of the plane, such that neither A nor C is polar, but $A \cap C$ is polar. For example, A and C could be two circles, possibly intersecting. Suppose that X starts at a point which is not in $A \cap C$. Then the number of crossings $N_{AC}(t)$ is a.s. finite for every t and increases to ∞ as $t \to \infty$. A focal point of this paper is provided by the following proposition:

PROPOSITION 1.1. *For a planar Brownian motion X, started at $x \notin A \cap C$, as $t \to \infty$,*

$$\frac{2\pi N_{AC}(t)}{\log t} \xrightarrow{d} Cap(A, C) H,$$

where $\xrightarrow{d}$ *denotes convergence in distribution, $Cap(A, C)$ is the logarithmic capacity of A relative to the grounded set C, and H has the exponential distribution*

$$P(H \in dh) = e^{-h}\, dh, \quad h > 0.$$

Proposition 1.1 is a simple combination of results due to Kallianpur and Robbins (1953) and Maruyama and Tanaka (1959). See also section 6.8 of Itô and McKean (1965). We give a proof in Section 2 below, along with some variations and extensions for crossings of a recurrent Hunt process.

Section 3 shows how the results for crossings in Section 2 can be extended to processes counting various kinds of excursions. This development is related to the notion of a Palm measure associated with a stationary process of excursions, as described in Pitman (1987). See also Getoor and Steffens (1986) for some related developments of capacity theory.

Section 4 concerns what more can be said about the distribution of $N_{AC}(t)$ in the planar Brownian case when $Cap(A, C) = \infty$. Results obtained here are closely related to asymptotic laws for crossing numbers

† Research supported in part by National Science Foundation DMS85-02930

of Brownian motion on the sphere S^2, due to Lyons and McKean (1984), and similar results for planar Brownian motion and the symmetric Cauchy process on the line described in Pitman and Yor (1986a) and (1986b).

2. Crossings of a recurrent Hunt process.

Let $X = \{\Omega, \mathbf{F}, (\mathbf{F}_t), (X_t), (\Theta_t), (P^x)\}$ be a Hunt process, with state space E which is locally compact with a countable base. Assume further that the process is Harris recurrent, with a single recurrent class, and invariant reference measure m. For background and definitions of these terms, see Blumenthal and Getoor (1968), Azéma, Duflo and Revuz (1967).

Let A and C be two closed subsets of E such that $A \cap C$ is polar. The number of crossings $N_{AC}(t)$ from A to C by time t is P^x a.s. finite for every $x \notin A \cap C$. For if not, there would be an accumulation of crossings at some time s with $0 \le s \le t$, implying either X_{s-} or X_s was in $A \cap C$. But a Hunt process cannot touch a polar set even as a left limit: see Blumenthal and Getoor (1968), Proposition I.10.20.

We will now be precise about just when each crossing is counted. Say the path of X *touches* the set A at time t if either X_t or X_{t-} is in A. For times $u \le v$, say there is a *crossing* from A to C which *starts* at time u and *ends* at time v if X touches A at time u, then does not touch either A or C until time v when X touches C. The number of such crossings ended by time t gives a counting process which is adapted to $(\mathbf{F}_t)$, but not additive with respect to the shifts (Θ_t). The number of such crossings which have started by time t gives a counting process which is not $(\mathbf{F}_t)$ adapted but which is (Θ_t) additive. These two possible definitions of $N_{AC}(t)$ differ by at most one, which is negligible so far as asymptotics are concerned. We now adopt the convention that *crossings are counted as they start* and that a crossing which starts at time 0 is ignored. Thus $N_{AC}(t)$ means the number of crossings from A to C which start in the interval $(0, t]$. The counting process $N_{AC}(t)$ is P^x additive for every $x \notin A \cap C$, hence is P^m additive for the shift invariant measure P^m. Here we use the following:

DEFINITION. Let P be a measure on $(\Omega, \mathbf{F})$, not necessarily a probability. A process $(V_t, t \ge 0)$ with values in $[0, \infty]$ is P *additive* if each of the following properties holds except on a P null set, possibly depending on s and t in the case of (iii):

(i) $V_0 = 0$;

(ii) $(V_t, t \ge 0)$ is right continuous and increasing;

(iii) $V_{t+s} = V_t + V_s \circ \Theta_t, \quad s, t \ge 0.$

Note that we do not assume that V_t is $\mathbf{F}_t$-measurable. By the shift invariance of P^m, if (V_t) is P^m additive then

$$P^m V_t = t P^m V_1.$$

The most basic additive functional is

$$time(B,t) = \int_0^t 1_B(X_s)ds,$$

the time spent in B before t, for a Borel subset B of the state space. Much of the asymptotic behaviour of P^m additive processes is dictated by the following ergodic theorem, which is a variation and consequence of the results of Azéma, Duflo and Revuz (1967), (1969):

ERGODIC THEOREM. *Suppose that (V_t) is P^m additive, and that $P^m(V_1 = \infty) = 0$. Then*

(i) *$P^x(V_t < \infty$ for all $t) = 1$, for all x except perhaps a polar set;*

(ii) *provided x does not belong to this exceptional set*

$$\lim_{t\to\infty} \frac{V_t}{time(B,t)} = \frac{P^m(V_1)}{m(B)} \quad P^x \ a.s.$$

for every Borel subset B of E with $0 < m(B) < \infty$.

REMARK. The exceptional set of starting points x cannot always be eliminated, even in the case

$$V_t = \int_0^t F(X_s)ds$$

for a positive function F with $\int F dm < \infty$. This is due to the possibility that V may explode immediately on starting at points x where F has a singularity. See Pitman and Yor (1986c) for an example with planar Brownian motion. The statement of Remarque 1) on p.170 and part 3)(ii) of the Théorème on p.181 of Azéma, Duflo and Revuz (1967) should be modified accordingly to allow for an exceptional polar set of starting points x.

As an immediate consequence of the ergodic theorem, we obtain the following:

PROPOSITION 2.1. *For closed subsets A and C such that $A \cap C$ is polar,*

$$\lim_{t\to\infty} \frac{N_{AC}(t)}{time(B,t)} = \frac{P^m N_{AC}(1)}{m(B)} \quad P^x \ a.s. \tag{2.1}$$

for every $x \notin A \cap C$, and every Borel set B with $0 < m(B) < \infty$.

For many processes X, there is a limit law for occupation times of the form

$$\frac{time(B,t)}{g(t)} \xrightarrow{d} m(B)H \quad as \quad t \to \infty \tag{2.2}$$

for some deterministic function $g(t)$ and limiting variable H, no matter what the starting point x. In particular, for complex Brownian motion X with m the area measure in the plane, Kallianpur and Robbins (1953) showed that (2.2) holds with $g(t) = (2\pi)^{-1}\log t$ and H exponentially distributed as in Proposition 1.1. Section 7.17 of Itô-McKean (1965) and Section 4 of Pitman and Yor (1986a) offer alternative proofs. Other results of this general form may be found in Darling and Kac (1957). See also Kasahara and Kotani (1979), Kasahara (1977), (1982), Bingham (1971) for various extensions and refinements. Touati (1988) gives a recent survey.

Now (2.1) combines with (2.2) to yield

$$(2.3)\qquad \frac{N_{AC}(t)}{g(t)} \xrightarrow{d} P^m[N_{AC}(1)]H \quad as \quad t \to \infty.$$

To complete the proof of Proposition 1.1, it only remains to be seen that the constant $P^m[N_{AC}(1)]$, the equilibrium rate of starts of A to C crossings per unit time, is identical to a capacity in the sense of potential theory. To this end, consider the measure λ_{AC} on the state space E of X associated with the P^m additive process $N_{AC}(t)$ by the formula

$$(2.4)\qquad P^m \int_0^t 1_B(X_{s-})\, dN_{AC}(s) \;=\; t\,\lambda_{AC}(B),$$

as in Azéma, Duflo and Revuz (1967). Thus $\lambda_{AC}(B)$ is the equilibrium rate per unit time of A to C crossings starting with a left limit in B. Assume now that X has a dual process $\hat{X}$ relative to the invariant reference measure m. Then the work of Azéma, Duflo and Revuz (1969) and Getoor and Sharpe (1973) shows that (2.4) extends to

$$(2.5)\qquad P^x \int_0^T 1_B(X_{s-})\, dN_{AC}(s) \;=\; \int_B \frac{P^x[time(dy,T)]}{m(dy)}\,\lambda_{AC}(dy)$$

for initial points $x \notin A \cap C$ and all stopping times T. In particular, for $T = T_C$ the hitting time of C, $dN_{AC}(t) \equiv 0$ on $[0, T_C]$ unless $T_A < T_C$, in which case $dN_{AC}(t)$ on $[0, T_C]$ is the unit mass at time $t = L_{AC}$, the last time that X was in A before T_C. Now (2.5) becomes the last exit formula of Getoor and Sharpe (1973) and Chung (1973):

$$(2.6)\qquad P^x(T_A < T_C\ ;\ X(L_{AC}-) \in dy) \;=\; U_C(x,y)\lambda_{AC}(dy),$$

where $U_C(x,y)$ is the density with respect to $m(dy)$ of the potential kernel for X killed on first hitting C. Thus λ_{AC} is the capacitary measure or equilibrium distribution on A in the potential theory of X killed on hitting C. And the total mass of λ_{AC}, which is the constant $P_m[N_{AC}(1)]$ appearing in (2.1) and (2.3), is the capacity of A in this potential theory. In particular, it is well known that the potential theory of Brownian motion killed when it hits C is the logarithmic potential theory with ground C. This yields Proposition 1.1.

The above arguments are closely related to results of Maruyama and Tanaka (1959), Ueno (1960) and McKean (1965), all of which generalize easily to the present setting. These results will now be mentioned briefly. Their proofs are straightforward applications of the ergodic/potential theory of X and $\hat{X}$ as presented above, and the ergodic theory of Harris recurrent Markov chains as presented in Revuz (1975).

Let G^n_{AC} denote the time at which the nth crossing from A to C starts, so

$$N_{AC}(t) \;=\; \sum_{n=1}^{\infty} 1(G^n_{AC} \le t).$$

By repeated last exit decompositions, $(X(G^n_{AC}-),\ n = 1, 2, \cdots)$ is a Markov chain. By the ergodic theorem applied to the process counting visits of this chain to B, as in (2.4), this chain is Harris recurrent with invariant measure the capacitary measure λ_{AC}, which is σ-finite and concentrated on A. Let D^n_{AC} denote the first time after G^n_{AC} that X returns to A. Then $(X(D^n_{AC}),\ n = 1,2,\cdots)$ is also a Harris recurrent Markov chain with invariant measure ρ_{AC} which is concentrated on A. A duality argument identifies ρ_{AC} as the equilibrium measure on A in the potential theory of the dual process killed on first hitting C. According to Ueno and McKean, if the original process X is started with distribution ρ_{AC}, then $X(G^n_{AC}-)$ has distribution λ_{AC} and $X(D^n_{AC})$ has distribution ρ_{AC} for every n. Finally, there is the Maruyama-Tanaka-Ueno formula

$$P^{\rho_{AC}} \int_0^{D^1_{AC}} 1_B(X_s)ds = m(B), \tag{2.7}$$

where

$$\rho_{AC}(E) = \lambda_{AC}(E) = Cap(A,C) = P^m N_{AC}(1). \tag{2.8}$$

These results become very natural in terms of a two-sided stationary Markov process with time parameter t varying over $\mathbb{R}$. Then λ_{AC}, ρ_{AC} and $P^{\rho_{AC}}$ may be interpreted as Palm distributions associated with the alternating stationary point processes which count the beginnings and ends of excursions away from A which reach C. See for example Geman and Horowitz (1973), Neveu (1968) (1976), Pitman (1987). As shown in the next section, these results for crossings admit extensions to numerous other processes counting excursions of various kinds.

EXAMPLE 2.2. BROWNIAN CROSSINGS BETWEEN CIRCLES. In case X is planar Brownian motion, and A and C are nonintersecting circles, calculation of the capacitary measure λ_{AC} is a classical problem. See for example Morse and Feshbach (1953), p. 1210 for a solution in terms of bipolar coordinates. The formulae below can be obtained from those in Morse and Feshbach by a change of variables. Another approach is to first solve the simplest case of two concentric circles. Then the capacitary distributions are uniform, and it is easy to show that

$$Cap(A,C) = \pi|\log(a/c)|^{-1},$$

where a and c are the radii of A and C. The invariance of hitting and capacitary distributions under conformal mapping, in this case a Möbius transformation (see for example Cohn (1967), Kellogg (1929)), can be exploited to show the following:

For non-intersecting circles A and C with centers at distance $d > 0$, there is a unique pair of points x_∞ and y_∞, such that x_∞ is inside A and the circle B with diameter $[x_\infty, y_\infty]$ is orthogonal to both A and C,

$$\lambda_{AC}(\cdot) = Cap(A,C)H_A(x_\infty,\cdot) = Cap(A,C)H_A(y_\infty,\cdot)$$

where $H_A(x,\cdot)$ is the hitting distribution on A starting from x, and

$$Cap(A,C) = \pi\left|\log\left|\frac{d^2-a^2-c^2-\sqrt{\Delta}}{2ac}\right|\right|^{-1}, \quad \textit{where } \Delta = (d-a-c)(d-a+c)(d+a-c)(d+a+c).$$

The density of $H_A(x,\cdot)$ with respect to length measure on A is given by the well known Poisson kernel. A more detailed discussion of this example and related results in higher dimensions are given in Sections 3 and 4 of Burdzy, Pitman and Yor (1987).

3. Excursions

We continue in this section in the framework of a recurrent Hunt process introduced in Section 2. Let M be a closed homogeneous optional subset of $[0,\infty)$, as in Maisonneuve (1975), for example the closure of $\{t : X_t \in A\}$ for a Borel subset A of the state space E. Let

$$R = \inf M \cap (0,\infty), \quad R_t = R \circ \Theta_t, \quad G = \{t>0 : R_{t-} = 0,\, R_t > 0\},$$

so G is the set of left end points of maximal intervals in the complement of M. We assume Ω is the space of paths in E that are right continuous with left limits. For $g \in G$, define the *excursion* $\varepsilon_g \in \Omega$ to be the path started at time g and stopped at time $g + R_g$ which is the right end of the interval of M^c whose left end is g:

$$X_s \circ \varepsilon_g = X_{s \wedge R} \circ \Theta_g = X_{g+(s\wedge R_g)},\ s \geq 0.$$

Note that the excursions are stopped rather than killed at their right ends, so the right end point is a function of the excursion:

$$X_R \circ \varepsilon_g = X_R \circ \Theta_g = X_{g+R_g},\ g \in G.$$

For $B \in \mathbf{F}$, define

$$N_B(t) = \#\{g : 0 < g \leq t,\ g \in G,\ \varepsilon_g \in B\}$$

the number of excursions of type B which have started by time t. Let $(N_B\ \mathit{finite})$ denote the event that $N_B(t)$ is finite for all t. The additivity property

$$N_B(t+s) = N_B(s) + N_B(t) \circ \Theta_s, \qquad s,\, t \geq 0,$$

and the assumption that X is Harris recurrent with invariant reference measure m, imply

either

(3.1) $\qquad P^x(N_B\ \mathit{finite}) = 0$ for *every* $x \in E$

or

(3.2) $\qquad P^x(N_B\ \mathit{finite}) = 1$ *except for* x *in a polar set.*

By the additivity property of $N_B(t)$ as t varies

$$P^m N_B(t) \;=\; t\, Q(B),$$

for some $Q(B)$. Call $Q(B)$ the *equilibrium rate of excursions of type* B. This defines a measure Q on $(\Omega, \mathbf{F})$, which is the rate measure of the stationary marked point process $(\varepsilon_g, g \in G)$ under P^m. Call Q the *equilibrium excursion law*. Clearly, if $Q(B) < \infty$, then we are in case (3.2) above. But $Q(B) < \infty$ is not necessary for this, as shown by the examples of Section 4. Essentially, Q is a Palm distribution and Q is σ-finite by general properties of Palm measures. See Pitman (1987) for further discussion and references to related work.

If $Q(B) = 0$, then it is clear that

$$P^x(N_B \equiv 0) = 1 \textit{ for all } x,$$

while if $Q(B) > 0$ then

$$P^x(\lim_{t\to\infty} N_B(t) = \infty) = 1 \textit{ for all } x,$$

due to the following special case of the ergodic theorem of Section 2:

For every $B \in \mathbf{F}$, *and Borel subsets* F *of the state space with* $0 < m(F) < \infty$

$$\lim_{t\to\infty} \frac{N_B(t)}{time\,(F,t)} = \frac{Q(B)}{m(F)} \quad P^x \; a.s. \tag{3.3}$$

for every $x \in E$ *such that* $P^x(N_B \textit{ finite}) = 1$.

On the set $(N_B(t) < \infty \textit{ for all } t \textit{ and } N_B(t) \to \infty \textit{ as } t \to \infty)$ define

$$G_B^1 < G_B^2 < \ldots.$$

to be the successive times $g \in G$ such that $\varepsilon_g \in B$, and let $\varepsilon_B^n = \varepsilon(G_B^n)$, the nth excursion of type B.

PROPOSITION 3.1. *Suppose that* $Q(B) > 0$, *let*

$$E_B = \{x : P^x(N_B \textit{ finite}) = 1\}$$

and assume that E_B *is not empty. Then* $E - E_B$ *is a polar set for* X. *For* $x \in E_B$, $D \in \mathbf{F}$ *define*

$$P_B(x,D) = P^x(\varepsilon_B^1 \in D),$$

the probability starting at x *that the first excursion of type* B *is in* D. *Then*

(i) *for every* x *in* E_B, *under* P^x *the sequence* (ε_B^n) *of excursions of type* B *is a Markov chain with state space* B *and transition function* $P_B(X_R(e), D)$ *where* $X_R(e)$ *is the right end in* E *of excursion* e.

(ii) *This Markov chain is Harris recurrent with invariant measure* Q_B *which is the restriction of* Q *to* B.

NOTE. Strictly speaking, the state space of the Harris chain is $B_0 = \{e \in B : X_{R(e)} \in E_B\}$, so that the definition of the transition function makes sense.

PROOF. (i) Let D_B^n be the right end point of the interval in M^c whose left end point is G_B^n. Then D_B^n is an $(\mathbf{F}_t)$ stopping time, for every $j \le n$, ε_B^j is $\mathbf{F}_{D_B^n}$ measurable,

$$X_R(\varepsilon_B^n) = X(D_B^n), \qquad \varepsilon_B^{n+1} = \varepsilon_B^1 \circ \Theta_{D_B^n}.$$

It follows at once from the strong Markov property of X at time D_B^n that, under P^x for $x \in E_B$, $(\varepsilon_B^n, \mathbf{F}_{D_B^n}, n = 1,2,\ldots)$ is a Markov chain with transition probabilities as stated.

(ii) For any $e \in B_0$, with $X_R(e) = x$ say, under P^x the sequence of excursions ε_B^n, $n = 1, 2, \cdots$ is distributed as a Markov chain with the specified transition function, as if this chain had started in state e at time $n = 0$. The ergodic theorem (3.3) above shows that for any $F \in \mathbf{F}$ with $Q(F \cap B) > 0$ there will be infinitely many excursions of type $F \cap B$, P^x a.s. It follows by Proposition 2.9 of Revuz (1975) that the chain of excursions is Harris recurrent with invariant measure $\tilde{Q}_B$ which is absolutely continuous with respect to Q_B, the restriction of Q to B. Comparison of the ergodic theorem above and the ergodic theorem for Harris recurrent chains (Theorem 3.6 of Revuz (1975)) now shows that $\tilde{Q}_B$ must be a constant multiple of Q_B.

REMARKS. (i) *Asymptotic behaviour of excursions of type B.* As a consequence of the above proposition, the asymptotic behavior of ε_B^n as $n \to \infty$ is governed by the measure Q_B in accordance with the ergodic theory of Harris recurrent Markov chains. See for example Chapter 6 of Revuz (1975). The chain of excursions of type B may be either null recurrent ($Q(B) = \infty$) or positive recurrent ($Q(B) < \infty$), and the chain may be either periodic or aperiodic. In particular, if $0 < Q(B) < \infty$ and the chain of excursions of type B is aperiodic, then for every x such that $P^x(N_B \ finite) = 1$, the P^x distribution of ε_B^n converges in total variation to $Q(\cdot \mid B)$ as $n \to \infty$, by the ergodic theory of Harris chains. To illustrate this point, let X be one-dimensional Brownian motion, $M = \{t : X_t = 0 \ or \ 1\}$. For $B = (R > 1)$, which picks out excursions away from 0 or 1 lasting longer than unit time, the chain of excursions of type B is aperiodic and positive recurrent. But for $B = (X_R \neq X_0)$, which picks out excursions crossing either from 0 to 1, or back from 1 to 0, the chain is obviously periodic with period 2, and positive recurrent.

(ii) *Nesting of sequences and Itô's excursion law.* If $B \subset C$, then the chain of excursions of type B is embedded in the chain of excursions of type C as the latter chain watched only when it hits B. If $M = \{t : X_t = a \ or \ X_{t-} = a\}$ is the set of return times to a single point $a \in E$, then the chain of excursions of type B is simply a sequence of independent random variables, identically distributed according to $Q(\cdot \mid B)$, whenever $0 < Q(B) < \infty$. In this case, Q is a multiple of Itô's excursion law. See Itô (1970), Maisonneuve (1975), Greenwood and Pitman (1980), Pitman (1987).

(iii) *Endpoints and Maisonneuve's excursion laws.* For B as in the Proposition 3.1, consider the left and right end positions

$$X(G_B^n) = \varepsilon_B^n(0), \qquad X(D_B^n) = \varepsilon_B^n(R)$$

of the chain of excursions of type B. The argument used to prove the proposition shows that the sequence of right ends is Markov with transition probability function on E_B given by

$$(x, H) \to P^x(X(D_B^1) \in H).$$

This chain is Harris recurrent with invariant measure the Q_B distribution of X_R. Also, last exit decompositions derived from Maisonneuve's formula show that the sequence of left ends is Markov with transition function on the state space $\bar{E}_B = \{x : Q^x(B) < \infty\}$ defined by

$$(x, H) \to Q^x(X(G_B^1) \in H \mid X^R \in B)$$

where X^R is X stopped at time R and Q^x is Maisonneuve's exit measure for excursions leaving from x.

This chain is Harris recurrent with invariant measure the Q_B distribution of X_0, which is

$$Q_B(X_0 \in H) = \int_H \beta(dx) Q^x(B)$$

where $\beta(dx)$ is the measure on the state space associated with the additive functional in the Maisonneuve exit system. From the present point of view, Maisonneuve's excursion laws Q^x give a disintegration of the equilibrium excursion law Q with respect to the starting point of excursions. With duality assumptions, there are nice disintegrations with respect to the end point as well. See for example Getoor and Sharpe (1982), Mitro (1984), Kaspi (1984) (1985), Fitzsimmons (1987).

(iv) *Occupation and Palm measures.* Let ρ_B be the Q_B distribution of X_R, which is the equilibrium distribution of final points of excursions of type B. The form of the transition function for excursions of type B shows that Q_B is the P^{ρ_B} distribution of ε_B^n for $n = 1$, and hence for every n. Thus, if the process X is started with initial distribution ρ_B, the sequence of excursions of type B is stationary. For example, the equilibrium distribution of initial points of excursions is the P^{ρ_B} distribution of $X(G_B^n)$ for every n. According to a standard result in the theory of Palm measures, the probability distribution invariant under shifts Θ_t can be recovered from the Palm measure of a stationary point process as the occupation measure between points. See Neveu (1977). In the present setting, this gives the formula

$$P^{\rho_B} \int_0^{D_B^1} f(X_t) dt = m(f).$$

In the case when M is the set of visits to one set, B the set of excursions which cross to another set, this becomes formula (2.7) above. Essentially this is Theorem 4.1 of Maruyama-Tanaka (1959). See also Ueno (1960) p. 121. The above formula holds just as well with ρ_B replaced by the Q_B distribution of X_0, described above as the invariant measure for the chain of left ends of excursions of type B, and D_B^1 replaced by G_B^1.

4. The Case of Infinite Capacity.

For a nice recurrent process, when $Cap(A, C)$ is finite, the asymptotics of the number of crossings $N_{AC}(t)$ as $t \to \infty$ is dictated by the asymptotics of occupation times, as explained in Section 2. But there are many interesting examples where $Cap(A, C)$ is infinite, $A \cap C$ is polar, so $N_{AC}(t)$ is a.s. finite for all starting points $x \notin A \cap C$. The question then arises of what governs the asymptotic distribution of $N_{AC}(t)$ as $t \to \infty$. Lyons and McKean (1984) give some particular results for Brownian motion on the sphere, as do Pitman-Yor (1986ab) for planar Brownian motion and the symmetric Cauchy process on the line. We do not know of any general theory governing the asymptotics of $N_{AC}(t)$ in this case. But we present here a number of results for crossings of planar Brownian motion, all closely related to windings of the Brownian path.

Suppose for the rest of this section that Z is planar Brownian motion.

PROPOSITION 4.1. *Suppose A is a closed connected set containing more than one point, and so is C. If* *either (i) $A \cap C$ is not empty*

or (ii) both A and C are unbounded

then $Cap(A, C) = \infty$.

PROOF. This is similar to the argument involving windings at the top of page 259 of Itô-McKean (1965). Consider first case (i). Without loss of generality, we suppose that $0 \in A \cap C$, and that both A and C intersect the unit circle. Let $D_n = \{z \in C : |z| = 1/2^n\}$, $n \geq 0$, $D = \bigcup_n D_n$. Let B_n denote the set of excursions from D which start in D_n and intersect themselves after making a loop around 0. Let $B = \bigcup_1^\infty B_n$. By a scaling argument, the rate of excursions of type B_n is the same for every n, and this rate is strictly positive, so the rate of excursions away from D of type B is infinite. But, since every excursion of type B contains at least one excursion from A hitting C, or at least one excursion from C hitting A,

$$N_B(t) \leq N_{AC}(t) + N_{CA}(t) \leq 2N_{AC}(t) + 1$$

for all t and ω, since $|N_{AC}(t) - N_{CA}(t)| \leq 1$. It now follows from the identification of $Cap(A, C)$ with $P^m[N_{AC}(1)]$ in Section 2, and the ergodic theorem as in (3.3), that $Cap(A, C) = \infty$. The result in case (ii) is deduced from case (i) by inversion, using the conformal invariance of capacity.

Asymptotic Distributions.

We propose now to study the asymptotic behaviour of $N_{AC}(t)$ in case A and C meet at a finite number of points, say $z_1, \ldots, z_n$, and perhaps also at ∞. We assume that at each of these points the sets meet at a finite number of well defined angles. To illustrate terminology, we say the positive real axis meets the positive imaginary axis at 0 at two angles, $\pi/2$ and $3\pi/2$. And these axes meet at ∞ at the same angles. Or let $A = \{x + iy : x = y\}$, $C = \{x + iy : x^2 = y\}$. Then A and C meet

at 0 at 4 angles $\pi/4$, $\pi/4$, $3\pi/4$, $3\pi/4$,

at $1 + i$ at 4 angles α, α, $\pi - \alpha$, $\pi - \alpha$, where $\alpha = \arctan(2) - \pi/4$,

at ∞ at 2 angles, $\pi/4$, $3\pi/4$.

The meaning of angles at a point of intersection is quite obvious for sets defined or bounded by smooth curves, such as the above. Angles at ∞ are defined by inversion. The definition of angles between sets A and C can be extended to some sets without smooth boundaries by squeezing. But we must require both A and C to be arcwise connected in some neighbourhoods of their meeting points. Note that the number of angles at any meeting point will always be even.

THEOREM 4.2. *Suppose $A \cap C = \{z_1, \ldots, z_n\}$, that A and C meet at z_i at a finite number of angles $\alpha_{i1}, \ldots, \alpha_{im_i}$, $i = 1, \ldots, n$, and that in case both A and C are unbounded that they meet at ∞ at angles $\alpha_{\infty 1}, \ldots, \alpha_{\infty m_\infty}$. Let*

$$\alpha_{i*} = \left(\sum_{j=1}^{m_i} \alpha_{ij}^{-1}\right)^{-1}, \quad i = 1, \ldots, n, \infty; \qquad \bar{\alpha} = \left(\sum_{i=1}^{n} (\alpha_{i*})^{-1/2}\right)^{-2}.$$

Then

$$\frac{16\pi N_{AC}(t)}{\log^2(t)} \xrightarrow{d} \frac{\sigma_-}{\bar{\alpha}} + \frac{\sigma_+}{\alpha_{\infty *}}, \tag{4.1}$$

where the coefficient of σ_+ *is zero in case either* A *or* C *is bounded, and* σ_- *and* σ_+ *are defined as follows in the terms of a one dimensional Brownian motion* $(\beta_t, t \geq 0)$ *up to the time* σ *that* β *first hits 1:*

σ_- *is the time* β *spends in* $(-\infty, 0)$ *before* σ,

σ_+ *is the time* β *spends in* $(0, \infty)$ *before* σ.

REMARKS. This result is a close relative of Theorem 2 of Pitman-Yor (1986b), and could be recast to give the *joint* asymptotic laws of variously classified crossing numbers as introduced in the following proof. These are all *log scaling limit laws* for planar Brownian motion, as considered in Pitman-Yor (1986a) and (1987). As a consequence, the above result holds jointly with Proposition 2.1 when the exponential variable H there is identified as half the local time L of β at 0 before σ. The Laplace transform and further information about the joint distribution of L, σ_- and σ_+ are given in the Introduction of Pitman-Yor (1986b).

PROOF. The basic idea is to decompose the total number of crossings from A to C as

$$N_{AC}(t) = \sum_{i \in S} \sum_{j=1}^{m_i} N_{ij}(t) + \varepsilon(t), \tag{4.2}$$

where $S = \{1, 2, \ldots, n, \infty\}$, for $i \in S$, $1 \leq j \leq m_i$, $N_{ij}(t)$ is the number of crossings across the jth angle between A and C near the point z_i ($z_\infty = \infty$), and $\varepsilon(t)$ is a remainder term of order $\log(t)$ in law. To be more precise, choose radii r_i so small that within radius r_i of z_i the region defining the *jth* angle between A and C is well approximated by a sector of angle α_j, for each $j = 1, \ldots, m_i$, and choose similarly a sufficiently large value of r_∞. It may be assumed in the first instance that, within each of these neighbourhoods of z_i, the complement of $A \cap C$ is simply a union of m_i sectors of angles α_j, $j = 1, \ldots, m_i$. Approximations for more general A and C can be justified later by squeezing arguments, using monotonicity of the crossing numbers as a function of angle. Define $N_{ij}(t)$ to be the number of crossings from A to C initiated before time t by excursions away from A, which leave A within the neighbourhood of z_i defined by the radius r_i, by entering the region of $(A \cap C)^c$ whose angle at z_i is α_{ij}. No matter what the choice of r_i, it follows from Proposition 1.1 and (3.3) that the neglected count $\varepsilon(t)$ will be of order $\log(t)$ in law, whereas each of the $N_{ij}(t)$ will be of larger order. Indeed, we claim the following:

LEMMA 4.3. *Provided* $\alpha_{ij} > 0$, *as* $t \to \infty$

$$\frac{16\pi N_{ij}(t)}{\log^2(t)} \cong \frac{4U_i(t)}{\alpha_{ij}\log^2(t)} \xrightarrow{d} \frac{\sigma_i}{\alpha_{ij}} \tag{4.2}$$

where the symbol $\cong$ *means the difference between the two random variables tends in probability to zero as* $t \to \infty$,

$$U_i(t) = \int_0^t \frac{ds}{|Z_s - z_i|^2} 1(|Z_s - z_i| \leq r_i), \quad i = 1, \ldots, n$$

$$= \int_0^t \frac{ds}{|Z_s|^2} 1(|Z_s| > r_\infty), \quad i = \infty,$$

and σ_i has the distribution of σ_- for $i = 1, \ldots, n$, and that of σ_+ for $i = \infty$. Moreover, as j varies and $t \to \infty$ there is convergence of finite dimensional distributions, to a limiting joint distribution of random variables $\sigma_1, \ldots, \sigma_n, \sigma_\infty$ defined on the same space as random variable L such that

(i) for each i, $(\sigma_i, L, \sigma_\infty)$ has the same joint distribution as (σ_-, L, σ_+), the time spent positive, the local time at 0, and the time spent negative by a Brownian motion β before $\sigma = \inf\{t : \beta_t = 1\}$;

(ii) the random variables $(\sigma_i, i = 1, \ldots, n, \infty)$ are mutually conditionally independent given L.

PROOF. The result on the joint asymptotic distribution of the processes $U_i(t)$ is due to Pitman-Yor (1986a), where these processes arise as the increasing processes associated with local martingales derived from windings. The comparison between $N_{ij}(t)$ and $U_i(t)$ is justified by a slight development of the argument used in Pitman-Yor (1986b). A minor variation of the results (3.6b) and (4.1) in that paper gives the first approximation in (4.2) in the case of two adjacent subintervals of a line, say $C = (-\infty, 0]$, $A = [0, \infty)$. Then $n = 1$, there are two angles at $z_1 = 0$ of $\alpha_{11} = \alpha_{12} = \pi$, and two angles at ∞, $\alpha_{\infty 1} = \alpha_{\infty 2} = \pi$. Now consider crossings from $A = [0, \infty)$ to $C = \{z : arg(z) = \alpha\}$ for some $0 < \alpha < \pi$, counting only counterclockwise crossings through the angle α, and counting only crossings initiated in $[0, r]$. The rate of such crossings is easily compared with the rate of clockwise crossings from $[0, \infty)$ to $(-\infty, 0]$ initiated in $[0, r]$, because these form disjoint sets of excursions away from $[0, \infty)$, and given that an excursion is of one of these types, the chance that it winds counterclockwise angle α before clockwise angle π is simply $\pi / (\pi + \alpha)$, independently of all previous excursions. The law of large numbers thus implies that crossings of the angle α occur relative to crossings of the angle $-\pi$ in the asymptotic ratio of π to α almost surely. This gives the required approximation (4.2) for crossings of an arbitrary angle α.

COMPLETION OF THE PROOF OF THEOREM 4.2. This theorem is an immediate consequence of the basic decomposition (4.2) and the asymptotics of the components in this decomposition described in Lemma 4.3. The coefficient α_{i*} comes from combining the counts N_{ij} for $j = 1, \ldots, m_i$. Then summing over $i = 1, \ldots, n$ gives the coefficient $\tilde{\alpha}$ for σ_-, using the fact that the joint distribution of $(\sigma_1, \ldots, \sigma_n, \sigma_\infty, L)$ described in Lemma 4.3 is identical to that of $(\hat{\sigma}_1 H^2, \ldots, \hat{\sigma}_n H^2, \sigma_\infty, 2H)$ where the variables $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ are all independent with the same stable (1/2) distribution as σ, independent of (σ_∞, L) and $H = L/2$. Thus the limit contribution from $i = 1$ to n is

$$\sum_{i=1}^n (\alpha_{i*})^{-1} \sigma_i \overset{d}{=} \sum_{i=1}^n (\alpha_{i*})^{-1} H^2 \hat{\sigma}_i .$$

But by scaling and convolution of the stable (1/2) laws,

$$\sum_{i=1}^n c_i^2 \hat{\sigma}_i \overset{d}{=} \Big(\sum_{i=1}^n c_i\Big)^2 \sigma, \quad \text{any } c_i > 0$$

and taking $c_i^2 = \alpha_{i*}^{-1}$, the limit law is seen to be that of

$$\Big(\sum_{i=1}^{n} (\alpha_{i*})^{-1/2}\Big)^2 H^2 \sigma \stackrel{d}{=} \sigma_- / \bar{\alpha}.$$

EXAMPLE 4.4. Suppose A and C are two circles which meet at angle α. Then letting z_1 and z_2 be the points of intersection, we get

$$\alpha_{11} = \alpha_{12} = \alpha, \ \alpha_{13} = \alpha_{14} = \pi - \alpha$$

$$\alpha_{21} = \alpha_{22} = \alpha, \ \alpha_{23} = \alpha_{24} = \pi - \alpha.$$

Thus

$$\alpha_{1*} = \alpha_{2*} = \Big(\frac{1}{\alpha} + \frac{1}{\alpha} + \frac{1}{\pi-\alpha} + \frac{1}{\pi-\alpha}\Big)^{-1} = \Big(\frac{2\pi}{\alpha(\pi-\alpha)}\Big)^{-1}$$

$$\bar{\alpha} = [2(\alpha_{1*})^{-1/2}]^{-2} = \frac{1}{4}\alpha_{1*} = \frac{\alpha(\pi-\alpha)}{8\pi}$$

That is to say, for circles intersecting at angle α

$$\frac{2\alpha(\pi-\alpha) N_{AC}(t)}{\log^2(t)} \xrightarrow{d} \sigma_-.$$

What if the circles touch tangentially? Then it is clear from the above that normalization by $\log^2(t)$ gives a limit of ∞. Our calculations suggest that in this case there do not exist normalizing constants a_t and b_t so that $(N_{AC}(t) - a_t)/b_t$ converges in law. Rather, no matter what the radii of the circles A and C, Mountford (1987) has shown

$$\frac{2\log(N_{AC}(t))}{\log t} \xrightarrow{d} -\inf_{0 \le s < \sigma} \beta_s$$

where β and σ are as in Theorem 4.2. This is yet another log scaling law (see Pitman-Yor (1986a), Sec.8) relative to the point of intersection. More generally, for two bounded arcs A and C which are tangent at a point the way the graph of $y = x^n$ is tangent to the x axis at $x = 0$, Mountford has shown that the above result holds with denominator $(n-1)\log t$ instead of $\log t$.

A similar result holds with $\int_0^t \frac{ds}{R_s^{n+1}}$ instead of $N_{AC}(t)$, where R_t is the radial part of the Brownian motion at time t. This is a much more elementary fact which can be derived using the methods leading to Table 1 in Pitman-Yor (1986a). Both results hold by comparison with $-\log(\inf_{0 \le s < t} R_s)$.

Contrast with results on the sphere.

Note how touching at ∞ produces a different component into the asymptotic law. Compare with the situation for Brownian motion on S^2, as discussed in Lyons-McKean (1984) — see also Le Gall-Yor (1986). On the sphere, if A and C do not touch, then

$$\frac{N_{AC}^{sphere}(t)}{t} \xrightarrow{a.s.} \mathrm{Cap}^{sphere}(A, C),$$

the capacity of A in the potential theory of $BM(S^2)$ killed on hitting C. But if A and C touch at points

$w_1, \ldots, w_n, w_\infty$ and angles α_{ij} defined in the obvious way, then the result becomes simply

$$\frac{N_{AC}^{sphere}(t)}{t^2} \xrightarrow{d} \frac{K\sigma}{\alpha^{sphere}}$$

where K is a universal constant, $\sigma = \sigma_- + \sigma_+$, and

$$\alpha^{sphere} = \Big(\sum_{i=1}^{n} (\alpha_{i*})^{-1/2} + (\alpha_{\infty *})^{-1/2}\Big)^{-2}.$$

After stereographic projection of the sphere onto the plane, this corresponds to a variation of Theorem 4.2, namely

$$\frac{4\pi N_{AC}^{plane}(\tau_h)}{h^2} \xrightarrow{d} \frac{\sigma}{\alpha^{sphere}}$$

as $h \to \infty$, where $N_{AC}^{plane}(t)$ is the number of crossings up to time t between the planar images of A and C on the sphere, and τ_h is the inverse of a planar additive functional L_t with $\| L_t \| = 1$. If z_i is the image of w_i and ∞ the image of w_∞ at the north pole of the sphere, this result may be compared to Theorem 4.2 above. Then

$$\alpha^{sphere} = [(\bar{\alpha})^{-1/2} + (\alpha_{\infty *})^{-1/2}]^2.$$

See Pitman-Yor (1986a) for further discussion, explaining in particular how these results hold *jointly* if instead of σ we substitute $\Lambda^{-1}(1)$, where Λ is the common local time process at 0 of the Brownian motion β in terms of which σ_- and σ_+ are defined, and the substitution $t = e^{2h}$ is made in Theorem 4.2.

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