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# MONODROMY AND THE KOWALEVSKAYA TOP.

J. P. Francoise

We consider algebraically completely integrable Hamiltonian systems which are separable [A-M], [Mo], [Moe] and [Mu]. For these systems, we prove that the symplectic form can be reduced to a simple expression involving Abelian forms. We use then Arnol'd's method [A] to define the Actions. The determination of the Actions turns out to be equivalent to a monodromy computation. The Actions are not given, in general, by simple functions of the first integrals. But we can write the corresponding Picard-Fuchs equations. We consider in detail the Kowalevskaya Top and we write down the 4-th order differential equation which is involved in this case.

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## 1. Algebraically Completely Integrable Hamiltonian System

We see here a completely integrable Hamiltonian System  $(H, \omega)$  as an algebraic mapping  $\underline{H} = (H_1, \dots, H_m): V^{2m} \rightarrow \mathbb{C}^m$  which is submersive on a non-empty Zariski open set  $V^* = V^{2m} \setminus S$ , where  $V^{2m}$  is a symplectic algebraic variety, and such that the fibers of  $\underline{H}$  are Lagrangian for the symplectic form  $\omega$ .

Definition. A completely integrable Hamiltonian system is algebraically separable if

- i) there is a family of hyperelliptic curves

$$\underline{C} = \{(z, w) \in \mathbb{C}^2 / z^2 = \phi_{\underline{C}}(w)\}$$

such that the fiber  $\underline{H}^{-1}(\underline{c})$  is the affine part of  $\text{Jac}(\underline{C}_{\underline{c}})$  and constants  $v_i \in \mathbb{C}^m$  such that

$$(1.1) \quad \sum_{k=1}^m \frac{w_k^{j-1} \{H_i, w_k\}}{\sqrt{\Phi(w_k)}} = v_i \delta_{ij}$$

$\{ \}$  is the Poisson bracket for  $\omega$  and  $v_i \neq 0$  for all  $i = 1, \dots, m$ .

Let us consider a Hamiltonian system  $(\underline{H}, \omega)$  which is a complexification of a real mapping  $\underline{H}: \mathbb{R}^{2m} \rightarrow \mathbb{R}$ . If the fibers  $\underline{H}^{-1}(\underline{c})$ ,  $\underline{c} \in \mathbb{R}^m$  are compact, the connected components of the general fiber are real tori (Arnol'd-Liouville).

The system is said to be algebraically completely integrable when the fibers are affine part of Abelian varieties [A-M]. Most of the interesting completely integrable Hamiltonian systems have this property. For instance, the three cases of integrability of the motion of a rigid body about a fixed point and their extensions [R], [R-M], the Toda Lattice and its extensions by Kostant [K] the examples of J. Moser [Mo] ect. Furthermore for all these examples, the Abelian varieties are Jacobians of Riemann surfaces  $\underline{C}_{\underline{c}}$ ,  $\text{Jac}(\underline{C}_{\underline{c}}) = H^0(\underline{C}, \Omega_{\underline{C}}^1) / H_1(\underline{C}, \mathbb{Z})$ .

Let us recall that if  $z^2 = \Phi(w)$  is an equation for a hyperelliptic curve  $\underline{C}$ ,  $\Phi$  being a polynomial of degree  $2g$  or  $2g + 1$ , the Abelian forms of the first kind  $(\frac{w^j dw}{\sqrt{\Phi(w)}}, j = 0, \dots, g - 1)$  generate a basis of  $H^0(\underline{C}, \Omega_{\underline{C}}^1)$ .

So the equation (1.1) means that the Hamiltonian flows of the functions  $H_i$  are linearized on the Jacobian and that they have a constant velocity  $v_i$  relatively to the basis of the Abelian forms of the first kind.

The velocities  $v_i$  are usually independent of  $\underline{H}$  in the classical examples. The algebraic polarization of the complex tori is not given by the expected one (Projective embedding of the fibers  $\underline{H}^{-1}(\underline{c})$  by homogenization) but is provided by the existence of Laurent developments for the solutions of the  $\underline{H}$  [A.M.]. In this sense, following Torelli's theorem the smooth curves  $\underline{C}_{\underline{c}}$  are uniquely determined by the couple  $(\underline{H}, \omega)$ .

We consider now an example.

## 2. The Integration of Kowalevskaya Top.

Euler's equations governing the motion of a rigid body about a fixed point are given by the following

$$\begin{aligned}
 (2.1) \quad & A\dot{p} + (C - B)qr = mg(y_0\gamma_3 - z_0\gamma_2) \\
 & B\dot{q} + (A - C)pr = mg(z_0\gamma_1 - x_0\gamma_1) \\
 & C\dot{r} + (B - A)pq = mg(x_0\gamma_2 - y_0\gamma_1) \\
 & \dot{\gamma}_1 = r\gamma_2 - q\gamma_3 \\
 & \dot{\gamma}_2 = p\gamma_3 - r\gamma_1 \\
 & \dot{\gamma}_3 = q\gamma_1 - p\gamma_2
 \end{aligned}$$

It is natural to restrict the vector field that they define to the algebraic variety  $V_R^4 \subset R^6$  given by the equations

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0$$

$$2(p\gamma_1 + q\gamma_2) + r\gamma_3 - 2\ell = 0,$$

$\ell \in R$  is fixed.

We get by restriction on  $V^4$  a Hamiltonian system for the symplectic form:

$$(2.2) \quad \omega = \frac{2}{\gamma_3} dp \wedge d\gamma_2 - \frac{2}{\gamma_3} dp \wedge d\gamma_1 - \frac{r}{\gamma_3^2} d\gamma_1 \wedge d\gamma_2$$

and the Hamiltonian is

$$(2.3) \quad H = p^2 + q^2 + \frac{r^2}{2} - c\gamma_1, \quad c = \frac{mgx_0}{c}$$

for the Kowalevskaya Top which corresponds to the values

$$A = B = 2C, \quad y_0 = z_0 = 0$$

of the parameters.

In that case, we have an extra integral  $K$ :

$$(2.4) \quad K = [(p + iq)^2 + c(\gamma_1 + i\gamma_2)][(p - iq)^2 + c(\gamma_1 - i\gamma_2)].$$

We define by  $\underline{H} = (K, H): V_{\mathbb{C}}^4 \rightarrow \mathbb{C}$  a completely integrable Hamiltonian system.

If we follow S. Kowalevskaya's computation [Ko], [Go], we choose  $x_1 = p + iq$ ,  $x_2 = p - iq$ ,  $\gamma_1$ ,  $\gamma_2$  as a system of coordinates on  $V^4$ . We introduce the polynomials

$$R(x) = -x^4 + 2Hx^2 + 4cx$$

$$R(x_1, x_2) = -x_1^2 x_2^2 + 2Hx_1 x_2 + 2c\ell(x_1 + x_2) + c^2 - K$$

$$\begin{aligned} R_1(x_1, x_2) = & -2Hx_1^2 x_2^2 - (c^2 - K)(x_1 + x_2)^2 - 4c\ell x_1 x_2 (x_1 + x_2) \\ & + 2H(c^2 - K) - 4c^2 \ell^2. \end{aligned}$$

We use then

$$w_1 = \frac{R(x_1 x_2) - \sqrt{R(x_1)R(x_2)}}{(x_1 - x_2)^2} \quad (2.5)$$

$$w_2 = \frac{R(x_1 x_2) + \sqrt{R(x_1)R(x_2)}}{(x_1 - x_2)^2}$$

and the polynomials

$$\phi(w) = (w + H)(w^2 + c^2 - K) - 2c^2 \ell^2 \quad (2.6)$$

$$\phi(w) = -2(w^2 - K)\phi(w).$$

Now an algebraic computation shows that the equations (2.1) are equivalent to

$$\dot{w}_1 = -\{H, w_1\} = \sqrt{\phi(w_1)}/w_1 - w_2 \quad (2.7)$$

$$\dot{w}_2 = -\{H, w_2\} = \sqrt{\Phi(w_2)}/w_1 - w_2$$

and so, we have

$$\frac{dw_1}{\sqrt{\Phi(w_1)}} + \frac{dw_2}{\sqrt{\Phi(w_2)}} = 0$$

(2.8)

$$\frac{w_1 dw_1}{\sqrt{\Phi(w_1)}} + \frac{w_2 dw_2}{\sqrt{\Phi(w_2)}} = -dt.$$

Let us introduce the hyperelliptic Riemann surface  $C$  defined by  $z^2 = \Phi(w)$  in  $\mathbb{CP}_2 = \{(z, w)\}$ . It is a compactification of a double cover of the plane minus three cuts and so it is a Hyperelliptic curve of genus 2. Let  $\text{Jac}(C) = H^0(C, \Omega_C^1)^* / H_1(C, \mathbb{Z})$  be the Jacobian variety of  $C$ . The forms  $(\frac{dw}{\sqrt{\Phi(w)}}, \frac{wdw}{\sqrt{\Phi(w)}})$  provide a basis of  $H^0(C, \Omega_C^1)$ . We can associate to  $(w_1, w_2)$  an element  $(z_1, w_1) - (z_2, w_2)$  of the Picard group  $\text{Pic}_0(C)$  where  $z_1^2 = \Phi(w_1)$  and  $z_2^2 = \Phi(w_2)$ . So the equations (2.8) describe a linear motion on  $\text{Jac}(C)$  and the real tori given by Arnol'd-Liouville are real part of Abelian varieties on which the motion is linear.

If we introduce

$$Q(w, \underline{H}, x_1, x_2) = (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2)$$

we deduce from (2.5) that

$$Q(w_1, \underline{H}, x_1, x_2) = Q(w_2, \underline{H}, x_1, x_2) = 0.$$

So we have an identity

$$\frac{\partial Q}{\partial w_i} dw_i + \frac{\partial Q}{\partial x_1} dx_1 + \frac{\partial Q}{\partial x_2} dx_2 + \frac{\partial Q}{\partial H} dH + \frac{\partial Q}{\partial K} dK \Big|_{w=w_i} = 0$$

for  $i = 1, 2$ .

We can deduce from this identity that

$$\begin{aligned} \{K, w_1\} &= 2w_2 \sqrt{\Phi(w_1)} / w_1 - w_2 \\ (2.9) \quad \{K, w_2\} &= 2w_1 \sqrt{\Phi(w_2)} / w_2 - w_1. \end{aligned}$$

So the condition (1.1) holds for the Kowalevskaya top with  $v_1 = +2$  and  $v_2 = 1$ .

### 3. Preparation of the Symplectic Form

Proposition 3.1. If  $(H, \omega)$  is algebraically separable, then there are functions  $\tilde{q}_j$  such that the forms  $d\tilde{q}_j \Big|_{\underline{H}^{-1}(\underline{c})}$  are sums of Abelian integrals and such that

$$\omega = \sum_{j=1}^m \tilde{dq}_j \wedge dH_j.$$

Proof. We start with the expression of the symplectic form  $\omega$  in the coordinates  $(H, w)$

$$\omega = \sum_{j, \ell} A_{j\ell} dH_j \wedge dH_\ell + B_{j\ell} dw_j \wedge dH_\ell + C_{j\ell} dw_j \wedge dw_\ell.$$



We have

$$(3.1) \quad -dH_i = \sum_{j,l} B_{jl} \{H_i, w_j\} dH_l + 2C_{jl} \{H_i, w_j\} dw_l.$$

It is convenient at this point to introduce a matrix notation. Let  $F$ ,  $B$ ,  $W$ ,  $V$  be the matrices whose general terms are:

$$(F)_{ij} = \{H_i, w_j\} \quad (B)_{ij} = B_{ij}$$

$$(W)_{ij} = \frac{w_i^{j-1}}{\sqrt{\phi(w_1)}} \quad (V)_{ij} = v_i \delta_{ij}.$$

Then, the equation (1.1) gives

$$F \cdot W = V.$$

From (3.1), we deduce that

$$F \cdot B = -1$$

and since  $\det(V) = \prod_{i=1}^m v_i \neq 0$ , that

$$B = -WV^{-1}$$

or

$$(3.2) \quad B_{j\ell} = -\frac{1}{V_\ell} w_j^{\ell-1} / \sqrt{\Phi(w_j)}.$$

Another consequence of (3.1) is

$$C_{j\ell} \{H_i, w_j\} = 0$$

and because  $\det F \neq 0$ , we have

$$(3.3) \quad C_{j\ell} = 0 \quad \text{for all } j, \ell.$$

We introduce now the pre-angles  $\tilde{q}_j$  in the following way. The symplectic form can be written:

$$\omega = \sum_{j=1}^m \eta_j \wedge dH_j$$

Let  $\tilde{\eta}$  be a one-form such that  $\omega = d\tilde{\eta}$  defined on an appropriate universal cover. We have

$$\tilde{\eta} = \sum_i \alpha_i dH_i + \beta_i dw_i$$

$$\frac{\partial \beta_i}{\partial w_k} = \frac{\partial \beta_k}{\partial w_i}.$$

Let us introduce a function  $\tilde{S}$  such that  $\beta_i = \frac{\partial \tilde{S}}{\partial w_i}$  and write:

$$\tilde{\eta}' = \eta - d\tilde{S} = \sum (\alpha_i - \frac{\partial \tilde{S}}{\partial H_i}) dH_i$$

$$\tilde{q}_i = \alpha_i - \frac{\partial \tilde{S}}{\partial H_i}$$

$$\omega = d\tilde{\eta}' = \sum d\tilde{q}_i \wedge dH_i.$$

#### 4. Arnol'd's Definition of the Actions

A system of Action-angles for  $(\underline{H}, \omega)$  can be defined following Arnol'd [A] when  $\underline{H}: V_R^{2m} \rightarrow R$  has compact fibers. In that case the connected components of  $\underline{H}^{-1}(\underline{c})$  are tori and we define an Action-angle coordinates system, relatively to a basis  $\gamma_j(\underline{c})$  of the homology of the real tori  $\underline{H}^{-1}(\underline{c})$ , as coordinates  $(\underline{p}, \underline{q})$  so that

$$i) \quad \omega = \sum_{j=1}^m dq_j \wedge dp_j$$

$$ii) \quad \underline{H} = \underline{H}(\underline{p}) \quad (\text{the first integrals do not depend on the angles})$$

$$iii) \quad \int_{\gamma_j(\underline{c})} dq_i = \delta_{ij}.$$

Basic references for Action-angles are Arnol'd [A], Nekhoroshev [N]. A nice example of R. Cushman of non-existence of global Action-angles is analyzed in [Du]. See also [F.M.]. Action-angles are very useful, for instance, for the quantization of classical mechanical systems [G-S].

For algebraically separable Hamiltonian systems, we have previously prepared the symplectic form

$$\omega = \sum_{j=1}^m \tilde{dq}_j \wedge dH_j.$$

So the Actions are determined as functions of  $\underline{H}$  by the periods

$$\psi_j(\underline{H}) = - \sum_{i,k} \int_{\gamma_j(\underline{H})} \frac{w_k^{i-1} dw_k}{v_k \sqrt{\phi(w_k)}}.$$

The periods are given by Abelian integrals of the first kind. Thus, their computation is a problem of Algebraic Geometry once we know explicitly how the Hyperelliptic curves  $\underline{C}_{\underline{c}}$  depend on  $\underline{H}$ .

## 5. Computation of the Angles

Proposition 5.1. The angles are given by

$$q_i = \sum_{j=1}^m T_{ij}(\underline{H}) \tilde{q}_j$$

where the matrix  $T: (T)_{ij} = T_{ij}$  is the inverse of  $\tilde{T}$ :

$$(\tilde{T})_{ij} = \int_{\gamma_j} \tilde{dq}_i.$$

In general, for a family of Hyperelliptic Riemann surfaces, it is not possible to compute explicitly the Abelian integrals as functions of the

parameters. But they are solutions of a Picard-Fuchs differential equation [D], [M].

The same situation appears for the Milnor fibration where the Gauss-Manin connection provides a Regular Singular Differential System which is very useful to study the local monodromy [D], [M], [Gr]. The integrals are in that case related to the Birkhoff series of Hamiltonian Systems [F], [V].

We make more explicit the computation of these differential equations for the Kowalevskaya Top.

#### 6. Picard-Fuchs Equations for the Kowalevskaya Top

We must, first of all, choose a system of generators for the homology of the real part of  $\underline{H}^{-1}(\underline{c})$ . The coordinates  $(w_1, w_2)$  represent a point on  $\underline{H}^{-1}(\underline{c})$ . If  $p, q; \gamma_1, \gamma_2$  are real, then  $x_1 = \bar{x}_2$  and (cf. (2.5))  $w_1, w_2 \in \mathbb{R}$  (in fact  $w_2 \in \mathbb{R}_+$ ). With  $(w_1, w_2)$  we can parametrize an element  $(z_1, w_1) - (z_2, w_2)$  of  $\text{Pic}_0(\underline{C}_{\underline{c}})$ .

The mapping  $h_{w_2} : \underline{C}_{\underline{c}} \rightarrow \text{Jac}(\underline{C}_{\underline{c}})$  defined by

$$h_{w_2} : (z_1, w_1) \mapsto (z_1, w_1) - (z_2, w_2),$$

where  $w_2$  is fixed, is a quasi-isomorphism.

So a system of generators for the Homology of  $\underline{H}^{-1}(\underline{c})$  can be prescribed by paths in the  $w_1$ -plane.

For our case, the polynomial  $\Phi$  (2.6) is of degree 5 and we know that  $+\sqrt{K}$  and  $-\sqrt{K}$  are two roots of  $\Phi$ . So we can explicitly compute the three other roots. They will be denoted  $(e_1, e_2, e_3)$ . The equation of the Discriminant locus of  $\underline{C}_{\underline{c}}$  is

$$(6.1) \quad \delta \delta' = 0$$

where

$$(6.2) \quad \delta = -K + (H - 2\ell^2)^2$$

$$\delta' = 4p^3 + 27q^2$$

with  $p = (c^2 - K - H^2/3)$  and  $q = \frac{2H}{3}(c^2 - K) - 2c^2\ell^2 + \frac{2H^2}{27}$ .

Thus the Discriminant locus is the union of a Parabola and of a singular sextic (with four singular points in its affine part).

We need a quick analysis of the respective localization of each roots of  $\phi$ . For instance if  $\ell = 0$  then (cf. (2.6))

$$(6.3) \quad \phi(w) = (w + H)(w^2 + c^2 - K).$$

If  $K$  is small enough, there is only one real root  $-H$ . If  $-H \ll -\sqrt{K}$ , let  $e$  be the real root of  $\phi$  which equals  $-H$  for  $\ell = 0$ ; for datas which are small perturbations of this situation, we get sign of  $\phi$



Hence, we can choose the segments  $[-\infty, e]$  and  $[\sqrt{K}, -\sqrt{K}]$  to have a basis of the real homology of  $\text{Jac}(\mathcal{C})$ . We are concerned with the four integrals

$$(6.4) \quad \begin{aligned} P_1 &= \int_{-\infty}^e \frac{dw}{\sqrt{\phi(w)}}, & P_2 &= \int_{-\infty}^e \frac{wdw}{\sqrt{\phi(w)}}, \\ Q_1 &= \int_{-\sqrt{K}}^{\sqrt{K}} \frac{dw}{\sqrt{\phi(w)}}, & Q_2 &= \int_{-\sqrt{K}}^{\sqrt{K}} \frac{wdw}{\sqrt{\phi(w)}} \end{aligned}$$

and their analytic extensions to any values of  $\underline{H} = (K, H)$ .

The Picard-Fuchs equation does not depend on the generator of the homology so we can restrict ourselves to the path  $\gamma$  defined by going from  $-\infty$  to  $e$  on the first sheet of  $\underline{\mathcal{C}}_{\underline{C}}$  then back from  $e$  to  $-\infty$  on the second sheet of  $\underline{\mathcal{C}}_{\underline{C}}$ . We have to deal with

$$P_i = \int_{\gamma} \frac{w^{i-1} dw}{\sqrt{\phi(w)}} \quad \text{for } i = 1, \dots, 4.$$

For the Kowalevskaya top there is a nice simplification of the monodromy computation because there is a vector field  $X_0$ :

$$(6.5) \quad X_0 = \frac{1}{2} \frac{\partial}{\partial H} - w \frac{\partial}{\partial K} - \frac{1}{2} \frac{\partial}{\partial w}$$

such that  $X_0 \cdot \phi(w) = 0$ .

From this and the relation

$$\begin{aligned}
 (6.6) \quad \int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi}} &= \int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi}} - \frac{1}{5} \int_{\gamma} \frac{\Phi'}{\sqrt{\Phi}} dw \\
 \int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi(w)}} &= -\frac{4}{5} H P_4 - \frac{3}{5} (c^2 - 2k) P_3 - \frac{2}{5} [(c^2 - 2k)H - 2c^2 k^2] P_2 \\
 &\quad + \frac{1}{5} K (c^2 - k) P_1.
 \end{aligned}$$

We get

$$\begin{aligned}
 (6.7) \quad \partial P_1 / \partial H &= 2 \partial P_2 / \partial K \\
 \partial P_i / \partial H &= 2 \partial P_{i+1} / \partial K - i P_{i-1} \quad \text{for } i = 1, 2, 3 \\
 \partial P_4 / \partial H &= \frac{8}{5} H \frac{\partial P_4}{\partial K} - \frac{6}{5} (c^2 - 2K) \frac{\partial P_3}{\partial K} - \frac{4}{5} [(c^2 - 2K)H - 2c^2 k^2] \frac{\partial P_2}{\partial K} \\
 &\quad + \frac{2}{5} K (c^2 - K) \frac{\partial P_1}{\partial K} - \frac{8}{5} P_3 + \frac{8}{5} H P_2 + \frac{1}{5} [2c^2 - 4K] P_1.
 \end{aligned}$$

This allows to separate simply the Picard-Fuchs equations into two parts involving respectively only the partial derivatives relatively to  $H$  or  $K$ .

Let us see now, for instance, the system for the partial derivatives relatively to  $H$ . We follow here the usual way [D].

We start with

$$(6.8) \quad \frac{\partial P_i}{\partial H} = -\frac{1}{2} \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi(w)}} \frac{\Phi'_H}{\Phi} dw = -\frac{1}{2} \int \frac{w^{i-1}}{\sqrt{\Phi}} \frac{\Phi'_H}{\Phi} dw$$



we denote by  $\chi$ ,  $\chi = w^2 - K$ , then  $\phi = 2\chi \cdot \phi$

$$(6.9) \quad \phi_H' = w^2 - K + c^2 = \chi + c^2.$$

We can check that

$$(6.10) \quad 1 = \lambda\phi + \mu\chi$$

with

$$(6.11) \quad \lambda = -\frac{w-(H-2\ell^2)}{c^2\delta}, \quad \mu = \frac{w^2+2\ell^2(w+H)-H^2+2\ell^2H+c^2}{c^2\delta}$$

so

$$(6.12) \quad \int_{\gamma} \frac{w^{i-1}}{\sqrt{\phi(w)}} \frac{\phi_H'}{\phi} dw = \int_{\gamma} \frac{w^{i-1}}{\sqrt{\phi(w)}} c^2 \lambda dw + \int_{\gamma} \frac{w^{i-1}}{\sqrt{\phi}} (1 + c^2 \mu) \frac{\chi}{\phi} dw.$$

Let us consider the first integral

$$(6.13) \quad - \int_{\gamma} \frac{w^i dw}{\delta \sqrt{\phi}} + \frac{(H-2\ell^2)}{\delta} \int_{\gamma} \frac{w^{i-1} dw}{\sqrt{\phi(w)}}$$

and so for  $i = 1, 2, 3$  we find it is

$$(6.14) \quad -\frac{1}{\delta} P_{i+1} + \frac{H-2\ell^2}{\delta} P_i.$$

For  $i = 4$ , with (6.6), we have

$$(6.15) \quad \frac{H-2\ell^2}{\delta} P_4 - \frac{1}{\delta} \int_{\gamma} \frac{w^4 dw}{\sqrt{\phi}} = \frac{9}{5} \frac{H-2\ell^2}{\delta} P_4 + \frac{3(c^2-2K)}{5\delta} P_3 + \frac{2((c^2-2K)H-2c^2\ell^2)}{5\delta} P_2 - \frac{1}{5} \frac{K(c^2-K)}{\delta} P_1.$$

The second integral in (6.12) is slightly harder to compute. First of all, we have:

$$(6.16) \quad 1 + c^2\mu = \frac{1}{\delta} [w^2 + 2\ell^2 w + 4\ell^4 - 2\ell^2 H + c^2 - K] = \frac{R}{\delta}.$$

With the notations of (6.3) and  $z = w + \frac{H}{3}$

$$(6.17) \quad \phi(z) = z^3 + Pz + q.$$

We write now

$$(6.18) \quad R = z^2 + uz + v$$

with

$$(6.19) \quad \begin{aligned} u &= -\frac{2H}{3} + 2\ell^2 \\ v &= \frac{H^2}{9} - \frac{8\ell^2 H}{3} + 4\ell^4 + c^2 - K. \end{aligned}$$

Then we have

$$(6.20) \quad R = L\phi + M\phi'$$

with

$$(6.21) \quad L = \frac{1}{\delta'}[-3rz - 3s]$$

$$M = \frac{1}{\delta'}[rz^2 + sz + t]$$

and

$$(6.22) \quad \begin{vmatrix} r \\ s \\ t \end{vmatrix} = \begin{vmatrix} 2p^2 & -9q & -6p \\ -3qp & -2p^2 & 9q \\ -9q^2 & -6pq & -4p^2 \end{vmatrix} \begin{vmatrix} 1 \\ -u \\ v \end{vmatrix}$$

and where  $\delta' = -4p^3 - 27q^2$  is the notation of (6.2).

So we can write

$$(6.23) \quad \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi(w)}} (1 + c^2 \mu) \frac{\chi}{\phi} dw = \frac{1}{\delta} \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi(w)}} (L\phi + M\phi') \frac{\chi}{\phi} dw$$

$$= \frac{1}{\delta} \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi}} L \chi dw + \frac{1}{\delta} \int_{\gamma} \frac{w^{i-1}}{\phi^{3/2}} M \chi' \phi' dw.$$

The first integral gives

$$\begin{aligned}
 (6.24) \quad & \frac{1}{\delta \delta'} \int_{\gamma} \frac{w^{i-1} (-3rw - rH - 3s)(w^2 - K + c^2)}{\sqrt{\Phi}} dw \\
 &= \frac{1}{\delta \delta'} \left[ -3r \int_{\gamma} \frac{w^{i+2}}{\sqrt{\Phi}} dw - (rH + 3s) \int_{\gamma} \frac{w^{i+1}}{\sqrt{\Phi}} dw \right. \\
 &\quad \left. - 3r(-K + c^2) \int_{\gamma} \frac{w^i}{\sqrt{\Phi}} dw - (rH + 3s)(c^2 - K) \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi}} dw \right].
 \end{aligned}$$

The second integral of (6.23) gives

$$\frac{1}{\delta} \int_{\gamma} \frac{2[(i-1)w^{i-2} M + w^{i-1} M'] \chi + w^{i-1} M \chi'}{\sqrt{\Phi}} dw$$

and then

$$\begin{aligned}
 (6.25) \quad & \frac{1}{\delta \delta'} \left[ 2(i+2)r \int_{\gamma} \frac{w^{i+2}}{\sqrt{\Phi}} dw + 2(i+1) \left( \frac{2Hr}{3} + s \right) \int_{\gamma} \frac{w^{i+1}}{\sqrt{\Phi}} dw \right. \\
 & \quad \left. + \left[ 2i \left( r \frac{H^2}{9} + \frac{sH}{3} + t \right) + 2(i+1)r(-K - c^2) \right] \int_{\gamma} \frac{w^i}{\sqrt{\Phi(w)}} dw \right. \\
 & \quad \left. + 2i \left( \frac{2Hr}{3} + s \right) (-K + c^2) \int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi}} dw \right. \\
 & \quad \left. + 2(i-1) \left[ r \frac{H^2}{9} + s \frac{H}{3} + t \right] (-K + c^2) \int_{\gamma} \frac{w^{i-2}}{\sqrt{\Phi}} dw \right].
 \end{aligned}$$

Now we have to express the integrals

$$\int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi}}, \quad \int_{\gamma} \frac{w^5 dw}{\sqrt{\Phi}}, \quad \int_{\gamma} \frac{w^6 dw}{\sqrt{\Phi}}$$

as a combination of the Abelian integrals of the first and the second kinds. This is a classical computation.

For the first one, we have the formula (6.6). For the others, we use the formula

$$(6.26) \quad \frac{b_0 w^m + b_1 w^{m-1} + \dots + b_m}{\sqrt{\Phi}} - \frac{b_0}{k + \frac{5}{2}} \frac{d}{dw} [w^k \sqrt{\Phi}] \frac{S}{\sqrt{\Phi}}$$

where  $S$  is of degree lower than  $m$ , choosing a  $k$  in such a way, that  $m = k + 4$ .

So if we denote

$$\Phi(w) = w^5 + \sigma_1 w^4 + \sigma_2 w^3 + \sigma_3 w^2 + \sigma_4 w + \sigma_5,$$

we find

$$(6.27) \quad \int_{\gamma} \frac{w^6 dw}{\sqrt{\Phi}} = -\frac{2}{9} [4\sigma_1 \int_{\gamma} \frac{w^5 dw}{\sqrt{\Phi}} + \frac{7}{2}\sigma_2 \int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi}} + 3\sigma_3 P_4 + \frac{5}{2}\sigma_4 P_3 + 2\sigma_5 P_2]$$

and

$$(6.28) \quad \int_{\gamma} \frac{w^5 dw}{\sqrt{\Phi}} = -\frac{2}{7} [3\sigma_1 \int_{\gamma} \frac{w^4 dw}{\sqrt{\Phi}} + \frac{5}{2}\sigma_2 P_4 + 2\sigma_3 P_3 + \frac{3}{4}\sigma_4 P_2 + \sigma_5 P_1].$$

Finally, if we put together (6.14), (6.15), (6.24), (6.25) and (6.6), (6.26), (6.27), we find explicitly the Picard-Fuchs equation in the form

$$(6.29) \quad \frac{\partial P_i}{\partial H} = \sum_{j=1}^4 = \left[ \frac{\alpha_{ij}}{\delta} + \frac{\beta_{ij}}{\delta \delta'} \right] P_j$$

where  $\alpha_{ij}$  are given by (6.14) and (6.15) and  $\beta_{ij}$  are derived from (6.24), (6.25). The  $\alpha_{ij}$  and  $\beta_{ij}$  are simple polynomial expressions of  $\underline{H} = (K, H)$ .

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