KLAUS HULEK

Projective geometry of elliptic curves

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PROJECTIVE GEOMETRY OF ELLIPTIC CURVES

Klaus HULEK

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Introduction

In this treatise we want to discuss some old and new topics concerning the projective geometry of elliptic curves embedded in some projective space $\mathbb{P}_n$. To be more precise, we want to study three different aspects of elliptic curves in $\mathbb{P}_n$, namely

1. The symmetries of elliptic normal curves
2. The Horrocks-Mumford vector bundle
3. The normal bundle of elliptic curves of degree 5.

These three subjects are closely related to each other and it is exactly this interrelation which we want to study. In order to give the reader some idea about what we intend to do, we want to outline the contents of the individual chapters.

In chapter I we shall study the symmetries of elliptic normal curves $C_n \subseteq \mathbb{P}_{n-1}$ of degree n. Translation by n-torsion points and involution of the curve C_n define $2n^2$ transformations of the projective curve C_n into itself, and they all lift to projective transformations of $\mathbb{P}_{n-1}$. We shall first define a suitable embedding (by means of specially chosen theta-functions which are products of translates of the Weierstrass σ-function), such that these symmetries take on a particularly simple form. This leads us to the Heisenberg group H_n in its Schrödinger representation. The material of this chapter is classically well known and the results can be traced back as far as to L. Bianchi [3] and A. Hurwitz [13].

If $n = p \geq 3$ is a prime number, then the symmetries of an elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$ lead to a special configuration of hyperplanes and projective subspaces of dimension $\frac{1}{2}(p-3)$. This configuration is of type $({p^2}_{p+1}, p(p+1)_p)$ and generalizes the classical configuration of the points of inflection of a plane cubic (the case $p = 3$). We shall study this configuration in chapter II. Although it can already be found in a paper by C. Segre [16], it was only fairly recently that I discovered this. I first heard about this configuration from W. Barth. In any case, our construction is quite different from C. Segre's. We shall briefly come back to C. Segre's point of view in chapter IX.

In chapter III we shall discuss elliptic normal curves of degree 3,4 and 5 in order to illustrate the results of the previous chapters.

Chapter IV deals with the quadric hypersurfaces which go through a fixed elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$. We shall first give a simple proof of a special case of a theorem of Mumford [15] on abelian varieties. We shall show that every elliptic curve $C_n \subseteq \mathbb{P}_{n-1}$ of degree $n \geq 4$ is the scheme-theoretic intersection of quadrics of rank 3. Then we shall use the symmetries of elliptic normal quintics $C_5 \subseteq \mathbb{P}_4$ to find quadratic equations for these curves. The rest of this chapter deals with the singular quadrics through a given elliptic normal quintic. The main result is, that there exists a 1-dimensional family of rank 3 quadrics through C_5 whose singular lines form a ruled surface F of degree 15. The surface F is birational to the second symmetric product S^2C of C_5 and we shall construct an explicit map between S^2C and F. The methods used here go back to Ellingsrud and Laksov [6]. Finally we shall briefly explain the relation between the curves C_5 and Shioda's modular surface S(5).

The normal bundle N_C of an elliptic normal quintic $C_5 \subseteq \mathbb{P}_4$ is the main object of chapter V. We shall first prove that N_C is indecomposable. It is then an easy consequence of Atiyah's classification [1] of vector bundles over an elliptic curve to describe the normal bundle N_C explicitly. We shall use this to give another proof of a vanishing result originally due to Ellingsrud and Laksov [6]. This vanishing result will be essential for chapter VIII.

In chapter VI we shall return to the Heisenberg group H_n and study its natural operation on the space of homogeneous forms of degree n in n variables. For every prime number $p = n \geq 3$ we shall determine the dimension of the space of invariant forms. In particular, if $p = 5$, we find that

$$\dim \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)) = 6$$

a result which was first proved by Horrocks and Mumford in [9] where it played an essential role in the study of the Horrocks-Mumford bundle. One can easily give a basis of the space of invariant quintic forms in terms of the configuration studied in chapter II.

For application in chapter VIII we shall finally construct a basis of the 3-dimensional space of invariant quintic forms whose corresponding hypersurfaces are singular along C_5.

In chapter VII we shall explain the relation between the Horrocks-Mumford bundle F on $\mathbb{P}_4$ and elliptic normal quintics. We shall prove, that, if $C_5 \subseteq \mathbb{P}_4$ is an elliptic normal quintic embedded as described in chapter I, then there exists a unique section $s \in \Gamma(F)$ whose zero-set is (scheme-theoretically) the tangent surface Tan C_5. In other words, the Horrocks-Mumford bundle can be reconstructed from the tangent developable of C_5 by means of the Serre-construction. This makes the statement of [9, p. 79(a)] precise and supplies a proof at the same time.

The main objective of chapter VIII is the study of the normal bundle of elliptic space curves of degree 5. Every such curve is the projection of an elliptic normal curve $C_5 \subseteq \mathbb{P}_4$. The normal bundle of these curves was first classified by Ellingsrud and Laksov in their paper [6] which was the starting point for this work. The main point is that their classification uses a certain 1-parameter family of quintic hypersurfaces $Y_M \subseteq \mathbb{P}_4$. (For a precise statement see (VIII. 2.7)). We shall first recall the results of Ellingsrud and Laksov and then turn to the hypersurfaces Y_M. To describe and understand this family was my original motive for this work. We shall see that the Y_M form a linear family of quintic hypersurfaces, whose base locus consists of the union of the tangent surface Tan C_5 and the ruled surface F which we have studied in chapter IV. This enables us to characterize the 2-dimensional space $U \subseteq \Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$ which belongs to the linear family Y_M. We shall first of all see that the elements of U are invariant under the Heisenberg group H_5. Moreover, U consists exactly of those H_5-invariant quintic forms which vanish on the tangent surface Tan C_5 and whose associated hypersurfaces are singular along C_5, i.e.

$$U = \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5)) \cap \Gamma_H(\mathcal{J}_C^2(5)).$$

We shall then relate this description to the Horrocks-Mumford vector bundle. Finally we shall describe U explicitly as a subspace of

$\Gamma_H(\mathcal{J}_C^2(5))$ using the basis of this space which we have found in chapter VI.

In chapter IX we shall discuss the normal bundle of elliptic space curves of degree 5 from a more geometric point of view. In order to say more precisely what we want to do let $C \subseteq \mathbb{P}_3$ be a smooth elliptic quintic. Then the maximal degree of a line subbundle of the normal bundle $N_{C/\mathbb{P}_3}$ of C in $\mathbb{P}_3$ is 10, and there always exists at least one such subbundle M. Our aim is to realize every maximal subbundle geometrically by a surface S of small degree which contains C. In order to define a subbundle M of degree 10, the surface S must have k singularities along C, where

$$k = 5 \cdot \deg S - 10.$$

We shall prove the following result: Every maximal subbundle $M \subseteq N_{C/\mathbb{P}_3}$ can be represented by a quartic surface $S \subseteq \mathbb{P}_3$, which is the projection of the complete intersection of two quadric hypersurfaces in $\mathbb{P}_4$, and which is singular in 10 points (counting multiplicities) of C. We shall also discuss special cases where a maximal subbundle M can be represented by a ruled cubic surface S which is singular in 5 points of C.

Throughout we shall work over the ground field $\mathbb{C}$. Many results, however, are also valid in positive characteristic.

I should like to thank all those mathematicians who discussed this subject with me. I am particularly indebted to W. Barth, J. Harris and A. Van de Ven, whose ideas and help were very important for me during the preparation of this manuscript.

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I. The elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$

In this chapter we want to collect some material concerning the symmetries of elliptic normal curves $C_n \subseteq \mathbb{P}_{n-1}$. Practically all of this was classically known. A very readable reference is an article by Bianchi [3] which was published in Mathematische Annalen in 1880. There Bianchi mainly treats the case of a plane cubic and of an elliptic quintic in $\mathbb{P}_4$ but he also looks at the general case of an elliptic normal curve of odd degree. The even degree case was treated by A. Hurwitz in [13]. Although at a first glance his formulas look somewhat different from ours, both treatments are, nevertheless, very similar.

1. Preliminaries

(I.1.1) Let C be an elliptic curve with fixed origin $\mathcal{O}$. Moreover, let

$$\Gamma = \{n_1\omega_1 + n_2\omega_2;\ n_1, n_2 \in \mathbb{Z}\}$$

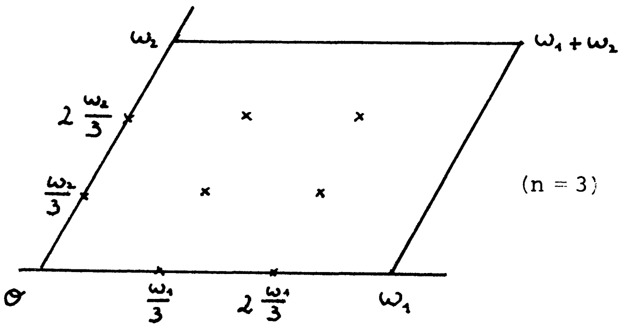
be a lattice such that $C = \mathbb{C}/\Gamma$. The n-torsion points of (the group) C are then given by

$$P_{pq} = \frac{-p\omega_1 + q\omega_2}{n} \quad (p, q \in \mathbb{Z}).$$

By abuse of notation we write $p, q \in \mathbb{Z}_n$. The n-torsion points form a subgroup $G_n \subseteq C$ and by identifying $-\frac{\omega_1}{n}$ with $(1,0)$ and $\frac{\omega_2}{n}$ with $(0,1)$ we fix an isomorphism

$$G_n \cong \mathbb{Z}_n \times \mathbb{Z}_n.$$

The following picture shows the group of 3-torsion points:



(I.1.2) Recall that the Weierstrass σ-function is defined by

$$\sigma(z) := z \prod_{\omega\in\Gamma-\{0\}} \left(1-\frac{z}{\omega}\right) e^{\left(\frac{z}{\omega}+\frac{z^2}{2\omega^2}\right)}$$

It is an entire function with simple zeroes exactly at the points of the lattice. Moreover, it is an odd function, i.e.

$$\sigma(-z) = -\sigma(z).$$

With respect to translation by ω_1 and ω_2 the following fundamental formulas hold:

$$(1) \qquad \sigma(z+\omega_1) = -e^{\eta_1\left(z+\frac{\omega_1}{2}\right)}\sigma(z)$$

$$(2) \qquad \sigma(z+\omega_2) = -e^{\eta_2\left(z+\frac{\omega_2}{2}\right)}\sigma(z).$$

Here η_1 and η_2 denote the period constants of the Weierstrass ζ-function. The above formulas can be combined to give the more general formula

$$(3) \qquad \sigma(z+k\omega_1+\ell\omega_2) = (-1)^{k\ell+k+\ell}\, e^{(k\eta_1+\ell\eta_2)\left(z+\frac{k\omega_1+\ell\omega_2}{2}\right)}\sigma(z)$$

Finally recall for later reference the important Legendre-Weierstrass relation which reads

$$(4) \qquad \eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

2. The symmetries of elliptic normal curves

Here we shall describe explicitly a set of functions embedding C as a linearly normal curve of given degree. These functions are chosen in such a way that the symmetries of the embedded curve take on a particularly simple form.

(I.2.1) For what follows we shall have to distinguish the case of odd and even degree. So let us first fix an odd integer $n \geq 3$. For $p,q \in \mathbb{Z}$ we set

$$\sigma_{pq}(z) := \sigma\left(z - \frac{p\omega_1 + q\omega_2}{n}\right).$$

Moreover we define the following constants

$$\omega := -e^{-\frac{n-1}{2}\frac{\eta_2\omega_1}{n}}, \quad \theta := e^{-\frac{\eta_1\omega_1}{2n}}$$

Finally we define functions x_m, $m \in \mathbb{Z}$ by setting

$$x_m(z) := \omega^m \theta^{m^2} e^{m\eta_1 z} \sigma_{m,0}(z) \cdot \ldots \cdot \sigma_{m,n-1}(z).$$

Next let $n \geq 4$ be an <u>even</u> integer. Then we set

$$\tilde{\sigma}_{pq}(z) := \sigma\left(z - \frac{p\omega_1 + q\omega_2}{n} - \frac{1}{2}\left(\omega_1 + \frac{\omega_2}{n}\right)\right).$$

Similarly as above we define constants

$$\tilde{\omega} := e^{-\frac{1}{2}(\eta_1\omega_1 + \eta_2\omega_1)}, \quad \tilde{\theta} := \theta = e^{-\frac{\eta_1\omega_1}{2n}}$$

which give rise to functions

$$x_m(z) := \tilde{\omega}^m \tilde{\theta}^{m^2} e^{m\eta_1 z} \tilde{\sigma}_{m,0}(z) \cdot \ldots \cdot \tilde{\sigma}_{m,n-1}(z).$$

We first note the following

(I.2.2) <u>Lemma</u>: <u>For a fixed integer</u> n <u>and for all</u> m <u>we have</u>

$$x_{n+m} = x_m.$$

We shall postpone the proof of this lemma to section (I.3). In any case we have now defined a set of n functions $\{x_m;\ m \in \mathbb{Z}_n\}$ which are a product of suitably adjusted σ-functions. The choice of these functions is justified by the following theorem.

(I.2.3) Theorem: The functions x_m define n linearly independent sections $x_m \in \Gamma(\mathcal{O}_C(n\mathcal{O}))$ and the map

$$z \mapsto (x_0(z) : \ldots : x_{n-1}(z))$$

embeds C as a linearly normal curve $C_n \subseteq \mathbb{P}_{n-1}$ of degree n. If $\varepsilon = e^{\frac{2\pi i}{n}}$ then the following formulas hold:

(i) $\quad x_m(-z) \sim (-1)^n x_{-m}(z)$

(ii) $\quad x_m\left(z - \frac{\omega_1}{n}\right) \sim x_{m+1}(z)$

(iii) $\quad x_m\left(z + \frac{\omega_2}{n}\right) \sim \varepsilon^m x_m(z).$

Here $\sim$ means that equality holds up to a common nowhere vanishing function independent of m. Moreover, at the origin one has

(iv) $\quad x_m(0) = (-1)^n x_{-m}(0).$

We shall prove this theorem in the next section.

(I.2.4) We want to rephrase the above result in a slightly different terminology. To do this we consider the vector space

$$V = \mathbb{C}^n$$

and denote its standard basis by $\{e_m\}_{m \in \mathbb{Z}_n}$. We define elements $\sigma, \tau \in GL(V)$ by

$$\sigma(e_m) := e_{m-1}$$

$$\tau(e_m) := \varepsilon^m e_m.$$

The automorphisms σ and τ do not commute but one finds

$$[\sigma, \tau] = \varepsilon \cdot \mathrm{id}_V.$$

Definition: The subgroup $H_n \subseteq GL(V)$ generated by σ and τ is called the Heisenberg group of dimension n. The representation defined by the inclusion is called the Schrödinger representation of H_n.

Remarks: (i) For a more general definition of the Heisenberg group and its Schrödinger representation see Igusa's book [14, p.10]. Instead of an arbitrary locally compact group we have just considered $\mathbb{Z}_n$ here.

(ii) The centre of the Heisenberg group H_n equals

$$\mu_n = \{\varepsilon^m \mathrm{id}_V ; m \in \mathbb{Z}\}$$

and the group H_n is a central extension

$$1 \to \mu_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 1$$

where σ and τ are mapped to $(1,0)$ and $(0,1)$ respectively. The order of H_n is n^3. In fact if $n = p \geq 3$ is a prime number then H_p is the unique group of order p^3 with exponent p.

(iii) The Schrödinger representation of H_n is an irreducible representation. Moreover if $n = p$ is a prime number it is not difficult to describe all irreducible representations of H_p. To do this let

$$\rho : H_p \to GL(V)$$

be the Schrödinger representation. We shall denote the corresponding H_p-module by V^1. The Schrödinger representation gives rise to p-1 irreducible H_p-modules V^i, $i = 1,\dots,p-1$ of dimension p in the following way:

$$\rho^i : H_p \to GL(V)$$

$$\rho^i(\sigma) := \sigma$$

$$\rho^i(\tau) := \tau^i.$$

In addition $\mathbb{Z}_p \times \mathbb{Z}_p$ and hence also H_p has p^2 characters which we shall denote by $V^{k\ell}$ with $k,\ell \in \mathbb{Z}_p$. Since the sum over the squares of dimensions of the irreducible representations described is

$$(p-1)p^2 + p^2 = p^3 = |H_p|$$

this is a complete list of irreducible H_p-modules.

In order to rephrase our result we finally consider the <u>involution</u>

$$\iota : V \to V$$
$$e_m \mapsto e_{-m}.$$

<u>Remark</u>: Note that the subgroup $\tilde{H}_n \subseteq GL(V)$ generated by H_n and ι has order $2n^3$. In fact it is a semi-direct product of H_n by $\mathbb{Z}_2 = \langle id, \iota \rangle$.

Now theorem (I.2.3) can be expressed as follows.

(I.2.5) <u>Theorem</u>: (i) <u>The involution</u> ι <u>leaves the elliptic normal curve</u> $C_n \subseteq \mathbb{P}_{n-1}$ <u>invariant (as a curve) and operates on it as the involution with respect to the origin</u> $\mathcal{O}$.

(ii) <u>Similarly the Heisenberg group</u> H_n <u>leaves the curve</u> $C_n \subseteq \mathbb{P}_{n-1}$ <u>invariant and operates on it by translation with</u> n-<u>torsion points</u>.

<u>Remark</u>: We can look at the situation from a somewhat more abstract point of view. The group $G_n \cong \mathbb{Z}_n \times \mathbb{Z}_n$ of n-torsion points operates on C by translation. This defines an operation of G_n on $\mathbb{P}(\Gamma(\mathcal{O}_C(n\mathcal{O})))$ in the following way

$$\sum_{i=1}^{n} P_i \mapsto \sum_{i=1}^{n} (P_i + P)$$

where + denotes addition on the elliptic curve. Identifying $\mathbb{P}_{n-1}$ with $\mathbb{P}(\Gamma(\mathcal{O}_C(n\mathcal{O})))$ we can say that we have extended the operation of the group G_n on C_n to projective space $\mathbb{P}_{n-1}$. We thus get a projective representation

$$\rho : G_n \to PGL(n, \mathbb{C}).$$

On the other hand the Heisenberg group H_n is a representation group of $G_n \cong \mathbb{Z}_n \times \mathbb{Z}_n$, i.e. each projective representation of G_n can be lifted to a linear representation of H_n and vice versa. Theorem (I.2.3) then tells us that the above projective representation ρ of G_n lifts to the Schrödinger representation of H_n.

3. Computations

In this section we want to give proofs for lemma (I.2.2) and theorem (I.2.3)

(I.3.1) Proof of lemma (I.2.2): This is a straightforward calculation which goes in the case of n odd as follows:

$$x_{m+n}(z) = \omega^{m+n}\theta^{(m+n)^2} e^{(m+n)\eta_1 z} \sigma_{m+n,0}(z)\cdot\ldots\cdot\sigma_{m+n,n-1}(z)$$

$$= \left(\omega^m\theta^{m^2} e^{m\eta_1 z}\right)\left(\omega^n\theta^{2mn+n^2} e^{n\eta_1 z}\right)\cdot$$

$$\sigma\left(z - \frac{(m+n)\omega_1}{n}\right)\cdot\ldots\cdot\sigma\left(z - \frac{(m+n)\omega_1 + (n-1)\omega_2}{n}\right)$$

$$= \left(\omega^m\theta^{m^2} e^{m\eta_1 z}\right)\left(\omega^n\theta^{2mn+n^2} e^{n\eta_1 z}\right).$$

$$\sigma\left(z - \frac{m\omega_1}{n} - \omega_1\right)\cdot\ldots\cdot\sigma\left(z - \frac{m\omega_1 + (n-1)\omega_2}{n} - \omega_1\right)$$

$$\overset{(3)}{=} x_m(z)\,\omega^n\theta^{2mn+n^2} e^{n\eta_1 z}(-1)^n$$

$$e^{-\eta_1\left(z - \frac{m\omega_1}{n} - \frac{\omega_1}{2}\right)}\cdot\ldots\cdot e^{-\eta_1\left(z - \frac{m\omega_1 + (n-1)\omega_2}{n} - \frac{\omega_1}{2}\right)}$$

$$= x_m(z)\,\omega^n\theta^{2mn+n^2}(-1)^n e^{-\eta_1\left(-m\omega_1 - \frac{1}{2}(n-1)\omega_2 - \frac{n}{2}\omega_1\right)}$$

$$= x_m(z)(-1)^{2n} e^{\frac{-(n-1)}{2}\eta_2\omega_1} e^{\frac{-\eta_1\omega_1}{2n}(2mn+n^2)}$$

$$e^{\eta_1[(m+\frac{n}{2})\omega_1+\frac{1}{2}(n-1)\omega_2]}$$

$$= x_m(z)\; e^{-\frac{1}{2}(n-1)[\eta_2\omega_1-\eta_1\omega_2]} \overset{(4)}{=} x_m(z)$$

The proof for n even is very much the same and since we are mostly concerned with the case n odd, anyway, we shall omit it.

(I.3.2) <u>Proof of theorem</u> (I.2.3): Again we shall limit ourselves to the case n odd since the case n even is very similar. The proof consists of several steps.

(i) We first have to see that the map

$$z \mapsto (x_0(z):\ldots:x_{n-1}(z))$$

is well defined. To see this we shall check that the functions x_m have the same automorphy factor.

$$x_m(z+k\omega_1+\ell\omega_2) = \omega^m\theta^{m^2} e^{m\eta_1(z+k\omega_1+\ell\omega_2)}\cdot$$

$$\cdot\sigma_{m,0}(z+k\omega_1+\ell\omega_2)\cdot\ldots\cdot\sigma_{m,n-1}(z+k\omega_1+\ell\omega_2)$$

$$= \omega^m\theta^{m^2} e^{m\eta_1 z} e^{m\eta_1(k\omega_1+\ell\omega_2)}$$

$$\sigma\left(z-\frac{m\omega_1}{n}+k\omega_1+\ell\omega_2\right)\cdot\ldots\cdot\sigma\left(z-\frac{m\omega_1+(n-1)\omega_2}{n}+k\omega_1+\ell\omega_2\right)$$

$$\overset{(3)}{=} x_m(z) e^{m\eta_1(k\omega_1+\ell\omega_2)}(-1)^{n(k\ell+k+\ell)}$$

$$e^{(k\eta_1+\ell\eta_2)\left(z-\frac{m\omega_1}{n}+\frac{k\omega_1+\ell\omega_2}{2}\right)}\cdot\ldots\cdot$$

$$e^{(k\eta_1+\ell\eta_2)\left(z-\frac{m\omega_1+(n-1)\omega_2}{n}+\frac{k\omega_1+\ell\omega_2}{2}\right)}$$

$$= x_m(z)(-1)^{(k\ell+k+\ell)} e^{m\eta_1(k\omega_1+\ell\omega_2)}$$

$$e^{(k\eta_1+\ell\eta_2)[nz - m\omega_1 - \frac{1}{2}(n-1)\omega_2 + \frac{n}{2}(k\omega_1+\ell\omega_2)]}$$

$$= x_m(z)(-1)^{(k\ell+k+\ell)} e^{m\ell[\eta_1\omega_2 - \eta_2\omega_1]}$$

$$e^{(k\eta_1+\ell\eta_2)[nz + \frac{n}{2}(k\omega_1+\ell\omega_2) - \frac{1}{2}(n-1)\omega_2]}$$

$$\overset{(4)}{=} x_m(z)(-1)^{(k\ell+k+\ell)} e^{(k\eta_1+\ell\eta_2)[nz + \frac{n}{2}(k\omega_1+\ell\omega_2) - \frac{1}{2}(n-1)\omega_2]}$$

This proves that the automorphy factor does not depend on m. In particular, this implies that the x_m define sections in the same line bundle L on C. Since

$$\sum_{i=0}^{n-1} \frac{m\omega_1 + i\omega_2}{n} \equiv 0 \bmod \Gamma.$$

We have $L = \mathcal{O}_C(n\mathfrak{o})$. By construction the x_m have no common zero and our map is well defined.

(ii) The next step is to check formulas (i) to (iv). We shall restrict ourselves to the most interesting case which is (i). This will give us a proof of (iv) at the same time.

$$x_m(-z) = \omega^m \theta^{m^2} e^{-m\eta_1 z} \sigma_{m,0}(-z)\cdot\ldots\cdot\sigma_{m,n-1}(-z)$$

$$= \omega^m \theta^{m^2} e^{-m\eta_1 z} \sigma\left(-z - \frac{m\omega_1}{n}\right)\cdot\ldots\cdot\sigma\left(-z - \frac{m\omega_1 + (n-1)\omega_2}{n}\right)$$

$$= (-1)^n \omega^m \theta^{m^2} e^{-m\eta_1 z} \sigma\left(z + \frac{m\omega_1}{n}\right)$$

$$\sigma\left(z+\frac{m\omega_1+\omega_2}{n}\right)\cdot\ldots\cdot\sigma\left(z+\frac{m\omega_1+(n-1)\omega_2}{n}\right)$$

$$= (-1)^n\omega^m\theta^{m^2}e^{-m\eta_1 z}\sigma\left(z-\frac{(n-m)\omega_1}{n}+\omega_1\right)$$

$$\sigma\left(z-\frac{(n-m)\omega_1+(n-1)\omega_2}{n}+\omega_1+\omega_2\right)\cdot\ldots\cdot\sigma\left(z-\frac{(n-m)\omega_1+\omega_2}{n}+\omega_1+\omega_2\right)$$

$$\overset{(3)}{=} \omega^m\theta^{m^2}e^{-m\eta_1 z}\sigma_{n-m,0}(z)\cdot\ldots\cdot\sigma_{n-m,n-1}(z)$$

$$e^{\eta_1[z-\frac{(n-m)}{n}\omega_1+\frac{\omega_1}{2}]}e^{(\eta_1+\eta_2)[z-\frac{(n-m)\omega_1+(n-1)\omega_2}{n}+\frac{\omega_1+\omega_2}{2}]}$$

$$\cdot\ldots\cdot e^{(\eta_1+\eta_2)[z-\frac{(n-m)\omega_1+\omega_2}{n}+\frac{\omega_1+\omega_2}{2}]}$$

$$= \omega^m\theta^{m^2}e^{-m\eta_1 z}\sigma_{n-m,0}(z)\cdot\ldots\cdot\sigma_{n-m,n-1}(z)$$

$$e^{\eta_1[nz+(m-\frac{n}{2})\omega_1]}e^{\eta_2[(n-1)z+(m-\frac{n}{2}-\frac{m}{n}+\frac{1}{2})\omega_1]}$$

$$= x_{n-m}(z)\,\omega^{2m-n}\theta^{-n^2+2mn}e^{\eta_2(n-1)z}e^{\eta_1\omega_1(m-\frac{n}{2})}$$

$$e^{\eta_2\omega_1(m-\frac{n}{2}-\frac{m}{n}+\frac{1}{2})}$$

$$= x_{-m}(z)e^{\eta_2(n-1)z}(-1)^{2m-n}e^{-\frac{(n-1)(2m-n)}{2}\frac{\eta_2\omega_1}{n}}$$

$$e^{\frac{n\eta_1\omega_1}{2}}e^{-m\eta_1\omega_1}e^{\eta_1\omega_1(m-\frac{n}{2})}e^{\eta_2\omega_1(m-\frac{n}{2}-\frac{m}{n}+\frac{1}{2})}$$

$$= - x_{-m}(z)e^{\eta_2(n-1)z} \sim -x_{-m}(z) \ .$$

(iii) In order to finish the proof we have to show that the x_m form a basis of $\Gamma(\mathcal{O}_C(n0))$. This is equivalent to saying that the image C_n of C spans $\mathbb{P}_{n-1}$. But the latter follows from (ii) together with the fact that the Schrödinger representation is irreducible, i.e. the orbit of any point under G_n spans $\mathbb{P}_{n-1}$.

II. An abstract configuration

Throughout this chapter let $n = p \geq 3$ be a prime number. We want to describe an abstract configuration in $\mathbb{P}_{p-1}$ which can be associated to the Heisenberg group H_p and the involution ι. In case $p=3$ this will turn out to be nothing but the well-known "Wendepunkts-konfiguration" of a plane cubic (see Chapter III).

I should mention here that I first heard about this configuration from W.Barth. We both did not know then that it bad been classically known and it was only by chance that I found C.Segre's paper [16] where he gives a different description of the same configuration.

1. The invariant hyperplanes

(I.1.1) Recall that there are exactly p+1 subgroups $\mathbb{Z}_p \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$. They are generated by (0,1) and $(1,\ell)$, $\ell \in \mathbb{Z}_p$ respectively. Before we can describe the configuration we shall first have to determine all hyperplanes $H \subseteq \mathbb{P}_{p-1}$ which are invariant (as hyperplanes) under one of these subgroups.

Let us start with the subgroup generated by (0,1). Clearly

$$\tau(H) = H$$

if and only if H is one of the hyperplanes

$$H_k = \{x_{-k} = 0\}.$$

Note that

$$H_k = \sigma^k(H_0).$$

Next we shall determine all hyperplanes H such that

$$\tau^\ell \sigma(H) = H.$$

We first remark that, because of σ, the equation of any such H must be of the form

$$x_0 + \sum_{m=1}^{p-1} \lambda_m x_m = 0.$$

It is easy to check that invariance under $\tau^{\ell}\sigma$ is equivalent to

$$\lambda_1^p = 1$$

$$\lambda_m = \lambda_1^m \cdot \varepsilon^{\frac{1}{2} m (m-\ell)} \qquad \text{for } m = 2,\ldots,p-1.$$

Hence we can set

$$\lambda_1 = \varepsilon^{-\frac{1}{2}(p-1)\ell - k}$$

for some $k \in \mathbb{Z}_p$ and the other λ_m's then become

$$\lambda_m = \varepsilon^{\frac{m}{2}(m-p)\ell - mk} .$$

It follows that the p hyperplanes

$$H_{k\ell} = \{\sum_{m=0}^{p-1} \varepsilon^{\frac{m}{2}(m-p)\ell - mk} x_m = 0\}; \qquad k = 0,\ldots,p-1$$

are exactly the hyperplanes invariant under $\tau^{\ell}\sigma$. Note that

$$H_{k\ell} = \tau^k (H_{0\ell}) .$$

We can sum this up as follows:

(I.1.2) Proposition: For each of the p+1 subgroups $\mathbb{Z}_p \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$ there are exactly p hyperplanes which are invariant under this subgroup.

2. The configuration

(I.2.1) At this point we want to return to the involution ι which was defined in (I.2). Recall that it is given by

$$\iota : \mathbb{C}^p \to \mathbb{C}^p$$

$$e_m \mapsto e_{-m} .$$

It defines a decomposition of $\mathbb{C}^p$ into eigenspaces, namely

$$\mathbb{C}^p = E^+ \oplus E^-$$

where

$$E^+ = \langle e_0, e_1 + e_{p-1}, \dots, e_{\frac{p-1}{2}} + e_{\frac{p+1}{2}} \rangle$$

$$E^- = \langle e_1 - e_{p-1}, \dots, e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}} \rangle$$

Clearly $\dim E^+ = \frac{1}{2}(p+1)$ and $\dim E^- = \frac{1}{2}(p-1)$.

(II.2.2) <u>Lemma</u>: $E^- = H_0 \cap H_{00} \cap \dots \cap H_{0,p-1}$.

<u>Proof</u>: (i) We shall first prove that E^- is contained in this intersection. Clearly $E^- \subseteq H_0$. Furthermore recall that $H_{0\ell}$ is given by

$$\sum_{m=0}^{p-1} \lambda_m^\ell x_m = 0$$

where

$$\lambda_m^\ell = \varepsilon^{\frac{1}{2} m\,(m-p)\,\ell}$$

The assertion now follows from

$$\lambda_{p-m}^\ell = \varepsilon^{\frac{1}{2}(p-m)\,(-m)\,\ell} = \varepsilon^{\frac{1}{2} m\,(m-p)\,\ell} = \lambda_m^\ell .$$

(ii) To finish the proof we have to show that $\frac{1}{2}(p+1)$ of the hypersurfaces H_0 and $H_{0\ell}$ are independent. To do this we have to examine the matrix

$$\begin{pmatrix} 1 & 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{p-1} \\ 0 & 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{p-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \lambda_{p-1} & \lambda_{p-1}^2 & \cdots & \lambda_{p-1}^{p-1} \end{pmatrix}$$

Using the well known formula for the Vandermonde determinant it will be sufficient to see that $\frac{1}{2}(p+1)$ of the λ_m's are different. Therefore we look at

$$\lambda_{2k} = \varepsilon^{k(2k-p)} = \varepsilon^{2k^2}$$

It suffices to see that

$$2k^2 \not\equiv 2\ell^2 \mod p$$

for $k \neq \ell \in \{0,\ldots,\frac{p-1}{2}\}$. But this is clearly so since otherwise

$$p \mid 2(k-\ell)(k+\ell)$$

which is impossible. This finishes the proof.

Remark: It follows immediately from the formulae given in (II.2.1) that E^- is not contained in any of the hyperplanes H_k or $H_{k\ell}$ unless $k = 0$.

Our next step is to define for all $k,\ell \in \mathbb{Z}_p$ subspaces

$$E_{k\ell} := \tau^k \sigma^\ell (E^-).$$

(II.2.3) Lemma: $E_{k\ell} \cap E_{k'\ell'} = 0$ if $(k,\ell) \neq (k',\ell')$.

Proof: It will be enough to show that

$$E_{00} \cap E_{-k,-\ell} = 0$$

for $(k,\ell) \neq (0,0)$. To see this assume that

$$x = \sum_{m=0}^{p-1} x_m e_m \in E_{00} \cap E_{-k,-\ell}$$

Since $x \in E_{00}$ it follows that

$$(1) \qquad x_m = -x_{-m}.$$

On the other hand, since $x \in E_{-k,-\ell}$ it follows that $\tau^k \sigma^\ell(x) \in E_{00}$. This is equivalent to

$$(2) \qquad x_{m+\ell}\, \varepsilon^{2mk} = -x_{-m+\ell} .$$

If $\ell = 0$ and $k \neq 0$ it follows immediately from (1) and (2) that $x = 0$. Hence assume $\ell \neq 0$. By (1) it follows that $x_0 = 0$. Setting $m = -\ell$ in (2) this implies $x_{2\ell} = 0$ which because of (1) leads to $x_{-2\ell} = 0$. Using (2) again, this time for $m = -3\ell$ we find $x_{4\ell} = 0$. Proceeding in this way one finds $x = 0$.

(II.2.4). We can now sum up the situation as follows: We have found $p(p+1)$ hyperplanes which we have denoted by H_k and $H_{k\ell}$ respectively. Moreover, we have constructed p^2 subspaces $E_{k\ell}$ of dimension $\frac{1}{2}(p-1)$. Now each of the spaces $E_{k\ell}$ is contained in exactly $p+1$ of the hyperplanes and is in fact their common intersection. On the other hand, each of the hyperplanes H_k and $H_{k\ell}$ contains exactly p of the subspaces $E_{k\ell}$ and is indeed spanned by any two of them. In particular we can say:

(II.2.5) <u>Proposition</u>: <u>The</u> $p(p+1)$ <u>hyperplanes</u> H_k <u>and</u> $H_{k\ell}$ <u>together with the</u> p^2 <u>subspaces</u> $E_{k\ell}$ <u>form a configuration of type</u> $(p^2{}_{p+1}\ ,\ p(p+1)_p)$.

(II.2.6) So far we have said nothing about the relation of this configuration to the elliptic normal curve C_p. Because of

$$x_m(O) = -x_{-m}(O)$$

it follows that the (projective) space $E_{OO} = E^-$ contains the origin $\mathcal{O}$. Hence each of the subspaces $E_{k\ell}$ goes through exactly one of the p-torsion points of C_p, namely $P_{k\ell} = \frac{-k\omega_1 + \ell\omega_2}{p}$. In fact this is the only point of intersection of $E_{k\ell}$ with C_p.

Since the hyperplanes H_k and $H_{k\ell}$ are invariant under some subgroup $\mathbb{Z}_p \subseteq G_p$ it follows that they each contain exactly p of the p-torsion points. On the other hand the hyperplanes are determined by these points. The exact relation is given by

$$H_k \ni \{mP_{01} + kP_{10}; m \in \mathbb{Z}_p\} = \{\frac{-k\omega_1 + m\omega_2}{p} \ ; \ m \in \mathbb{Z}_p\}$$

$$H_{k\ell} \ni \{mP_{1\ell} + kP_{01} \ ; \ m \in \mathbb{Z}_p\} = \{\frac{m(-\omega_1 + \ell\omega_2) + k\omega_2}{p}; \ m \in \mathbb{Z}_p\} .$$

We can summarize this as follows:

(II.2.7) <u>Proposition</u>: <u>Each of the hyperplanes</u> H_k <u>and</u> $H_{k\ell}$ <u>intersects</u> C_p <u>in exactly</u> p <u>of the</u> p-<u>torsion points. The union of all</u> p <u>hyperplanes belonging to a fixed subgroup</u> $\mathbb{Z}_p \subseteq G_p$ <u>contains all</u> p^2 <u>hyperosculating points of</u> C_p.

3. <u>The fundamental polyhedra</u>

(II.3.1) In this section we want to discuss some polyhedra which arise naturally from the above configuration and which will be useful later.

<u>Definition</u>: The <u>fundamental polyhedron</u> associated to a subgroup $\mathbb{Z}_p \subseteq G_p$ is the union of all hyperplanes which are invariant under this subgroup.

<u>Remarks</u>: (i) By what we have said before there are exactly p+1 fundamental polyhedra and each of them is the union of p

hyperplanes. The homogeneous forms describing these polyhedra are

$$Q_{-1} = \prod_{k=0}^{p-1} x_k$$

$$Q_\ell = \prod_{k=0}^{p-1} \left(\sum_{m=0}^{p-1} \varepsilon^{\frac{m}{2}(m-p)\ell-mk} x_m \right) ; \quad \ell = 0,\dots,p-1 .$$

(ii) Another way of describing the fundamental polyhedra is as follows. Since the operation of G_p on the space of hyperplanes is irreducible it follows that the p hyperplanes forming a fundamental polyhedron are independent. Hence any p-1 of them intersect in a point. In this way we get a p-simplex whose vertices are exactly the fixed points under the subgroup $\mathbb{Z}_p$ belonging to the fundamental polyhedron. The fundamental polyhedron itself consists of the (p-2)-dimensional faces of this simplex. For example if $\mathbb{Z}_p = \{\tau^m ; m \in \mathbb{Z}_p\}$ then the p-simplex in question is nothing but the simplex of reference.

(iii) Note that by (II.2.7) each fundamental polyhedron intersects the elliptic normal curve C_p exactly in the p^2 points of p-torsion.

(II.3.2) <u>Proposition</u>: <u>The</u> p+1 <u>homogeneous forms</u> Q_ℓ <u>are invariant under the operation of the Heisenberg group</u> H_p.

<u>Proof</u>: The assertion is clear for Q_{-1}. To prove it for Q_ℓ, $\ell = 0,\dots,p-1$ we show the following:

(i) $$\tau(Q_\ell) = \tau\left(\prod_{k=0}^{p-1} \sum_{m=0}^{p-1} \varepsilon^{\frac{m}{2}(m-p)\ell - mk} x_m \right)$$

$$= \prod_{k=0}^{p-1} \left(\sum_{m=0}^{p-1} \varepsilon^{\frac{m}{2}(m-p)\ell - m(k+1)} x_m \right)$$

$$= \prod_{k=1}^{p} \left(\sum_{m=0}^{p-1} \varepsilon^{\frac{m}{2}(m-p)\ell - mk} x_m \right)$$

$$= Q_\ell .$$

$$\text{(ii)} \qquad \tau^{\ell}\sigma(Q_{\ell}) = \tau^{\ell}\sigma\left(\prod_{k=0}^{p-1}\sum_{m=0}^{p-1}\varepsilon^{\frac{m}{2}(m-p)\ell-mk}x_m\right)$$

$$= \prod_{k=0}^{p-1}\left(\sum_{m=0}^{p-1}\varepsilon^{\frac{m}{2}(m-p)\ell-mk-(m-1)\ell}x_{m-1}\right)$$

$$= \prod_{k=0}^{p-1}\left(\sum_{m=-1}^{p-2}\varepsilon^{(\frac{m+1}{2})(m+1-p)\ell-(m+1)k-m\ell}x_m\right)$$

$$= \prod_{k=0}^{p-1}\left(\varepsilon^{(\frac{\ell}{2}(1-p)-k)}\cdot\sum_{m=-1}^{p-2}\varepsilon^{\frac{m}{2}(m-p)\ell-mk}x_m\right)$$

$$= Q_{\ell}.$$

Since τ and $\tau^{\ell}\sigma$ generate the Heisenberg group H_p this concludes the proof.

III. Examples

In this chapter we want to illustrate the results of the preceding two chapters by looking at elliptic normal curves of low degree. We shall treat the cases $n = 3,4$ and 5. In order to complete the picture in the case of the elliptic normal quintic we shall also state some results here whose proofs will be given in chapters IV and V.

1. The plane cubic

(III.1.1) Let $C_3 \subseteq \mathbb{P}_2$ be a plane cubic curve embedded as described in chapter I. Then C_3 is invariant (as a curve) under H_3 and the involution ι and the same holds for its equation - at least up to a multiplicative constant. Now the only homogeneous forms of degree 3 in 3 variables which have this property are of the form

$$a(x_0^3 + x_1^3 + x_3^3) + bx_0x_1x_2 = 0.$$

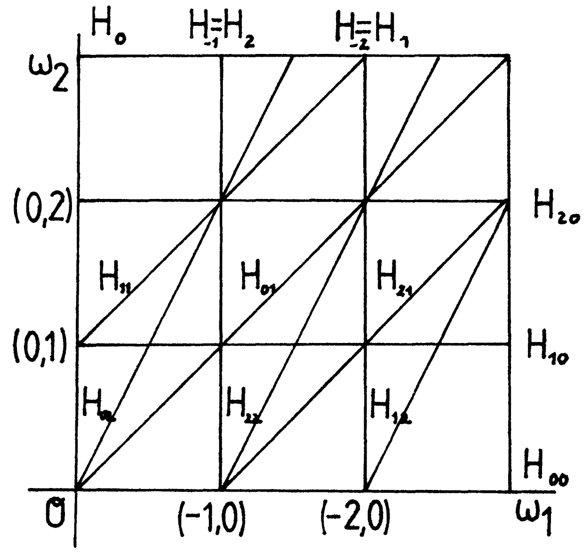
Since C_3 is irreducible it follows that $a \neq 0$ and the equation of C_3 is given by

$$x_0^3 + x_1^3 + x_2^3 + \frac{b}{a} x_0x_1x_2 = 0.$$

Hence we see that to embed the curve as described in chapter I implies that C_3 is already in Hesse normal form.

(III.1.2) Next we want to describe the configuration determined by H_3 and ι. Each of the 4 subgroups $\mathbb{Z}_3 \subseteq G_3$ gives rise to 3 invariant lines, i.e., to a triangle. Each of these lines intersects C_3 in 3 points of inflection and each of the triangles contains all 9 points of inflection. The 9 subspaces $E_{k\ell}$ have (affine) dimension 1 hence coincide with the 9 points of inflection. So the configuration we get is exactly the well known "Wendepunktskonfiguration" of a plane cubic. It is of type $(9_4, 12_3)$.

In the following picture we want to describe how the invariant lines are related to the 3-torsion points.



2. <u>The elliptic normal quartic</u>

(III.2.1) Let $C_4 \subseteq \mathbb{P}_3$ be an elliptic normal quartic embedded as before. Any such curve is the complete intersection of two quadric surfaces and the first point we want to make is that we can use the symmetries of the curve to determine this pencil of quadrics. To do this we look at the H_4-module

$$S^2V^* = H^0(\mathcal{O}_{\mathbb{P}_3}(2)).$$

As an H_4-module it has a decomposition

$$S^2V^* = \bigoplus_{i=1}^{5} V_i$$

where

$$V_1 = \langle x_0^2 + x_2^2 \, , \, x_1^2 + x_3^2 \rangle$$

$$V_2 = \langle x_0^2 - x_2^2 \, , \, x_1^2 - x_3^2 \rangle$$

$$V_3 = \langle x_1 x_3 \, , \, x_0 x_2 \rangle$$

$$V_4 = \langle x_0x_1 + x_2x_3 \, , \, x_1x_2 + x_0x_3 \rangle$$

$$V_5 = \langle x_0x_1 - x_2x_3 \, , \, x_1x_2 - x_0x_3 \rangle$$

Now $V_1 \cong V_3$ as H_4-modules whereas no other two of the direct summands are isomorphic. Clearly the pencil of quadrics which cuts out C_4 must be invariant under H_4. We first want to exclude that it is either V_2, V_4 or V_5. It cannot be V_2 since C_4 is not a plane curve. To exclude V_4 and V_5 it is sufficient to remark that $x_i(\frac{1}{2}(\omega_1 + \frac{\omega_2}{4})) = 0$ if and only if $i = 0$. Hence we find that

$$\Gamma(\mathcal{J}_C(2)) \subseteq V_1 \oplus V_3 \ .$$

Moreover, as an H_4-module $\Gamma(\mathcal{J}_C(2))$ is isomorphic to V_1 and V_3. Hence it has a basis Q_0, Q_1 with $\tau(Q_0) = Q_0$ and $Q_1 = \sigma^{-1}(Q_0)$. It follows that C_4 is the intersection of the quadrics

$$Q_0 = x_0^2 + x_2^2 + 2ax_1x_3$$

$$Q_1 = x_1^2 + x_3^2 + 2ax_0x_2$$

where

$$a = -\frac{x_2^2(z_0)}{2x_1(z_0)x_3(z_0)} \quad \text{with} \quad z_0 = \frac{\omega_1}{2} + \frac{\omega_2}{8}.$$

Note that $a \neq 0, \infty, \pm 1, \pm i$.

(III.2.2) Although we cannot associate a configuration similar to the Wendepunktskonfiguration to the curve C_4 we have still got an interesting geometric picture which we want to describe next. To do this note that the pencil of quadrics

$$Q = \lambda Q_0 + Q_1$$

counts four singular quadrics which are given by

$$\lambda = \pm\frac{1}{a}, \pm a.$$

The vertices of these quadric cones can be easily computed to be

$$S_1 = (0:1:0:-1)$$

$$S_2 = (0:1:0:1)$$

$$S_3 = (1:0:-1:0)$$

$$S_4 = (1:0:1:0).$$

Next note that the involution $\iota : e_m \mapsto e_{-m}$ defines a decomposition

$$V = E^- \oplus E^+$$

where

$$E^- = \langle e_1 - e_3 \rangle$$

$$E^+ = \langle e_0, e_2, e_1 + e_3 \rangle.$$

We see that S_1 is just the point defined by E^- whereas the other vertices span the plane determined by E^+. This has the following consequence. Projection from S_1 defined a 2:1 map

$$\pi : C_4 \to \overline{C} \cong \mathbb{P}_1$$

where $\overline{C}$ is a plane conic. The projection map π induces an isomorphism

$$\pi^* : \Gamma(\mathcal{O}_{\mathbb{P}_1}(2)) \cong E^+.$$

From this it follows that the branch points of π are the 2-torsion points of C. In other words, the vertex S_1 lies on the tangents through the origin and the points related to $\mathcal{O}$ by half-periods. In this way the 16 tangents at the 4-torsion points can be grouped into 4 sets of 4 lines all meeting in one of the vertices.

3. The elliptic normal quintic

(III.3.1) Unlike in the cases before an elliptic normal quintic $C_5 \subseteq \mathbb{P}_4$ is no longer a complete intersection. But it is still true that C_5 is cut out by quadric hypersurfaces. We shall see in the next chapter that

$$h^0(\mathcal{J}_C(2)) = 5$$

i.e. that there are 5 independent quadrics through C_5 and, moreover, that their common intersection is (scheme-theoretically) the elliptic normal quintic curve. As in the case of an elliptic quartic one can again make use of the symmetries of C_5 to determine the space

$$W := H^0(\mathcal{J}_C(2))$$

As a result one finds the following basis of W:

$$\bar{Q}_0 = x_0^2 + ax_2x_3 - \frac{1}{a}x_1x_4$$

$$\bar{Q}_1 = x_1^2 + ax_3x_4 - \frac{1}{a}x_2x_0$$

$$\bar{Q}_2 = x_2^2 + ax_4x_0 - \frac{1}{a}x_3x_1$$

$$\bar{Q}_3 = x_3^2 + ax_0x_1 - \frac{1}{a}x_4x_2$$

$$\bar{Q}_4 = x_4^2 + ax_1x_2 - \frac{1}{a}x_0x_3$$

where

$$a = -\frac{x_1(0)}{x_2(0)} .$$

(III.3.2) Next we want to discuss the configuration associated to the Heisenberg group H_5 and the involution ι. For each of the 6 subgroups $\mathbb{Z}_5 \subseteq \mathbb{Z}_5 \times \mathbb{Z}_5$ we get five invariant hyperplanes which together form the 6 fundamental pentahedra. On the other hand the 25 subspaces $E_{k\ell}$ define lines $L_{k\ell}$ whose equations are

$$x_{-k} = x_{1-k} + \varepsilon^{2\ell}x_{4-k} = \varepsilon^{\ell}x_{2-k} + x_{3-k} = 0.$$

Together they form a configuration of type $(25_6, 30_5)$.

Recall that we had seen in (II.3.2) that the quintic forms associated to the fundamental pentahedra

$$Q_{-1} = \prod_{k=0}^{4} x_k$$

$$Q_\ell = \prod_{k=0}^{4} \left(\sum_{m=0}^{4} \varepsilon^{\frac{m}{2}(m-5)\ell - mk} x_m \right) \qquad (\ell = 0,\dots,4)$$

are invariant under the Heisenberg group H_5. We shall see in chapter V that all quintic forms which are H_5-invariant form an affine 6-dimensional space

$$\Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)) \subseteq \Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$$

and that the quintic forms associated to the fundamental pentahedra form a basis of this space. We shall also see that the common intersection of these quintics are the 25 skew lines $L_{k\ell}$. We can summarize this as follows:

(III.3.3) Proposition: The six fundamental pentahedra determine a basis of the space $\Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$ of invariant quintic forms and the 25 skew lines $L_{k\ell}$ are the common intersection of these quintics.

(III.3.4) We want to conclude this section with a remark relating the curves C_4 and C_5. If one projects C_5 from the origin one gets an elliptic normal curve $C_4 \subseteq \mathbb{P}_3$. This projection is compatible with the involution. It maps $L_{00} = \mathbb{P}(E^-)$ to the point S_1 in $\mathbb{P}_3$ which is the 1-dimensional eigenspace of ι belonging to the eigenvalue -1. Hence S_1 is the vertex of a quadric cone through C_4. But this means that L_{00} is the singular line of a rank 3 quadric in $\mathbb{P}_4$ which contains C_5. Corresponding to the four quadric cones through C_4 there are 4 such lines. L_{00} is distinguished among these by the fact that it is contained in the osculating plane of C_5 at $\mathcal{O}$. An analogous statement holds, of course, for the other lines $L_{k\ell}$, too.

IV. Elliptic normal curves and quadric hypersurfaces

In this chapter we want to study the quadric hypersurfaces through an elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$. To start with we shall prove that the curves C_n are projectively normal. This is well known. For lack of a suitable reference, however, we want to include a proof. Next we shall show that an elliptic normal curve of degree at least 4 is the scheme-theoretic intersection of quadrics of rank 3. This is a special case of a result of Mumford on abelian varieties [15, theorem 10]. Here we shall give a simple proof for the case of elliptic curves.

In section 2 we shall return to the case $n = 5$ and determine the space of quadrics through an elliptic normal quintic.

Sections 3 and 4 finally deal with the singular quadrics through an elliptic normal curve of degree 5. Some of the results here are new, in particular the description of the locus of the singular lines of rank 3 quadrics through C_5 and its connection with the 25 skew lines $L_{k\ell}$.

1. The space of quadrics through C_n

As usual we denote by $C_n \subseteq \mathbb{P}_{n-1}$ an elliptic normal curve of degree n.

(IV.1.1) Lemma: *Let $P_1,\dots,P_k$ be k different points on C_n. Then these points are independent if $k \le n-1$, i.e. they span a subspace of dimension $k-1$.*

Proof: It will be enough to prove the lemma for $k = n-1$. Hence assume that the points $P_1,\dots,P_{n-1}$ lie in some subspace of dimension n-3. Then if $P \in C_n$ is any other point different from $P_1,\dots,P_{n-1}$ we can always choose a hyperplane H such that it cuts out the divisor $P + P_1 + \dots + P_{n-1}$ on C_n. This is a contradiction.

(IV.1.2) Proposition: *Every elliptic normal curve C_n of degree $n \ge 3$ is projectively normal.*

Proof: The assertion is clear for $n = 3,4$. Hence we assume $n \geq 5$. We have to show that the map

$$\Gamma(\mathcal{O}_{\mathbb{P}_{n-1}}(\ell)) \to \Gamma(\mathcal{O}_{C_n}(\ell))$$

is surjective for all $\ell \geq 0$. We want to give a proof by induction using the fact that C_n is linearly normal. To do this we choose a general hyperplane H such that the set

$$\Gamma := C_n \cap H$$

consists of n different points $P_1,\ldots,P_n$. Then there is a commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{J}_C(\ell) & \to & \mathcal{O}_{\mathbb{P}_{n-1}}(\ell) & \to & \mathcal{O}_C(\ell) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{J}_C(\ell+1) & \to & \mathcal{O}_{\mathbb{P}_{n-1}}(\ell+1) & \to & \mathcal{O}_C(\ell+1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{J}_\Gamma(\ell+1) & \to & \mathcal{O}_H(\ell+1) & \to & \sum_{i=1}^{n} \mathbb{C}_i & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{J}_\Gamma$ denotes the ideal sheaf of Γ in H. To prove the proposition it will then be enough to show that

$$h^1(\mathcal{J}_\Gamma(\ell+1)) = 0 \qquad \text{for} \quad \ell \geq 1.$$

But that follows if we can prove that the map

$$\Gamma(\mathcal{O}_H(\ell+1)) \to \sum_{i=1}^{n} \mathbb{C}_i$$

is surjective for $\ell \geq 1$. To see this, however, it will be enough to prove it for $\ell = 1$. To do so we want to show that for each P_i there exists a quadric Q in H which goes through all points of Γ with the exception of P_i. To simplify our notation we take $i = n$. Then the points $P_1,\dots,P_{n-2}$ span a hyperplane $H' \subseteq H$ which does not contain P_n. On the other hand, we can certainly choose a hyperplane H" through P_{n-1} which does not contain P_n. Therefore, we can choose Q as the union of H' and H".

Mumford [15, theorem 10] has shown that every abelian variety embedded by a complete linear system $|nH|$ where H is ample and $n \geq 4$ is the scheme-theoretic intersection of quadrics of rank ≤ 4. For elliptic curves a stronger statement holds:

(IV.1.3) Theorem: Every elliptic normal curve $C_n \subseteq \mathbb{P}_{n-1}$ of degree $n \geq 4$ is the scheme-theoretic intersection of the quadrics of rank 3 which contain it.

Remark: Since C_n spans $\mathbb{P}_{n-1}$ there are no quadrics of rank ≤ 2 containing the curve C_n.

Proof: We shall proceed by induction on n. For $n = 4$ the statement is true since every elliptic quartic $C_4 \subseteq \mathbb{P}_3$ is the complete intersection of two quadric cones (see (IV.1.2) and (III.2.2)). Now assume $n \geq 5$. We shall first show that C_n is a set-theoretic intersection of rank 3 quadrics. To do this let $P \notin C_n$ be an arbitrary point not lying on C_n. We can choose a point $P_o \in C_n$ such that the line $\overline{PP_o}$ is neither a secant nor a tangent of C_n. Otherwise projection from P_o would map C_n onto a curve $\bar{C} \subseteq \mathbb{P}_{n-2}$ of degree $\leq \frac{n}{2}$. Since $\bar{C}$ spans $\mathbb{P}_{n-2}$ this implies

$$\frac{n}{2} \geq n - 2$$

which is a contradiction to $n \geq 5$. Projecting from P_o we get an elliptic normal curve $C_{n-1} \subseteq \mathbb{P}_{n-2}$. The image $\bar{P}$ of P does not lie on C_{n-1}. By our induction hypothesis there is a rank 3 quadric Q' through C_{n-1} which does not contain $\bar{P}$. Let Q be the cone over Q' with vertex P_o. Then $C_{n-1} \subseteq Q$ but $P \notin Q$.

It remains to show that the rank 3 quadrics separate tangents. Let $P \in C_n$ be an arbitrary point on the curve C_n and let L be a line through P which is not tangent to C_n at P. Next choose a point $P_o \in C_n$ which does not lie on L. Projecting from P_o we get an elliptic normal curve $C_{n-1} \subseteq \mathbb{P}_{n-2}$. Let $\bar{P}$, resp. $\bar{L}$, be the image of P, resp. L. Using once more our induction hypothesis we can choose a quadric Q' through C_{n-1} such that Q' and the line $\bar{L}$ intersect transversally at $\bar{P}$. The cone Q over Q' with vertex P_o therefore intersects the line L at P transversally and we are done.

2. Quadratic equations for C_5

In this section we want to return to the case $n=5$ and determine the quadric hypersurfaces through an elliptic normal quintic thus providing a proof of our statemtnt in (III.3.1). Since for the rest of this chapter we shall restrict ourselves to the case of an elliptic normal quintic we shall frequently write C instead of C_5.

The following result can already be found in Bianchi's paper [3]:

(IV.2.1) Proposition: There exists a 5-dimensional space of quadric hypersurfaces through an elliptic normal quintic C_5. A basis of this space is given by

$$\bar{Q}_0 = x_0^2 + ax_2x_3 - \frac{1}{a}x_1x_4$$

$$\bar{Q}_1 = x_1^2 + ax_3x_4 - \frac{1}{a}x_2x_0$$

$$\bar{Q}_2 = x_2^2 + ax_4x_0 - \frac{1}{a}x_3x_1$$

$$\overline{Q}_3 = x_3^2 + ax_0x_1 - \frac{1}{a}x_4x_2$$

$$\overline{Q}_4 = x_4^2 + ax_1x_2 - \frac{1}{a}x_0x_3$$

<u>where</u>

$$a = -\frac{x_1(0)}{x_2(0)}$$

<u>Proof</u>: (i) It follows from the exact sequence

$$0 \to \mathcal{J}_C(2) \to \mathcal{O}_{\mathbb{P}_4}(2) \to \mathcal{O}_C(2) \to 0$$

and the fact that C_5 is projectively normal that

$$h^0(\mathcal{J}_C(2)) = h^0(\mathcal{O}_{\mathbb{P}_4}(2)) - h^0(\mathcal{O}_C(2))$$
$$= 15 - 10 = 5.$$

(ii) To determine the space $H^0(\mathcal{J}_C(2))$ we look at the H_5-module

$$S^2V^* = H^0(\mathcal{O}_{\mathbb{P}_4}(2)).$$

As an H_5-module S^2V^* has a decomposition

$$S^2V^* = \bigoplus_{i=0}^{2} V_i$$

where

$$V_0 = \langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2\rangle$$

$$V_1 = \langle x_2x_3, x_3x_4, x_4x_0, x_0x_1, x_1x_2\rangle$$

$$V_2 = \langle x_1x_4, x_2x_0, x_3x_1, x_4x_2, x_0x_3\rangle.$$

Note that the H_5-modules V_i are all mutually isomorphic. Indeed with respect to the given bases the operation of H_5 is given by

$$\sigma(e_i) = e_{i-1}$$

$$\tau(e_i) = \varepsilon^{-2i} e_i .$$

It follows that the H_5-module $H^0(\mathcal{J}_C(2))$ itself must be isomorphic to the V_i. Therefore we can choose a basis $\bar{Q}_0, \dots, \bar{Q}_4 \in H^0(\mathcal{J}_C(2))$ such that

$$\sigma(\bar{Q}_i) = \bar{Q}_{i-1}$$

$$\tau(\bar{Q}_i) = \varepsilon^{-2i} \bar{Q}_i .$$

In particular, $\bar{Q}_0$ must be invariant under τ, hence it must be of the form

$$\bar{Q}_0 = a' x_0^2 + b' x_2 x_3 + c' x_1 x_4 .$$

We first note that $a' \neq 0$ since $x_k\left(\frac{\ell\omega_1}{5}\right) = 0$ if and only if $k = \ell$. Hence we can rewrite $\bar{Q}_0$ as

$$\bar{Q}_0 = x_0^2 + a x_2 x_3 + b x_1 x_4$$

and it remains to determine the constants a and b. We find

$$a = -\frac{x_0^2\left(\frac{\omega_1}{5}\right)}{x_2\left(\frac{\omega_1}{5}\right) x_3\left(\frac{\omega_1}{5}\right)} = -\frac{x_4^2(0)}{x_1(0) x_2(0)} = -\frac{x_1(0)}{x_2(0)}$$

and

$$b = -\frac{x_0^2\left(\frac{2\omega_1}{5}\right)}{x_1\left(\frac{2\omega_1}{5}\right) x_4\left(\frac{2\omega_1}{5}\right)} = \frac{-x_3^2(0)}{x_4(0) x_2(0)} = \frac{x_2(0)}{x_1(0)} = -\frac{1}{a} .$$

(IV.2.2) It is known (see [4]) that one gets equations for C_5 by taking the 4×4-pfaffians of a suitable skew-symmetric 5×5-matrix whose entries are linear forms. We briefly want to mention how this fits into our picture. The first remark is that there are the following linear relations between the quadrics $\bar{Q}_i$:

$$(x_3\bar{Q}_1 - x_2\bar{Q}_4) + a(x_1\bar{Q}_2 - x_4\bar{Q}_3) = 0$$

$$(x_4\bar{Q}_2 - x_3\bar{Q}_0) + a(x_2\bar{Q}_3 - x_0\bar{Q}_4) = 0$$

$$(x_0\bar{Q}_3 - x_4\bar{Q}_1) + a(x_3\bar{Q}_4 - x_1\bar{Q}_0) = 0$$

$$(x_1\bar{Q}_4 - x_0\bar{Q}_2) + a(x_4\bar{Q}_0 - x_2\bar{Q}_1) = 0$$

$$(x_2\bar{Q}_0 - x_1\bar{Q}_3) + a(x_0\bar{Q}_1 - x_3\bar{Q}_2) = 0.$$

Indeed one can easily see that these are the only such relations. We can rewrite the above set of equations in the form

$$M\bar{Q}^t = 0$$

where

$$\bar{Q} = (\bar{Q}_0, \ldots, \bar{Q}_4)$$

and where M is the following 5 x 5-matrix:

$$M = \begin{pmatrix} 0 & -x_3 & -ax_1 & ax_4 & x_2 \\ x_3 & 0 & -x_4 & -ax_2 & ax_0 \\ ax_1 & x_4 & 0 & -x_0 & -ax_3 \\ -ax_4 & ax_2 & x_0 & 0 & -x_1 \\ -x_2 & -ax_0 & ax_3 & x_1 & 0 \end{pmatrix}$$

It is then easy to check that the quadrics $\bar{Q}_i$ are just the 4 x 4-pfaffians of the matrix M.

3. The singular quadrics through C_5

(IV.3.1) Here we want to study the singular quadrics through an elliptic normal quintic $C = C_5$. Recall that C is embedded by the line bundle

$$L = \mathcal{O}_C(5\mathcal{O}) = \mathcal{O}_C(H).$$

We denote its space of sections by

$$V^* := H^0(L) = H^0(\mathcal{O}_{\mathbb{P}_4}(1))$$

and we shall frequently identify a point $P \in \mathbb{P}_4$ with the hyperplane $V_P \subseteq V^*$ of linear forms vanishing in P.

Finally if P is any point of C we denote the (up to a scalar unique) section in the line bundle $\mathcal{O}_C(P)$ by

$$t_P \in H^0(\mathcal{O}_C(P)).$$

The following proposition is a slight generalization of a result of Ellingsrud and Laksov [6, prop. 2].

(IV.2.3) Proposition: Let $P_1, P_2 \in C$ be two points (possibly equal) and let L be the secant line (resp. tangent) of C through P_1 and P_2. Then for each point $P \in L-C$ the space curve C_P, i.e. the projection of C from P lies on a unique quadric Q_P. If $2P_1 + 3P_2 \not\sim H$ and $3P_1 + 2P_2 \not\sim H$ then there exists a unique point $P_0 \in L-C$ such that Q_{P_0} is singular, otherwise no such point exists.

Proof: If the quadric Q_P exists it must clearly be unique since C_P is a space curve of degree 5 and hence cannot lie on two different quadrics. To prove that the quadrics Q_P exist we want to distinguish between two different cases.

Case 1: Assume that $2P_1 + 3P_2 \not\sim H \not\sim 3P_1 + 2P_2$. We define points $Q, R \in C$ by the equations

$$Q \sim H - 2P_1 - 2P_2$$

$$R \sim 3P_1 + 3P_2 - H$$

Note that $Q \neq P_i \neq R$ and that

(1) $\quad 3Q + 2P \sim H.$

At this point it is convenient to introduce two more subcases.

Subcase 1: $2Q \not\sim P_1 + P_2$.

We first note that

$$(2) \qquad H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \cap H^0(L(-2R-Q))t_Q t_R^2 = 0.$$

Because otherwise we would have

$$Q + 2R + P_1 + P_2 \sim H$$

which together with (1) would imply

$$2Q \sim P_1 + P_2.$$

Now we fix some point $P \in L$, $P \neq P_i$. Then

$$H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \subseteq V_P \ .$$

Because of (2) it follows that

$$\dim(H^0(L(-2R-Q))t_Q t_R^2 \cap V_P) = 1.$$

This implies that there are (unique) points $R_1, R_2 \in C$ such that

$$t_Q t_R^2 t_{R_1} t_{R_2} \in V_P \ .$$

We then define sections $y_i \in V_P$ as follows:

$$y_1 := t_Q t_{P_1}^2 t_{P_2}^2 \ , \quad y_2 := t_Q^2 t_R t_{P_1} t_{P_2}$$

$$y_3 := t_Q t_R^2 t_{R_1} t_{R_2} \ , \quad y_4 := t_R t_{P_1} t_{P_2} t_{R_1} t_{R_2} \ .$$

The equation

$$y_1 y_3 - y_2 y_4 = 0$$

then defines - unless it is identically 0 - a quadric Q_P which contains C_P. We shall have to prove that Q_P has rank 4 for all points in L-C but one. To see this we first notice that y_1 and y_2 are linearly independent since $Q \neq P_i \neq R$ and that because of (2) the section y_3 is not in the span of y_1 and y_2. Hence any possible linear relation between the y_i must be of the form

$$(3) \qquad y_4 = \sum_{i=1}^{3} \alpha_i y_i \ .$$

This implies that Q is one of the points P_i , R_i or R. We know that $Q \neq P_i$ $(i=1,2)$. Moreover, $Q \neq R$ since otherwise

$$2Q \sim Q + R \sim P_1 + P_2$$

which was excluded. Hence we can assume $Q = R_1$. But then it follows from

$$2R + 2Q + R_2 \sim H \sim 2R + 3Q$$

that in fact $R_1 = R_2 = Q$. Hence (3) is possible only if P_0 is given by

$$V_{P_0} = H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \oplus \mathbb{C}t_R^2t_Q^3 \ .$$

Clearly $P_0 \neq P_i$ $(i = 1,2)$ and this implies that $P_0 \in L-C$ since C has no trisecants. Since, moreover, the quadric Q_{P_0} has rank 3 we are done in this case.

Subcase 2: $2Q \sim P_1 + P_2$

In this case we have $5Q \sim H$ and $Q = R$. But we have still $Q \neq P_i$ $(i=1,2)$ since $2P_1 + 3P_2 \not\sim H \not\sim 3P_1 + 2P_2$. Because of $Q \neq P_i$ it follows that

$$H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \cap H^0(L(-3Q))t_Q^3 = \mathbb{C}t_Q^3t_{P_1}t_{P_2} \ .$$

Again let $P \in L-C$ be some fixed point with corresponding hyperplane V_P. Since

$$\dim(V_P \cap H^0(L(-2Q)t_Q^2) \geq 2$$

there are points $R_1, R_2, R_3 \in C$ such that $t_Q^2 t_{R_1} t_{R_2} t_{R_3} \in V_P$ is independent from $t_Q^3 t_{P_1} t_{P_2}$. We define sections $y_i \in V_P$ by

$$y_1 := t_Q t_{P_1}^2 t_{P_2}^2 \quad , \qquad y_2 := t_Q^3 t_{P_1} t_{P_2}$$

$$y_3 := t_Q^2 t_{R_1} t_{R_2} t_{R_3} \quad , \quad y_4 := t_{P_1} t_{P_2} t_{R_1} t_{R_2} t_{R_3} .$$

As before the equation

$$y_1 y_3 - y_2 y_4 = 0$$

defines a quadric Q_P which contains C_P. It remains to see that Q_P has rank 4 for all points but one in L-C. By construction y_2 and y_3 are independent and y_1 is not in the span of y_2 and y_3 since $Q \neq P_i$. Hence there can only be a relation of the form

$$y_4 = \sum_{i=1}^{3} \alpha_i y_i .$$

In this case we can assume $Q = R_3$. If $\alpha_3 \neq 0$ the above relation implies $R_i = P_i$ $(i = 1,2)$ and hence $y_2 = y_3$ which is a contradiction. It follows that $\alpha_3 = 0$ and that the relation must be of the form

$$t_{R_1} t_{R_2} = \alpha_1 t_{P_1} t_{P_2} + \alpha_2 t_Q^2 .$$

We can apply the above argument once more to conclude that $\alpha_2 \neq 0$. Hence

$$t_Q^5 \in \langle t_Q^3 t_{R_1} t_{R_2} \, , \, t_Q^3 t_{P_1} t_{P_2} \rangle \subseteq V_P .$$

Hence Q_P can only be of rank 3 if P is the point P_0 given by

$$V_{P_0} = H^0(L(-P_1-P_2)) t_{P_1} t_{P_2} \oplus \mathbb{C} t_Q^5 .$$

As before we can see that $P_0 \in L-C$ and since Q_P has indeed rank 3 we are again done.

Case 2. $2P_1+3P_2 \sim H$ or $3P_1+2P_2 \sim H$.

After possibly interchanging P_1 and P_2 we can assume that

$$(4) \qquad 2P_1 + 3P_2 \sim H.$$

We can choose a point $S \neq P_i$ $(i=1,2)$ such that

$$2S \sim P_1 + P_2 .$$

First note that

$$(5) \qquad H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \cap H^0(L(-3S))t_S^3 = 0$$

since otherwise we would have

$$P_1 + P_2 + 3S \sim H$$

which one can rewrite as

$$S + 2P_1 + 2P_2 \sim H.$$

Together with (4) this would imply $S=P_2$ which we have excluded. Let $P \in L-C$ be some fixed point. Because of (4) we find

$$\dim(H^0(L(-3S)t_S^3 \cap V_P) = 1$$

i.e. there are (unique) points $R_1, R_2 \in C$ such that

$$t_S^3 t_{R_1} t_{R_2} \in V_P .$$

Once again we define 4 sections $y_i \in V_P$, this time by

$$y_1 := t_{P_1}^2 t_{P_2}^3 \quad , \quad y_2 := t_S^2 t_{P_1} t_{P_2}^2$$

$$y_3 := t_S^3 t_{R_1} t_{R_2} \ , \quad y_4 := t_S t_{P_1} t_{P_2} t_{R_1} t_{R_2} .$$

Then we can define Q_P as

$$y_1y_3 - y_2y_4 = 0.$$

In order to finish our proof it remains to see that Q_P has rank 4 for all points $P \in L-C$. Because of (5) we know that y_3 is not in the span of y_1, y_2 and y_4. Moreover, since $S \neq P_i$ $(i=1,2)$, it follows that y_1 is not in the span of y_2 and y_4. Hence any relation between the y_i must be of the form

$$\alpha_2 y_2 + \alpha_4 y_4 = 0.$$

In this case we can assume that $R_1 = S$ and $R_2 = P_2$. But then P is given by

$$V_P = H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \oplus \mathbb{C}t_S^4 t_{P_2} = V_{P_2}$$

i.e.

$$P = P_2 \in C.$$

This concludes the proof.

Remark: In the above proposition we have not said anything about what happens if we project from a point on the curve C. In this case C_P is an elliptic normal quartic and as such it is cut out by a pencil of quadrics. In particular, C_P lies on 4 different quadric cones, i.e. C itself lies on 4 rank 3 quadrics whose singular lines go through P.

In the above proof we have associated to each (unordered) pair (P_1, P_2) with $2P_1 + 3P_2 \not\sim H \not\sim 3P_1 + 2P_2$ a point P in $\mathrm{Sec}C - C$ together with a rank 3 quadric Q_P which contains C and whose singular line goes through P. We shall see later that the map which associates to (P_1, P_2) the point P is injective. The point which we want to make here is that the quadric Q_P only depends on the divisor class of $P_1 + P_2$.

(IV.3.3) Proposition: The quadric Q_P depends only on the divisor class of $P_1 + P_2$.

<u>Proof</u>: Let (P_1,P_2) and (P_1',P_2') be two (unordered) pairs with

$$P_1 + P_2 \sim P_1' + P_2' .$$

The quadrics Q_P and $Q_{P'}$ associated to these pairs are given by

$$Q_P = y_1y_3 - y_2^2 , \quad Q_{P'} = y_1'y_3' - (y_2')^2$$

where

$$y_1 = t_Q t_{P_1}^2 t_{P_2}^2 , \quad y_1' = t_Q t_{P_1'}^2 t_{P_2'}^2$$

$$y_2 = t_Q^2 t_R t_{P_1} t_{P_2} , \quad y_2' = t_Q^2 t_R t_{P_1'} t_{P_2'}$$

$$y_3 = y_3' = t_Q^3 t_R^2 .$$

Since

$$P_1' + P_2' \sim P_1 + P_2 \sim Q + R$$

and since $(P_1,P_2) \neq (Q,R)$ there is a linear relation

$$t_{P_1'} t_{P_2'} = \alpha_1 t_{P_1} t_{P_2} + \alpha_2 t_Q t_R .$$

Note that $\alpha_1 \neq 0$ since $(P_1',P_2') \neq (Q,R)$. Using this relation we find

$$y_2' = \alpha_1 y_2 + \alpha_2 y_3$$

$$y_1' = \alpha_1^2 y_1 + 2\alpha_1\alpha_2 y_2 + \alpha_2^2 y_3 .$$

But then

$$y_1'y_3' - (y_2')^2 = (\alpha_1^2 y_1 + 2\alpha_1\alpha_2 y_2 + \alpha_2^2 y_3)y_3 - (\alpha_1 y_2 + \alpha_2 y_3)^2$$

$$= \alpha_1^2 (y_1y_3 - y_2^2) .$$

The above proposition can be interpreted as follows: If we vary the secant line through the points P_1, P_2 in such a way that $P_1 + P_2$ remains in the same divisor class then the distinguished point P on the secant line moves along the singular line of the (fixed) quadric Q_P. We shall come back to this and make it more precise in the next section. But before doing so we want to conclude this section with an easy corollary to proposition (IV.3.2).

(IV.3.4) Corollary: The secant variety Sec C of C is the union of the singular loci of all quadrics of rank ≤ 4 through C.

Proof: We have already seen that if $P \in \mathrm{Sec}\, C$ then C_P lies on a quadric surface, i.e. P lies on the singular locus of the cone over this quadric surface. Now assume that $P \notin \mathrm{Sec}\, C$. In this case C_P is smooth and we have to see that an elliptic space curve of degree 5 cannot lie on a quadric surface Q_P. If Q_P were smooth then C_P would have to have bidegree (1,4) or (2,3). But these curves have genus $g = 0$ or $g = 2$ respectively. On the other hand, every smooth curve of degree 5 on a quadric cone also has genus $g = 2$. Since C_P is not a plane curve this proves the corollary.

(IV.3.5) Remark: From the above corollary it follows immediately that every quadric of rank ≤ 4 through C is one of those which we described in the proof of proposition (IV.3.2). In particular, every rank 3 quadric through C is of the form

$$Q = y_1 y_3 - y_2^2$$

where the y_i are as in the proof of proposition (IV.3.3).

4. The locus of singular lines

In this section we want to discuss the locus which is the union of the singular lines of rank 3 quadrics through C. Before we can do this, however, we shall have to collect some basic facts about the second symmetric product of C.

(IV.4.1) Let C be an elliptic curve with origin $\mathcal{O}$. The second symmetric product of C is defined as the variety

$$S^2C = {}^{C\times C}/_{\sim}$$

where $\sim$ is the equivalence relation given by

$$(P_1,P_2) \sim (P_2,P_1).$$

It is well known that S^2C is a smooth surface and the canonical map

$$\rho: C\times C \to S^2C$$

is a 2:1 covering of S^2C by $C\times C$ branded over the diagonal $\Delta \subseteq C\times C$. For points in S^2C we shall use the notation

$$\overline{(P_1,P_2)} := \rho(P_1,P_2).$$

(IV.4.2) To study the surface S^2C more closely we look at the map

$$\pi : S^2C \to C$$

$$\overline{(P_1,P_2)} \mapsto P_1+P_2.$$

Here + means addition on C. In other words we associate to each pair $\overline{(P_1,P_2)}$ the class of the divisor P_1+P_2 in the Jacobian of degree 2 and then use the origin $\mathcal{O}$ to identify Jac_2C and C. By means of the map π the surface S^2C becomes a $\mathbb{P}_1$-bundle over C itself. It is not hard to determine the rank 2 vector bundle (unique up to a twist with a line bundle) on C which gives rise to this ruled surface. In [1] Atiyah proved that

$$S^2C = \mathbb{P}(E)$$

where E is the vector bundle associated to the non-trivial extension class

$$1 \in \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(\mathcal{O}),\mathcal{O}_C)$$

i.e. E is given by a non-split extension

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(\mathcal{O}) \to 0.$$

(IV.4.3) The $\mathbb{P}_1$-bundle $\mathbb{P}(E)$ is well understood (see e.g. [8]).

It is easy to see that the minimal self-intersection number of a section $C_0 \subseteq \mathbb{P}(E)$ is 1. In fact this characterizes $\mathbb{P}(E)$. The Picard group of $S^2C = \mathbb{P}(E)$ is

$$\operatorname{Pic}(S^2C) = \mathbb{Z} \oplus \pi^* \operatorname{Pic} C$$

where we can choose a generator C_0 of $\mathbb{Z}$ such that $C_0^2 = 1$. In particular, the group of numerical equivalence classes is

$$\operatorname{Num} S^2C = \mathbb{Z} \oplus \mathbb{Z}$$

where the second copy of $\mathbb{Z}$ is generated by the class of a fibre f. The intersection pairing is given by

$$C_0^2 = C_0.f = 1$$

$$f^2 = 0.$$

Before leaving this general discussion we want to make one more remark. For every point $P \in C$ we set

$$C_P^1 := C \times \{P\} \subseteq C \times C$$

$$C_P^2 := \{P\} \times C \subseteq C \times C.$$

Then we get sections $C_P \subseteq S^2C$ by

$$C_P := \pi(C_P^1) = \pi(C_P^2).$$

One finds immediately that

$$C_P^2 = \frac{1}{2}(C_P^1 + C_P^2)^2 = 1.$$

In fact any section with self-intersection number 1 arises in this way. In the future we shall set

$$C_0 := C_{\mathcal{O}}.$$

Then we do not only know that $C_0^2 = 1$ but the above discussion also shows that we can write more precisely

$$\mathcal{O}(C_0)|C_0 = \mathcal{O}_{C_0}(\sigma) = \mathcal{O}_C(\sigma).$$

For future reference we shall make the following definition

$$D := \{\overline{(P_1,P_2)}\ ;\quad 2P_1 + 3P_2 \sim H\}.$$

By this we mean that we take all points $\overline{(P_1,P_2)} \in S^2C$ wich have a representative (P_1,P_2) with $2P_1+3P_2 \sim H$.

(IV.4.4) Lemma: D is a section of $S^2C = \mathbb{P}(E)$ and its linear equivalence class is

$$D \sim C_0 + 12f_\sigma$$

where $f_\sigma := \pi^{-1}(\sigma)$ denotes the fibre over σ.

Proof: We first want to show that D is a section. To see this let D_0 be a fixed divisor of degree 2 on C. Our assertion then follows from the fact that the two equations

$$P_1 + P_2 \sim D_0$$

$$2P_1 + 3P_2 \sim H$$

have exactly 1 common solution. To finish the proof of the lemma it remains to compute the intersection of D with C_0. It follows from the definition of D that as a divisor on C_0 one has

$$D.C_0 = \sum_{2P\sim 2\sigma} P + \sum_{3Q\sim 3\sigma} Q \sim 13\sigma.$$

This proves the lemma.

(IV.4.5) We can now return to our discussion of the rank 3 quadrics through an elliptic normal quintic. It is our aim to describe the locus of the union of the singular lines of these quadrics. We want to denote this union by F. In other words we set

$$F := \bigcup_{\substack{Q \supseteq C \\ \operatorname{rank} Q = 3}} \operatorname{sing} Q .$$

The first thing we want to do is to define a map

$$\phi : S^2C \to F$$

which we shall do as follows: If $2P_1 + 3P_2 \not\sim H \not\sim 3P_1 + 2P_2$ we set

$$\phi(\overline{P_1,P_2}) := H^0(L(-P_1-P_2))t_{P_1}t_{P_2} \oplus \mathbb{C}t_R^2t_Q^3 \subseteq V^* .$$

Here R,Q are defined as in the previous section, i.e. by

$$Q \sim H - 2P_1 - 2P_2$$

$$R \sim 3P_1 + 3P_2 - H$$

and as before we shall identify points in $\mathbb{P}_4$ with hyperplanes in V^*. Now if $2P_1 + 3P_2 \sim H$ or $3P_1 + 2P_2 \sim H$ we set

$$\phi(\overline{P_1,P_2}) := H^0(L(-P_2))t_{P_2} = P_2$$

or

$$\phi(\overline{P_1,P_2}) := H^0(L(-P_1))t_{P_1} = P_1$$

respectively. The first thing to notice is that ϕ is well-defined. This is clear if $2P_1 + 3P_2 \not\sim H \not\sim 3P_1 + 2P_2$. To see it in the other cases it is enough to remark that if simultaneously

$$2P_1 + 3P_2 \sim H$$

$$3P_1 + 2P_2 \sim H$$

then this implies that $P_1 = P_2$.

The fact that ϕ maps S^2C to F follows from the proof of proposition (IV.3.2) and the remark following it. Moreover, it follows from (IV.3.5) that ϕ is surjective.

(II.4.6) Proposition: The map

$$\phi : S^2C \to F$$

is a birational morphism. It is 1:1 outside the section

$$D = \phi^{-1}(C) = \{\overline{(P_1,P_2)};\ 2P_1 + 3P_2 \sim H\}$$

and the restriction of ϕ to D

$$\phi_D : D \to C$$

is a 4:1 covering. Moreover, ϕ maps the rulings of S^2C to the singular lines of the rank 3 quadrics through C.

Proof: We shall first verify the set-theoretic properties of ϕ. It follows immediately from the definition of ϕ that

$$\phi^{-1}(C) = D = \{\overline{(P_1,P_2)};\ 2P_1 + 3P_2 \sim H\}.$$

The restriction of ϕ to D is given by

$$\phi_D : D \to C$$

$$\overline{(P_1,P_2)} \mapsto P_2.$$

To see that this is a 4:1 map it is enough to remark that for each $P_2 \in C$ the equation

$$2P_1 \sim H - 3P_2$$

has exactly 4 different solutions. Next we want to show that ϕ is 1:1 outside D. To do this we first remark that C has only ordinary secants. Now assume that

$$\phi\overline{(P_1,P_2)} = \phi\overline{(Q_1,Q_2)}$$

where $\overline{(P_1,P_2)} \notin D$ and $\overline{(Q_1,Q_2)} \notin D$. This implies that the (different) secants (resp. tangents) through P_1,P_2 and Q_1,Q_2 meet in the point $\phi\overline{(P_1,P_2)} = \phi\overline{(Q_1,Q_2)} \notin C$. But this cannot be. Projecting from this point we would get a curve C_P of degree 5 being on a quadric cone Q_P. Moreover, C_P would have at least two singularities. On the other hand every degree 5 curve on a quadric cone has arithmetic genus 2. But since there are at least two singularities this would make C rational which is a contradiction.

Next we want to prove that ϕ maps the rulings of S^2C to the singular lines of the rank 3 quadrics through C. To do this we fix some divisor class D_O of degree 2 on C. Then for every point $(\overline{P_1,P_2}) \in \pi^{-1}(D_O) - D$ the image $\phi(\overline{P_1,P_2})$ lies on the singular line of the rank 3 quadric

$$Q_P = y_1y_3 - y_2^2$$

which is given by

$$L = \operatorname{sing} Q_P = \{y_1 = y_2 = y_3 = 0\}.$$

Here

$$y_1 = t_Q t_{P_1}^2 t_{P_2}^2 \ , \ y_2 = t_Q^2 t_R t_{P_1} t_{P_2} \ , \ y_3 = t_R^2 t_Q^3 \ .$$

We have seen in (IV.3.3) that Q_P, and hence in particular L, do not depend on $(\overline{P_1,P_2})$ but only on the divisor class D_O. On the other hand the above description shows that ϕ maps $\pi^{-1}(D_O) - D$ to L. We finally remark that

$$(\overline{R,Q}) = \pi^{-1}(D_O) \cap D$$

and that

$$\phi(\overline{R,Q}) = Q \in L.$$

This together with what we have said above then implies immediately that in fact ϕ defines an isomorphism

$$\phi \mid \pi^{-1}(D_O) \ : \ \pi^{-1}(D_O) \xrightarrow{\ \simeq\ } L.$$

We have also already seen that every singular line arises in this way.

What remains to be seen is that ϕ is a morphism on the whole of S^2C. Clearly

$$\phi^o := \phi | S^2C - D$$

is a morphism. From the above it follows that if ϕ^o has an extension as a morphism then this must necessarily be equal to ϕ. Therefore it is enough to see that the rational map ϕ^o has no points of indeterminacy. So let us assume that ϕ^o is defined outside the points $P_1, \ldots, P_n$. Then for a general hyperplane section

$$H_F := F \cap H$$

the inverse image $(\phi^o)^{-1}(H_F)$ is a section in S^2C not meeting the points P_i. But this implies that an open neighborhood of the points P_i is mapped into an affine set and therefore ϕ^o can be extended as a morphism to the whole of S^2C.

This finally concludes the proof.

Remarks: (i) That the map ϕ_D is 4:1 is easily understood. This is nothing else but the fact (which we have already pointed out) that the elliptic normal quartic C_P lies on exactly 4 quadric cones, i.e. that there are exactly 4 rank 3 quadrics through C whose singular lines go through P.

(ii) The space of quadrics through C has (projective) dimension 4. Hence one would expect that there is a 1-dimensional family of rank 3 quadrics through C. The above proposition shows that this is indeed the case and that the rank 3 quadrics through C are in fact parametrized by C itself.

(IV.4.7) Proposition: If H_F is the hyperplane section of F then

$$\phi^*(H_F) \sim C_o + 7f_{\mathcal{O}} .$$

In particular

$$\deg F = 15.$$

Proof: Since ϕ maps the rulings of S^2C isomorphically to lines in $\mathbb{P}_4$ it follows for its numerical class that

$$\phi^{*}(H_F) \equiv C_O + \lambda f.$$

In order to verify our claim it is therefore sufficient to compute the intersection of $\phi^{*}(H_F)$ with the section $D \cong C$. One finds

$$\mathcal{O}_{S^2C}(\phi^{*}(H_F)) \mid D = \phi_D^{*}(\mathcal{O}_C(H)) = \mathcal{O}_C(20\mathcal{O}).$$

From this it follows readily that

$$\phi^{*}(H_F) \sim C_O + 7f_{\mathcal{O}} .$$

But then the degree of F can be computed as

$$\deg F = \phi^{*}(H_F) \, . \, \phi^{*}(H_F) = 15.$$

(IV.4.8) We finally want to say a word about the relation between this picture and the configuration which we have associated to an elliptic normal quintic in chapter II. We have already seen in (III.3.4) that the 25 skew lines $L_{k\ell}$ are singular lines of rank 3 quadrics through C. Hence they must correspond to certain rulings in the $\mathbb{P}_1$-bundle S^2C. To find out which these are let $\bar{\Delta}$ be the image of the diagonal $\Delta \subseteq C \times C$ under the canonical projection onto S^2C. Then

$$\bar{\Delta} \, . \, D = 50$$

and set-theoretically

$$\bar{\Delta} \cap D = \{ \overline{(P,P)} ; \; 5P \sim H\}$$

where each of these 25 points has to be counted with multiplicity 2. Note that if $5P \sim H$ then

$$\phi\overline{(P,P)} = P \in C.$$

(IV.4.9) <u>Proposition</u>: <u>The image of the</u> 25 <u>rulings through the points</u>

$$\bar{\Delta} \cap D = \{ \overline{(P,P)} \; : \; 5P \sim H\}$$

under the map ϕ are the skew lines $L_{k\ell}$.

Proof: Let L_P be the image of the ruling through the point $\overline{(P,P)}$ under the map ϕ. We have already seen that

$$L_P = \{y_1 = y_2 = y_3 = 0\}$$

where

$$y_1 = t_P t_S^4 \,, \quad y_2 = t_P^3 t_S^2 \,, \quad y_3 = t_P^5 \,.$$

Here $S \in C$ is a point with $2S \sim 2P$ but $S \neq P$. To see that L_P is one of the lines $L_{k\ell}$ we have to show that L_P lies in all those hyperplanes H_k and $H_{k\ell}$ of our configuration which go through P. But any such hyperplane is of the form

$$H = \{y = 0\}$$

where

$$y = t_P t_{P+P'} \cdot \ldots \cdot t_{P+4P'}$$

for some 5-torsion point $P' \neq \mathcal{O}$. We have to see that y is in the span of the sections y_i. Since $P \neq S$ the sections $t_P^2, t_S^2 \in H^0(\mathcal{O}_C(2P))$ form a basis of this space. Hence there is a linear relation

$$t_{P+P'} t_{P+4P'} = \alpha_1 t_S^2 + \alpha_2 t_P^2 \,.$$

It then follows that

$$y_4 := \alpha_1 y_2 + \alpha_2 y_3 = t_P^3 t_{P+P'} t_{P+4P'} \in \langle y_2 \,, y_3 \rangle$$

Similarly one sees that

$$y_5 := t_P t_S^2 t_{P+P'} t_{P+4P'} \in \langle y_1 \,, y_2 \rangle$$

Applying the same argument once again, this time to y_4 and y_5, we finally find that

$$y \in \langle y_4 \,, y_5 \rangle \subseteq \langle y_1 \,, y_2 \,, y_3 \rangle$$

and we are done.

5. Shioda's modular surface S(5)

(IV.5.1) Here we want to touch briefly on the relation between the results of the previous chapters and Shioda's modular surface S(5). We shall only state facts without giving proofs. This has several reasons: One is that the remainder of this treatise is only concerned with one fixed elliptic curve and the question what happens if one varies this curve does not play a role here. The second is that the results which follow are not due to myself only. They are joint work with W.Barth and R.Moore and were only worked out after the first version of this paper was written up. Moreover some of the proofs are rather long and technical and would lengthen this text out of proportion. For details the reader is referred to [18] , [19].

Recall that every smooth elliptic quintic $C_5 \subseteq \mathbb{P}_4$ embedded as in chapter I is the intersection of five quadrics

$$Q_i = Q_i(a) = x_i^2 + ax_{i+2}x_{i+3} - \frac{1}{a}x_{i+1}x_{i+4} \qquad (i=0,\ldots,4).$$

We write

$$C_a = C_{5,a} = \bigcap_{i=0}^{4} Q_i(a).$$

We can now ask what happens if we vary the parameter $a \in \mathbb{P}_1$. The first result is this:

(IV.5.2) Proposition: For every value of $a \in \mathbb{P}_1$ the set C_a is a curve in $\mathbb{P}_4$. If $a \in \mathbb{P}_1 - \{0, \infty, -\frac{1}{2}(1 \pm \sqrt{5})\varepsilon^k\}$ where $\varepsilon = e^{2\pi i/5}$; $k = 0,\ldots,4$ the curve C_a is a smooth elliptic quintic. Otherwise C_a is a connected cycle of 5 lines, i.e. a pentagon.

Proof: See [18, theorem 1].

(IV.5.3) Remarks: (i) By varying the parameter a we get, therefore, a 1-dimensional family of elliptic quintics. This family contains precisely 12 singular curves, namely the 12 pentagons.

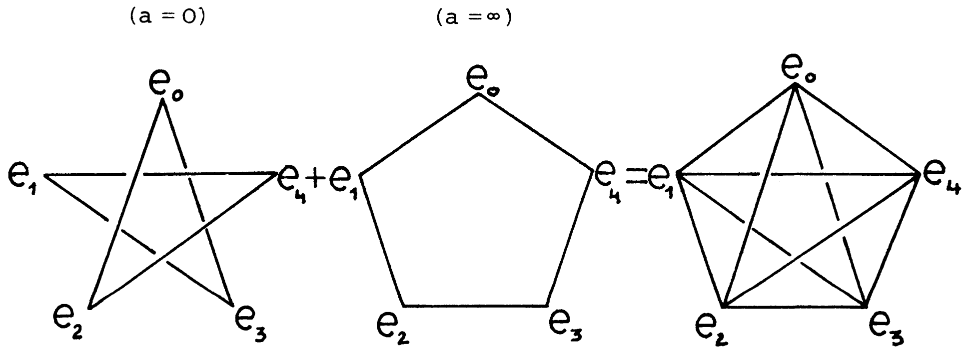
(ii) If one identifies $\mathbb{P}_1$ with the 2-sphere $S^2 \subseteq \mathbb{R}^3$ then the 12 points $\Lambda = \{0,\infty, -\frac{1}{2}(1 \pm \sqrt{5})\varepsilon^k\}$ can be identified with the 12 vertices of an icosahedron sitting inside $S^2 \cong \mathbb{P}_1$.

Another important property of the elliptic family C_a is the following.

(IV.5.4) Proposition: If a and a' are different parameters then the two curves C_a and $C_{a'}$ do not intersect unless a and a' belong to opposite vertices of the icosahedron Λ. In this case the two singular curves have common vertices and thus form a complete pentagon.

Proof: See [18, proposition 2].

One can easily envisage the geometric situation by means of the following picture:



Let us now consider the union

$$S_{15} := \bigcup_{a \in \mathbb{P}_1} C_a$$

of the family C_a of elliptic curves. Clearly S_{15} is a surface in $\mathbb{P}_4$. We collect some of the properties of S_{15} by quoting the following:

(IV.5.6) Proposition: (i) S_{15} is a determinantal surface. More precisely

$$S_{15} = \{x \in \mathbb{P}_4 \; ; \; \text{rank} \begin{pmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_2x_3 & x_3x_4 & x_4x_0 & x_0x_1 & x_1x_2 \\ x_1x_4 & x_2x_0 & x_3x_1 & x_4x_2 & x_0x_3 \end{pmatrix} \le 2\}.$$

(ii) The surface S_{15} is irreducible and has degree 15.

(iii) S_{15} is smooth outside the 30 vertices of the 6 complete pentagons formed by the singular curves C_a, $a \in \Lambda$. There, two smooth branches meet transversely.

(iv) The normalization $\tilde{S}_{15}$ of S_{15} has in a natural way the structure of an elliptic surface over $\mathbb{P}_1$. The fibres are the curves C_a.

Proof: Can be found in [18, section 4].

(IV.5.7) Our next aim is to give an interpretation of the surface S_{15} (resp. its normalization $\tilde{S}_{15}$). To do this we have to make a slight detour. Recall that the modular group

$$\Gamma := SL(2,\mathbb{Z})$$

operates on the upper half plane

$$\mathcal{H} := \{z \in \mathbb{C} \; ; \; \operatorname{Im} z > 0\}$$

by

$$z \mapsto \frac{az+b}{cz+d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. The quotient

$$\mathcal{H}/\Gamma \cong \mathbb{C}$$

parametrizes the (isomorphism classes of) elliptic curves.

For every integer $n \ge 2$ one defines the principal congruence subgroup of level n as follows:

$$\Gamma(n) := \{\gamma \in \Gamma \; ; \; \gamma \equiv \mathbf{1} \bmod n\}.$$

As a subgroup of Γ it also operates on $\mathcal{H}$ and the quotient is an open Riemann surface

$$X'(n) = \mathcal{H}/\Gamma(n).$$

$X'(n)$ parametrizes the (smooth) elliptic curves with a <u>level n structure</u>. Recall that a level n structure is (up to isomorphism) given by the choice of a pair of generators $(\bar{\sigma},\bar{\tau})$ of the group $G_n = \mathbb{Z}_n \times \mathbb{Z}_n$ of n-torsion points such that for the Weil pairing $(\ ,\ )$ on G_n one has

$$(\bar{\sigma},\bar{\tau}) = 1.$$

For details see e.g. [18, section 4.1].

By adding a finite number of <u>cusps</u> one can compactify $X'(n)$ and gets a complete curve

$$X(n) = \overline{X'(n)} = \overline{\mathcal{H}/\Gamma(n)}$$

called the <u>modular curve</u> of <u>level</u> n. The number of cusps needed to compactify $X'(n)$ is

$$t(n) = \frac{1}{2}n^2 \prod_{p|n} (1 - \frac{1}{p^2})$$

and the genus of the resulting curve is

$$g(n) = 1 + \frac{n-6}{12} t(n).$$

Note that in particular

$$t(5) = 12$$

and that

$$g(5) = 0.$$

Hence

$$X(5) \cong \mathbb{P}_1.$$

is a rational curve. Moreover we have a diagram

$$\begin{tikzcd} & \overline{\mathcal{H}/\Gamma(5)} \cong \mathbb{P}_1 \arrow[d, "\pi"] \\ \mathcal{H} \arrow[ur] \arrow[r] & \overline{\mathcal{H}/\Gamma} \cong \mathbb{P}_1 \end{tikzcd}$$

Since $-\mathbb{1} \in \Gamma$ operates trivially on $\mathcal{H}$ and $-\mathbb{1} \notin \Gamma(5)$ the quotient map π is given by the group

$$PSL(2,\mathbb{Z})/_{\Gamma(5)} = PSL(2,\mathbb{Z}_5) = A_5$$

where $\Gamma(5)$ is identified with its image under the canonical map $SL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z})$. The 12 cusps of $X(5)$ form the minimal orbit of the action of A_5 on $\mathbb{P}_1 = X(5)$. They are mapped under π to the unique cusp $\infty \in \mathbb{P}_1 = X(0)$. Moreover they can be identified with the vertices of the icosahedron Λ in $\mathbb{P}_1$.

(IV.5.8) It is well known that no universal elliptic curve exists. However, if $n \geq 3$ then there exists a universal elliptic curve with level-n structure. For this we define the semi-direct product $\Gamma \ltimes (\mathbb{Z} \times \mathbb{Z})$ by

$$(\gamma,(m_1,m_2)) \cdot (\gamma',(m_1',m_2')) = (\gamma\gamma',(m_1,m_2)\gamma' + (m_1',m_2')).$$

It operates on $\mathcal{H} \times \mathbb{C}$ by

$$(\gamma,(m_1,m_2)) : (z,\zeta) \mapsto (\gamma z,(\zeta + m_1 z + m_2)(cz+d)^{-1})$$

For $n \geq 3$ the subgroup $\Gamma(n) \ltimes (\mathbb{Z} \times \mathbb{Z})$ operates without fixed points. The resulting quotient

$$S'(n) = \mathcal{H} \times \mathbb{C}/_{\Gamma(n) \ltimes (\mathbb{Z} \times \mathbb{Z})}$$

is smooth and admits in a canonical way a projection

$$S'(n) \to X'(n)$$

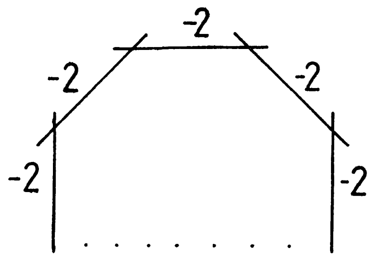
which makes it an elliptic fibration. The surface $S'(n)$ has n^2 sections which intersect each fibre in its n-torsion points. Shioda has constructed a natural compactification

$$S(n) = \overline{S'(n)}$$

of $S'(n)$. The surface $S(n)$ (called Shioda's modular surface of level n) comes together with a natural map

$$S(n) \rightarrow X(n)$$

which is an extension of the map $S'(n) \rightarrow X'(n)$. In this way $S(n)$ becomes an elliptic surface. Its singular fibres lie over the cusps of $X(n)$ and are of type I_n, i.e. they are cycles of the following form:



In particular $S(5)$ has 12 singular fibres of type I_5, i.e. 12 pentagons.

The sections of the surface $S'(n)$ extend to n^2 sections of $S(n)$. After choosing a zero-section they form a group $\mathbb{Z}_n \times \mathbb{Z}_n$ and we shall denote the sections by $L_{k\ell}$ where $(k,\ell) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

We can now formulate the connection between $S(5)$ and the surface S_{15}.

(IV.5.9) Theorem: (i) The normalization $\tilde{S}_{15}$ of the surface S_{15} is isomorphic to Shioda's modular surface $S(5)$.

(ii) There exists a unique divisor $I \in \operatorname{Pic} S(5)$ such that

$$5I \sim \sum_{(k,\ell)} L_{k\ell}$$

The complete linear system $|I + 2F|$ defines an immersion of $S(5)$ into

$\mathbb{P}_4$ whose image is the surface S_{15}.

(iii) Under this map the sections $L_{k\ell}$ of the surface $S(5)$ are mapped to the 25 skew lines of the configuration described in (III.3)

Proof: See [18, section 4].

(IV.5.10) The reader ought to be warned that the relation between the curves C_n and the surface $S(n)$ is not always as straight forward as in the case $n = 5$. For example if $n = 4$ every elliptic quartic is given by two quadratic equations (see (III.2.1)):

$$Q_0 = x_0^2 + x_2^2 + 2ax_1x_3$$

$$Q_1 = x_1^2 + x_3^2 + 2ax_0x_2.$$

Varying a one gets a 1-dimensional family of disjoint elliptic curves which contains 4 singular members. Together they sweep out the surface

$$x_0x_2(x_0^2 + x_2^2) = x_1x_3(x_1^2 + x_3^2)$$

which is projectively equivalent to the Fermat quartic $F_4 \subseteq \mathbb{P}_3$.

But Shioda's modular surface is not isomorphic to the Fermat quartic. There is, however, an isogeny $F \to S(4)$ of degree 4. For details see [19, section IV].

V. The normal bundle of C_5

The principal purpose of this chapter is to discuss the normal bundle $N_{C/\mathbb{P}_4}$ of an elliptic normal quintic $C = C_5 \subseteq \mathbb{P}_4$. Our main result is that the normal bundle is indecomposable. This was also known to G. Sacchiero and appears to be the first case of a curve in $\mathbb{P}_4$ where this has been proved. Using Atiyah's classification of vector bundles on elliptic curves [1] it is then straightforward to describe the normal bundle $N_{C/\mathbb{P}_4}$ explicitly.

In section 2 we want to recall a vanishing result due to Ellingsrud and Laksov [6] which we shall use later. We shall give two proofs of this. The first uses the indecomposability of the normal bundle whereas the second proof (which is the one given by Ellingsrud and Laksov) uses the quadratic equations of C.

1. Indecomposability of the normal bundle

(V.1.1) First of all we want to recall the definition of normal bundle. Let X and Y be smooth, irreducible varieties and let

$$f : X \to Y$$

be a morphism whose differential is injective at every point of X. Then the normal bundle of the map f is defined as the quotient

$$N_f := f^*T_Y/T_X .$$

It is a vector bundle with rank equal to the codimension of X in Y.

If $f : X \to Y$ is an embedding we shall frequently write $N_{X/Y}$, or even N_X, rather than N_f. In this case one can give an equivalent definition of the normal bundle as follows: Let $\mathcal{J}_X \subseteq \mathcal{O}_Y$ be the ideal sheaf of X in Y. Then $\mathcal{J}_X/\mathcal{J}_X^2$ is a locally free $\mathcal{O}_X$-module whose rank is just the codimension of X in Y. It can be identified with the conormal bundle of X in Y, i.e.

$$N_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_X/\mathcal{J}_X^2, \mathcal{O}_X).$$

(V.1.2) Proposition: The normal bundle $N_{C/\mathbb{P}_4}$ of an elliptic normal quintic is indecomposable.

Proof: The proof is based on the following observation which can be proved in exactly the same way as [10, theorem 1.6]. Assume that the vector bundle $N_C(-1)$ has a quotient line bundle

$$N_C(-1) \to L \to 0$$

of degree ℓ. Since N_C is a quotient of $T_{\mathbb{P}_4}|C$ we can define a rank 3 subbundle $U \subseteq T_{\mathbb{P}_4}|C$ by

$$U := \ker(T_{\mathbb{P}_4}|C \to L(1)).$$

The Gauss map of the bundle U is the map

$$g : G \to \mathbb{P}_4^*$$

which sends a point P to the unique hyperplane which is tangent to $U_P \subseteq T_{\mathbb{P}_4,P}$. We set

$$D := g(C).$$

Then it follows from the Nakano sequence that

$$\ell = \deg(g) \cdot \deg D.$$

The relation between C and D is as follows: If D is a point or a line then C lies necessarily in a hyperplane. If D is a plane curve (space curve) then C lies on the cone over the plane dual curve $D^\vee$ (the dual surface $D^\vee$) whose vertex is the dual of the linear space spanned by D. Finally if D is not contained in a hyperplane then C lies on the dual threefold $D^\vee$ of D.

The rest of the proof then goes as follows: We shall show that if N_C decomposes then $N_C(-1)$ has a quotient line bundle L of degree $\ell \leq 3$. This will then give rise to a contradiction in the following way. Since $\ell \leq 3$ and since C is elliptic D must either be a point or a line or a smooth plane cubic. The first two cases can be excluded since C is not contained in a hyperplane. If D is a smooth plane cubic then its dual curve $D^{\vee}$ has degree 6. On the other hand C has to lie on the cone (with vertex a line) over $D^{\vee}$. Since C has degree 5 this is impossible.

Hence it remains to prove that if N_C splits then $N_C(-1)$ has a quotient line bundle L of degree $\ell \leq 3$. It follows from the exact sequence

$$0 \to T_C \to T_{\mathbb{P}_4}|C \to N_C \to 0$$

that

$$\deg N_C = \deg T_{\mathbb{P}_4}|C - \deg T_C = 25.$$

Hence

$$\deg N_C(-1) = 10.$$

We want to distinguish between three cases.

<u>Case 1</u>: We assume that N_C splits into three line bundles, i.e.

$$N_C(-1) = \bigoplus_{i=1}^{3} L_i.$$

Since $\deg N_C(-1) = 10$ it follows that at least one of the line bundles L_i has degree ≤ 3.

<u>Case 2</u>: Here we assume that

$$N_C(-1) = E \oplus L$$

where L is a line bundle and where E is an indecomposable rank 2 bundle of even degree, say

$$\deg E = 2k.$$

If $2k \geq 8$ then $\deg L \leq 2$ and we are done. Hence we can assume that $2k \leq 6$. By Atiyah's classification E is given by an extension

$$0 \to M \to E \to M \to 0$$

where

$$\deg M = k \le 3.$$

Hence $N_C(-1)$ also has a quotient line bundle of degree ≤ 3.

Case 3: It remains to treat the case

$$N_C(-1) = E \oplus L$$

where E is an indecomposable rank 2 bundle of odd degree

$$\deg E = 2k+1.$$

If $2k+1 \ge 7$ then $\deg L \le 3$. Hence we can assume $2k+1 \le 5$. In this case E is given by an extension

$$0 \to M^{-1} \otimes \det E \to E \to M \to 0$$

where

$$\deg M = k+1 \le 3.$$

This concludes the proof.

(V.1.3) Using Atiyah's classification of vector bundles over an elliptic curve it is now easy to describe the normal bundle N_C. It follows from the exact sequence

$$0 \to T_C \to T_{\mathbb{P}_4}|C \to N_C \to 0$$

that

$$\det N_C = \det T_{\mathbb{P}_4}|C = \mathcal{O}_C(5) = \mathcal{O}_C(25\mathcal{O}).$$

Hence the bundle

$$E_C := N_C(-8\mathcal{O}).$$

has determinant

$$\det E_C = \mathcal{O}_C(\mathcal{O}).$$

Since E_C is indecomposable and since its rank and degree are relatively prime it follows from [1, p.343] that E_C is uniquely determined by its determinant. Moreover, it is given by the (unique) non-split extension

(1) $$0 \to \mathcal{O}_C \to E_C \to E'_C \to 0$$

where E'_C is the unique indecomposable rank 2 bundle with determinant $\mathcal{O}_C(\mathfrak{o})$. I.e. E'_C itself is given by the non-split extension

(2) $$0 \to \mathcal{O}_C \to E'_C \to \mathcal{O}_C(\mathfrak{o}) \to 0.$$

We can summarize this as follows.

(V.1.4) <u>Corollary</u>: <u>The normal bundle</u> $N_{C/\mathbb{P}_4}$ <u>of an elliptic normal quintic is</u>

$$N_{C/\mathbb{P}_4} = E_C(8\mathfrak{o})$$

<u>where</u> E_C <u>is the unique indecomposable rank</u> 3 <u>bundle on</u> C <u>with determinant</u> $\mathcal{O}_C(\mathfrak{o})$.

2. A vanishing result

Here we want to prove the following vanishing result.

(V.2.1) <u>Proposition</u>: <u>For any line bundle</u> $M \in \mathrm{Pic}^0(C)$ <u>of degree</u> 0 <u>one has</u>

$$h^1(N_C^*(2) \otimes M) = 0.$$

<u>Proof 1</u>: It follows from the description of N_C in the previous section that there is an exact sequence

$$0 \to (E'_C)^* \otimes \mathcal{O}_C(2\mathfrak{o}) \otimes M \to N_C^*(2) \otimes M \to \mathcal{O}_C(2\mathfrak{o}) \otimes M \to 0.$$

Since

$$h^1(\mathcal{O}_C(2\mathfrak{o}) \otimes M) = h^0(\mathcal{O}_C(-2\mathfrak{o}) \otimes M^{-1}) = 0$$

it suffices to prove that

$$h^1((E_C')^* \otimes \mathcal{O}_C(2\sigma) \otimes M) = 0.$$

On the other hand it follows from (2) that there is an exact sequence

$$0 \to \mathcal{O}_C(\sigma) \otimes M \to (E_C')^* \otimes \mathcal{O}_C(2\sigma) \otimes M \to \mathcal{O}_C(2\sigma) \otimes M \to 0.$$

From this our assertion follows immediately.

Proof 2: We saw in (IV.1.3) that C is the scheme-theoretic intersection of the quadrics containing it. Hence the natural map

$$H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_{\mathbb{P}_4} \to \mathcal{J}_C(2)$$

is surjective. Together with the exact sequence

$$0 \to \mathcal{J}_C^2(2) \to \mathcal{J}_C(2) \to N_C^*(2) \to 0$$

this gives rise to a surjective map

$$H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C \xrightarrow{\alpha} N_C^*(2) \to 0$$

In order to determine the kernel of this map we recall from (IV.2.2) that the matrix

$$M = \begin{pmatrix} 0 & -x_3 & -ax_1 & ax_4 & x_2 \\ x_3 & 0 & -x_4 & -ax_2 & ax_0 \\ ax_1 & x_4 & 0 & -x_0 & -ax_3 \\ -ax_4 & ax_2 & x_0 & 0 & -x_1 \\ -x_2 & -ax_0 & ax_3 & x_1 & 0 \end{pmatrix}$$

gives the linear relations between the quadrics $\bar{Q}_i$ which form a basis of $H^0(\mathcal{J}_C(2))$. We claim that the sequence

$$H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C(-1) \xrightarrow{M} H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C \xrightarrow{\alpha} N_C^*(2) \to 0$$

is exact. It follows from the construction of M that

$$\mathrm{im}(M) \subseteq \ker(\alpha).$$

On the other hand M has rank 2 over each point of C and this proves exactness. Dualising the above sequence we get

$$(3) \qquad 0 \to N_C(-2) \xrightarrow{\alpha^t} H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C \xrightarrow{M^t} H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C(1).$$

To prove the proposition it is enough to show that

$$h^0(N_C(-2) \otimes M^{-1}) = 0$$

since by Serre duality

$$h^1(N_C^*(2) \otimes M) = h^0(N_C(-2) \otimes M^{-1}).$$

For $M \neq \mathcal{O}_C$ this follows immediately from (3). For $M = \mathcal{O}_C$ it also follows from (3) since M^t is injective on global sections.

An immediate consequence of the above result is:

(V.2.2) <u>Corollary</u>: <u>For all line bundles</u> $M \in \mathrm{Pic}^0 C$ <u>of degree</u> 0 <u>one has</u>

$$h^0(N_C^*(2) \otimes M) = 5.$$

<u>Proof</u>: This follows immediately from the above proposition and from Riemann-Roch.

VI. The invariant quintics

Let H_n denote the Heisenberg group in dimension n. As before we shall consider the Schrödinger representation:

$$\rho : H_n \to GL(V)$$

which was defined in (I.2). Here we are interested in the induced representation $S^n\rho^*$ of H_n on the space of homogeneous forms of degree n in n variables. We shall first determine the dimension of the space of H_n-invariant forms if $n = p \geq 3$ is a prime number. Then we shall turn to the case $n = 5$. This case was first treated by Horrocks and Mumford [9] where it plays an important role in the construction of the Horrocks-Mumford bundle. We shall come back to this in chapter VII. Next we shall discuss the space of invariant quintics through an elliptic normal curve $C_5 \subseteq \mathbb{P}_4$ and finally we shall determine the invariant quintics which are singular along C_5. These results will be useful in chapter VIII.

1. Some invariant theory

(VI.1.1) Recall that the Heisenberg group in dimension n is defined as the subgroup $H_n \subseteq GL(V)$ which is generated by the elements

$$\sigma : e_i \mapsto e_{i-1}$$

$$\tau : e_i \mapsto \varepsilon^i e_i$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$ and where the $\{e_i\}_{i \in \mathbb{Z}_n}$ denote the standard basis of $V = \mathbb{C}^n$. The representation

$$\rho : H_n \to GL(V)$$

given by the inclusion is called the Schrödinger representation of H_n. By $S^n\rho^*$ we denote the induced representation of H_n on the space

$$S^nV^* = \Gamma(\mathcal{O}_{\mathbb{P}_{n-1}}(n))$$

of homogeneous polynomials of degree n in n variables. In this section we want to determine the dimension of the space of H_n-invariant forms for a prime number $p = n \geq 3$. In order to simplify our notation we shall sometimes drop the index n and write H instead of H_n.

(VI.1.2) Before we can give the main result of this section we shall first have to prove a lemma on symmetric functions. Let $n \geq 2$ be a fixed integer. For each integer $m \geq 1$ one can define symmetric polynomials of degree m as follows:

$$s_m(x) := s_m(x_1,\dots,x_n) := \sum_{i=1}^{n} x_i^m$$

$$h_m(x) := h_m(x_1,\dots,x_n) := \sum_{i_1+\dots+i_n=m} x_1^{i_1}\cdot\ldots\cdot x_n^{i_n} .$$

The sets of polynomials $\{s_m ; 1 \leq m \leq n\}$ and $\{h_m ; 1 \leq m \leq n\}$ both generate the ring of symmetric polynomials. In particular there is a relation

$$h_n(x) = \sum_{i_1+\dots+ni_n=n} \alpha_{i_1\dots i_n} s_1^{i_1}\cdot\ldots\cdot s_n^{i_n}$$

(VI.1.3) Lemma: $\alpha_{0\dots01} = \frac{1}{n}$.

Proof: We consider the polynomial

(1) $$P(t) = \prod_{i=1}^{n} (1-x_i t) .$$

Then

(2) $$\frac{1}{P(t)} = 1 + \sum_{m=1}^{\infty} h_m(x)t^m .$$

Taking logarithms in (1) we get

(3) $$\log \frac{1}{P(t)} = -\sum_{i=1}^{n} \log(1-x_i t)$$

$$= \left(\sum_{i=1}^{n} x_i\right) t + \frac{1}{2} \left(\sum_{i=1}^{n} x_i\right) t^2 + \dots$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} s_m(x) t^m .$$

We can now combine (2) and (3) to get

$$(4) \qquad 1 + \sum_{m=1}^{\infty} h_m(x) t^m = \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} s_m(x) t^m \right)$$

$$= 1 + \left(\sum_{m=1}^{\infty} \frac{1}{m} s_m(x) t^m \right) + \frac{1}{2!} \left(\sum_{m=1}^{\infty} s_m(x) t^m \right)^2 + \ldots$$

Comparing coefficients in (4) finally gives the result

$$\alpha_{0\ldots01} = \frac{1}{n} .$$

We are now ready to state and prove the main result of this section.

(VI.1.4) <u>Proposition</u>: <u>Let</u> $p \geq 3$ <u>be a prime number. Then the dimension of the space of</u> H_p-<u>invariant</u> p-<u>forms is given by</u>

$$\dim \Gamma_H(\mathcal{O}_{\mathbb{P}_{p-1}}(p)) = \frac{1}{p^2} \binom{2p-1}{p} + \frac{1}{p^2}(p^2-1)$$

<u>Proof</u>: We set

$$N(p) := \dim \Gamma_H(\mathcal{O}_{\mathbb{P}_{p-1}}(p)) .$$

Then

$$N(p) = \frac{1}{|H_p|} \sum_{g \in H_p} \chi_{S^p \rho^*}(g) = \frac{1}{p^3} \sum_{g \in H_p} \chi_{S^p \rho^*}(g) .$$

On the other hand

$$\chi_{S^p \rho^*}(g) = \sum_{i_1 + \ldots + p i_p = p} \alpha_{i_1 \ldots i_p} \chi_{\rho^*}(g)^{i_1} \cdot \ldots \cdot \chi_{\rho^*}(g^p)^{i_p} .$$

Since p is a prime number it follows that

$$\chi_{\rho*}(g^i) = 0$$

unless g is in the centre of H_p or $i = p$. For $p \geq 3$ the group H_p has exponent p, hence

$$\chi_{\rho*}(g^p) = \chi_{\rho*}(id) = p$$

for all $g \in H_p$. This implies

$$N(p) = \frac{1}{p^3} \sum_{g \in Z(H)} \chi_{S^p\rho*}(g) + \frac{1}{p^3}\Big(\sum_{g \in H-Z(H)} \alpha_{0\ldots 0p}\chi_{\rho*}(g^p)\Big)$$

$$= \frac{1}{p^2} \dim \Gamma_{Z(H)}(\mathcal{O}_{\mathbb{P}_{p-1}}(p)) + \frac{1}{p^3}(p^3 - p)$$

Since the centre $Z(H_p)$ operates trivially on the p-forms it follows that

$$\dim \Gamma_{Z(H)}(\mathcal{O}_{\mathbb{P}_{p-1}}(p)) = h^0(\mathcal{O}_{\mathbb{P}_{p-1}}(p)) = \binom{2p-1}{p}$$

and this concludes the proof.

(VI.1.5) <u>Example</u>: For $p = 3$ the above formula gives

$$N(3) = 2.$$

In fact it is easy to find a basis for the H_3-invariant forms:

$$\Gamma_H(\mathcal{O}_{\mathbb{P}_2}(3)) = \langle x_0^3 + x_1^3 + x_2^3 \ , \ x_0x_1x_2 \rangle.$$

2. <u>The case</u> $n = 5$

If we apply the above proposition to the case $p = 5$ we find immediately that

$$\dim \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)) = 6.$$

This was in fact first observed by Horrocks and Mumford [9]. In their paper the space of H_5-invariant quintic forms plays an important role in the construction of the Horrocks-Mumford bundle. The following two propositions can also be found in [9].

(VI.2.1) Proposition: The following quintic forms define a basis of the space $\Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$:

$$Q'_{-1} := \prod_{i=0}^{4} x_i \qquad Q'_0 := \sum_{i=0}^{4} \sigma^{-i}(x_0^3 x_1 x_4)$$

$$Q'_1 := \sum_{i=0}^{4} \sigma^{-i}(x_0^3 x_2 x_3) \qquad Q'_2 := \sum_{i=0}^{4} \sigma^{-i}(x_0^2 x_1^2 x_3)$$

$$Q'_3 := \sum_{i=0}^{4} \sigma^{-i}(x_0^2 x_2^2 x_1) \qquad Q'_4 := \sum_{i=0}^{4} x_i^5 .$$

The scheme-theoretic intersection of the H_5-invariant quintics is the union of the 25 reduced skew lines $L_{k\ell}$.

Proof: It is clear that $Q'_{-1},\dots,Q'_4$ form a basis of the space of invariant quintics. Hence it remains to determine their common intersections. Recall that the 25 skew lines $L_{k\ell}$ have the equation

$$x_{-k} = x_{1-k} + \varepsilon^{2\ell} x_{4-k} = \varepsilon^{\ell} x_{2-k} + x_{3-k} = 0 \qquad (0 \le k,\ell \le 4).$$

Note that each of these lines is contained in one of the hyperplanes $x_{-k} = 0$. Because of Q'_{-1} it is therefore sufficient to consider the intersection of the quintics $Q'_0,\dots,Q'_4$ with those hyperplanes. We shall restrict ourselves to $x_0 = 0$. The other cases then follow from symmetry considerations. Hence we have to look at the equations

$$(1) \qquad x_3 x_2 (x_3^2 x_4 + x_2^2 x_1) = 0$$

$$(2) \qquad x_1 x_4 (x_4^2 x_2 + x_1^2 x_3) = 0$$

$$(3) \qquad x_1 x_4 (x_3^2 x_4 + x_2^2 x_1) = 0$$

$$(4) \qquad x_2 x_3 (x_4^2 x_2 + x_1^2 x_3) = 0$$

$$(5) \qquad x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0.$$

We first assume that all $x_i \neq 0$. It then follows from (1) that

$$(1)' \qquad x_1 = \frac{-x_3^2 x_4}{x_2^2}$$

Together with (2) this gives

$$(2)' \qquad x_2^5 + x_3^5 = \prod_{\ell=0}^{4} (\varepsilon^{\ell} x_2 + x_3) = 0.$$

Combining (1)' and (2)' we find the equations

$$(3)' \qquad \varepsilon^{\ell} x_2 + x_3 = x_1 + \varepsilon^{2\ell} x_4 = 0.$$

These are just the equations for $L_{0\ell}$ in $x_0 = 0$. One checks immediately that every point which fulfills (3)' also fulfills (1) to (5). It remains to treat the case where some $x_i = 0$. Let (i,j,k,ℓ) be a permutation of $(1,2,3,4)$ and assume $x_i = 0$. Then at least one other coordinate say $x_j = 0$. From (5) it then follows that

$$x_k^5 + x_{\ell}^5 = 0.$$

This gives - apart from the cases covered by (3)' - 20 more points which are precisely the points of intersection of the lines $L_{k\ell}$ where $k \neq 0$ with the hyperplane $x_0 = 0$. It remains to see that the $L_{k\ell}$ are the scheme-theoretic intersection of the quintics $Q'_{-1}, \dots, Q'_4$. Again it will be enough to look at just one line, say L_{00}, which is given by

$$x_0 = x_1 + x_4 = x_2 + x_3 = 0.$$

If $P \in L_{00}$ is a point with $x_i(P) \neq 0$ for $i = 1, \dots, 4$ then the quintics Q'_{-1}, Q'_0 and Q'_1 intersect transversally in P. It remains to consider the points $P_1 = (0:1:0:0:-1)$ and $P_2 = (0:0:1:-1:0)$. In the case of P_1 the quintics Q'_1, Q'_3 and Q'_4 intersect transversally, in the case of P_2 the same holds for Q'_0, Q'_2 and Q'_4.

Remark: Note that the H_5-invariant quintics Q_i' are also invariant under the operation induced by the involution ι introduced in (I.2).

(VI.2.2) In section (II.2) we gave the definition of a fundamental pentahedron. Recall that there are precisely 6 fundamental pentahedra, one for each subgroup $\mathbb{Z}_5 \subseteq \mathbb{Z}_5 \times \mathbb{Z}_5$. In fact the fundamental pentahedron associated to a given subgroup $\mathbb{Z}_5$ is the (up to a scalar unique) quintic form whose zero-set is the union of the 5 hyperplanes which are invariant under this subgroup. Recall that these forms are given by

$$Q_{-1} = \prod_{i=0}^{4} x_i$$

$$Q_\ell = \prod_{k=0}^{4} \left(\sum_{m=0}^{4} \varepsilon^{\frac{m}{2}(m-5)\ell - mk} x_m\right) \qquad (\ell = 0,\ldots,4).$$

(VI.2.3) Proposition: The fundamental pentahedra Q_i; $i=-1,\ldots,4$ also form a basis of the space $\Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$.

Proof: We have already seen in (II.3.2) that the Q_i are H_5-invariant. Hence it remains to prove that they are linearly independent. To see this we want to relate the Q_i to the basis Q_i'. We first note that $Q_{-1} = Q'_{-1}$. It is somewhat tedious but not difficult to check that the following relations hold modulo Q'_{-1}.

$$Q_0 \equiv -5Q_0' - 5Q_1' + 5Q_2' + 5Q_3' + Q_4'$$

$$Q_1 \equiv -5\varepsilon Q_0' - 5\varepsilon^4 Q_1' + 5\varepsilon^3 Q_2' + 5\varepsilon^2 Q_3' + Q_4'$$

$$Q_2 \equiv -5\varepsilon^2 Q_0' - 5\varepsilon^3 Q_1' + 5\varepsilon Q_2' + 5\varepsilon^4 Q_3' + Q_4'$$

$$Q_3 \equiv -5\varepsilon^3 Q_0' - 5\varepsilon^2 Q_1' + 5\varepsilon^4 Q_2' + 5\varepsilon Q_3' + Q_4'$$

$$Q_4 \equiv -5\varepsilon^4 Q_0' - 5\varepsilon Q_1' + 5\varepsilon^2 Q_2' + 5\varepsilon^3 Q_3' + Q_4'$$

Hence we have to see that the matrix

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 \\ 1 & \varepsilon^4 & \varepsilon^3 & \varepsilon^2 & \varepsilon \\ 1 & \varepsilon^3 & \varepsilon & \varepsilon^4 & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon & \varepsilon^3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

is non-singular. But this follows immediately from the fact that A is a Vendermonde matrix.

Remarks: (i) The last two propositions together give a proof for (III.3.3).

(ii) At this point we do want to point out that the case $p = 5$ is special in so far that the above proposition cannot be generalized to other prime numbers. Already for $p = 7$ we have

$$\dim \Gamma_H(\mathcal{O}_{\mathbb{P}_6}(7)) = 36$$

whereas there are only 8 fundamental polyhedra. In general the function $N(p)$ grows much faster than $p + 1$.

Now let $C = C_5 \subseteq \mathbb{P}_4$ be an elliptic normal curve of degree 5. For later applications we shall be particularly interested in the space of invariant quintics which are singular along C. Before we turn to this problem, however, we want to prove the following result.

(VI.2.4) Proposition: There are 5 independent invariant quintics through an elliptic normal curve $C_5 \subseteq \mathbb{P}_4$, i.e.

$$\dim \Gamma_H(\mathcal{J}_C(5)) = 5.$$

Every invariant quintic which does not contain C_5 intersects the elliptic normal curve precisely in its 25 hyperosculating points.

Proof: First note that the fundamental pentahedra do not contain the curve C. Hence

$$\dim \Gamma_H(\mathcal{J}_C(5)) \leq 5 .$$

Since the fundamental pentahedra intersect C in the 25 hyperocsulating points it remains to prove that the above inequality is in fact an equality. To see this let $P \in C$ be a general point, i.e. not a 5-torsion point. By

$$W_P \subseteq \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$$

we denote the space of invariant quintics through P. To go through one point imposes one condition, i.e.

$$\dim W_P = 5.$$

We claim that

$$W_P = \Gamma_H(\mathcal{J}_C(5)).$$

To see this recall that any quintic $Q \in W_P$ is invariant under both the Heisenberg group H_5 and the involution ι. Hence Q contains the orbit of P under the action of the group generated by H_5 and ι. This orbit consists of 50 points and by Bezout's theorem this implies that C is contained in Q. This concludes the proof.

(VI.2.5) It is possible to improve the above results to some degree. Recall from (I.2) that the Heisenberg group H_5 has exactly 4 irreducible representations V^i of dimension 5 and 24 characters $V^{k\ell}$. The H_5-module $\Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$ has dimension

$$h^0(\mathcal{O}_{\mathbb{P}_4}(5)) = 126$$

It is not difficult to determine the canonical decomposition of this module. The result is

(VI.2.6) <u>Proposition</u>: <u>The canonical decomposition of the</u> H_5-<u>module</u> $\Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$ <u>is</u>

$$\Gamma(\mathcal{O}_{\mathbb{P}_4}(5)) = \bigoplus_{(k,\ell)\neq(0,0)} 5V^{k\ell} \oplus \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)).$$

Since we shall not use this result in what follows we shall omit its proof. We only want to remark that it is easy to find an explicit basis for each of the components $5V^{k\ell}$.

(VI.2.7) A similar result also holds for the H_5-module $\Gamma(\mathcal{J}_C(5))$. From the exact sequence

$$0 \to \mathcal{J}_C(5) \to \mathcal{O}_{\mathbb{P}_4}(5) \to \mathcal{O}_C(5) \to 0$$

and the fact that C is projectively normal it follows that

$$\begin{aligned} h^0(\mathcal{J}_C(5)) &= h^0(\mathcal{O}_{\mathbb{P}_4}(5)) - h^0(\mathcal{O}_C(5) \\ &= 126-25 \\ &= 101. \end{aligned}$$

We can then prove

(VI.2.8) <u>Proposition</u>: <u>The canonical decomposition of the</u> H_5-<u>module</u> $\Gamma(\mathcal{J}_C(5))$ <u>is</u>

$$\Gamma(\mathcal{J}_C(5)) = \bigoplus_{(k,\ell)\neq(0,0)} 4V^{k\ell} \oplus \Gamma_H(\mathcal{J}_C(5)).$$

As before we want to omit the proof. It is enough to remark that it is easy to find elements $Q \in 5V^{k\ell}$ which do not contain C.

3. <u>The</u> H_5-<u>module</u> $H^0(\mathcal{J}_C^2(5))$

In this section we want to study the space of quintic hypersurfaces which are singular along an elliptic normal curve $C = C_5 \subseteq \mathbb{P}_4$. Our first result is

(VI.3.1) <u>Lemma</u>: $h^0(\mathcal{J}_C^2(5)) = 51$.

<u>Proof</u>: We consider the short exact sequence

$$0 \to \mathcal{J}_C^2(2) \to \mathcal{J}_C(2) \to N_C^*(2) \to 0.$$

Since C is not a plane curve it follows that

$$h^0(\mathcal{J}_C^2(2)) = 0.$$

But then it follows from (IV.2.1) and (V.2.2) that

$$H^0(\mathcal{J}_C(2)) \cong H^0(N_C^*(2))$$

since both these vector spaces have dimension 5. Moreover, it follows from the proof of proposition (V.2.1) that we have an exact sequence

$$0 \to F \to H^0(\mathcal{J}_C(2)) \otimes \mathcal{O}_C \xrightarrow{\alpha} N_C^*(2) \to 0$$

where

$$F = \ker(\alpha) = \mathrm{im}(M)$$

is a rank 2 bundle with

$$\Lambda^2 F = \mathcal{O}_C(-1)$$

$$h^0(F) = 0.$$

Then F is either a direct sum of line bundles

$$F = M_1 \oplus M_2$$

with

$$-5 \leq \deg M_i \leq 0$$

or is given by a non-split exact sequence

$$0 \to \mathcal{O}_C(-3\sigma) \otimes M \to F \to \mathcal{O}_C(-2\sigma) \otimes M \to 0$$

where M is a theta-characteristic, i.e. $M^2 = \mathcal{O}_C$. In any case we find that

$$h^1(F(3)) = 0.$$

This implies that we have a surjective map

$$H^0(\mathcal{J}_C(2)) \otimes H^0(\mathcal{O}_C(3)) \to H^0(N_C^*(5)) \to 0$$

Since C is projectively normal this implies that the natural map

$$H^0(\mathcal{J}_C(5)) \to H^0(N_C^*(5))$$

is surjective. It therefore follows from the exact sequence

$$0 \to \mathcal{J}_C^2(5) \to \mathcal{J}_C(5) \to N_C^*(5) \to 0$$

that

$$h^0(\mathcal{J}_C^2(5)) = h^0(\mathcal{J}_C(5)) - h^0(N_C^*(5)).$$

We have already seen that

$$h^0(\mathcal{J}_C(5)) = 101.$$

Moreover it follows from (V.1.4) that

$$h^0(N_C^*(5)) = 50.$$

Hence

$$h^0(\mathcal{J}_C^2(5)) = 101 - 50 = 51.$$

We are now ready to describe the canonical decomposition of the H_5-module $\Gamma(\mathcal{J}_C^2(5))$.

(VI.3.2) Proposition: The canonical decomposition of the H_5-module $\Gamma(\mathcal{J}_C^2(5))$ is

$$\Gamma(\mathcal{J}_C^2(5)) = \bigoplus_{(k,\ell)\neq(0,0)} 2V^{k\ell} \oplus \Gamma_H(\mathcal{J}_C^2(5)).$$

In particular the space of invariant quintics which are singular along C has dimension 3, i.e.

$$\dim \Gamma_H(\mathcal{J}_C^2(5)) = 3.$$

Proof: Recall from (IV.2.1) that the space $H^0(\mathcal{J}_C(2))$ has a basis

$$\bar{Q}_0,\dots,\bar{Q}_4 \in H^0(\mathcal{J}_C(2))$$

where

$$\bar{Q}_0 = x_0^2 + ax_2x_3 - \frac{1}{a}x_1x_4$$

$$\bar{Q}_i = \sigma^{-i}(\bar{Q}_0).$$

We can use this basis to describe the canonical decomposition of $H^0(\mathcal{J}_C^2(5))$. In order to keep this proof to a reasonable length we shall simply give a basis for each of the components. We leave it to the reader to verify that the quintics given are linearly independent. In any case this is straightforward though somewhat tiresome. Counting dimensions then concludes the proof.

Case 1: A basis for the space of invariant quintics which are singular along C is given by

$$\Gamma_H(\mathcal{J}_C^2(5)) = \langle Q_0, Q_1, Q_2 \rangle$$

where

$$Q_0 = \sum_{i=0}^{4} x_i\bar{Q}_i^2 = \sum_{i=0}^{4} \sigma^{-i}(x_0\bar{Q}_0^2)$$

$$Q_1 = \sum_{i=0}^{4} \sigma^{-i}(x_0\bar{Q}_1\bar{Q}_4)$$

$$Q_2 = \sum_{i=0}^{4} \sigma^{-i}(x_0\bar{Q}_2\bar{Q}_3).$$

Case 2: For $k \neq 0$ one finds

$$2V^{k0} = \langle Q_0^k , Q_1^k \rangle$$

with

$$Q_0^k = \sum_{i=0}^{4} \varepsilon^{ik} x_i\bar{Q}_i^2 = \sum_{i=0}^{4} \varepsilon^{ik}\sigma^{-i}(x_0\bar{Q}_0^2)$$

$$Q_1^k = \sum_{i=0}^{4} \varepsilon^{ik}\sigma^{-i}(x_0\bar{Q}_1\bar{Q}_4).$$

Case 3: Finally if $\ell \neq 0$ one finds

$$2V^{k\ell} = \langle Q_0^{k\ell}, Q_1^{k\ell} \rangle$$

where

$$Q_0^{k\ell} = \sum_{i=0}^{4} \varepsilon^{ik} \sigma^{-i} (\bar{Q}_0 \bar{Q}_{2\ell})$$

$$Q_1^{k\ell} = \sum_{i=0}^{4} \varepsilon^{ik} \sigma^{-i} (\bar{Q}_{3\ell} \bar{Q}_{4\ell})$$

VII. The Horrocks-Mumford bundle and elliptic quintics

In this chapter we want to explain the relation between the Horrocks-Mumford bundle F on $\mathbb{P}_4$ and elliptic normal curves of degree 5. The bundle F was first constructed by Horrocks and Mumford in their famous paper [9] and is still essentially the only known indecomposable rank 2 bundle over $\mathbb{P}_4$. The construction given in [9] is to exhibit the bundle F as the cohomology of a certain monad. On the other hand a general section of F has an abelian surface Z of degree 10 as its zero-set and F can be reconstructed from Z by means of the Serre-construction. In fact this is how the bundle F was found in the first place.

In [9, p.79(a)] Horrocks and Mumford state (without proof) that the abelian surface Z can degenerate into the tangent surface of an elliptic normal quintic. In this chapter we want to make this statement precise and supply a proof at the same time. We shall see that if $C \subseteq \mathbb{P}_4$ is an elliptic normal curve, embedded as in chapter I, and if Tan C is its tangent developable then there is a section $s \in \Gamma(F)$ whose zero-set is the tangent surface Tan C, i.e. F can be reconstructed from Tan C by means of the Serre-construction.

1. A property of tangent developables

In this section we shall first prove a general result about the desingularization of the tangent developable of a smooth curve C in $\mathbb{P}_n$. We shall then apply this result to describe some properties of the tangent surface of an elliptic normal quintic.

(VII.1.1) Let $C \subseteq \mathbb{P}_n$ be a smooth curve and let

$$f : C \to Gr(1,n)$$

be the map which sends each point $P \in C$ to the tangent of C at P. If U is the universal subbundle on Gr(1,n) we set

$$E := f^*(U) \otimes \mathcal{O}_C(1).$$

Then the canonical projection

$$p : \tilde{X} := \mathbb{P}(E) \to \text{Tan } C$$

is birational and finite (see [5, prop.3]).

(VII.1.2) Lemma: The bundle E is the unique rank 2 bundle on C associated to $1 \in \mathrm{Ext}^1_{\mathcal{O}_C}(T_C, \mathcal{O}_C)$.

Proof: There is an exact and commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_C \arrow[r, equal] \arrow[d] & \mathcal{O}_C \arrow[d] & & \\
0 \arrow[r] & E \arrow[r] \arrow[d] & \mathbb{C}^{n+1} \otimes \mathcal{O}_C(1) \arrow[r] \arrow[d] & N_{C/\mathbb{P}_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_C \arrow[r] \arrow[d] & T_{\mathbb{P}_n}|C \arrow[r] \arrow[d] & N_{C/\mathbb{P}_n} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

To see that the extension on the left hand side is non-trivial we look at the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_n}}(T_{\mathbb{P}_n}, \mathcal{O}_{\mathbb{P}_n}) \arrow[r, "\mathrm{rest}"] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_C}(T_{\mathbb{P}_n}|C, \mathcal{O}_C) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathcal{O}_C}(T_C, \mathcal{O}_C) \arrow[d, equal] \\
H^1(\Omega_{\mathbb{P}_n}) \arrow[r] \arrow[rr, bend right, "d"'] & H^1(\Omega_{\mathbb{P}_n}|C) \arrow[r] & H^1(\Omega_C).
\end{tikzcd}$$

It is now sufficient to observe that the map d under suitable identifications is nothing but the degree of C (e.g. [8, Ex. III.7.4]) and, therefore, in particular non-zero.

(VII.1.3) We now return to the case of an elliptic normal quintic.

Here $\tilde{X} = \mathbb{P}(E)$ where E is the indecomposable rank 2 bundle on C given by a non-split extension

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0.$$

As a ruled surface $\mathbb{P}(E)$ has invariant $e = 0$ and there is a unique section $C_0 \subseteq \mathbb{P}(E)$ with $C_0^2 = 0$. In fact, to be more precise, one has

$$\mathcal{O}_{\mathbb{P}(E)}(C_0) \,|\, C_0 = \mathcal{O}_{C_0}.$$

The Picard group of $\mathbb{P}(E)$ is of the form

$$\mathrm{Pic}(\mathbb{P}(E)) = \mathbb{Z} \oplus \pi^*(\mathrm{Pic}\ C)$$

where $\pi : \mathbb{P}(E) \to C$ is the projection map. Here $\mathbb{Z}$ is generated by the section C_0. As before we shall denote the fibre over the origin $\mathcal{O} \in C$ by $f_{\mathcal{O}}$.

Finally the group of numerical equivalence classes is given by

$$\mathrm{Num}(\mathbb{P}(E)) = \mathbb{Z} \oplus \mathbb{Z}$$

where the two copies of $\mathbb{Z}$ are generated by the classes of C_0 and a fibre f respectively.

As before we denote by

$$p : \tilde{X} = \mathbb{P}(E) \to \mathrm{Tan}\ C \subseteq \mathbb{P}_4$$

the canonical projection map. Let H be the hyperplane section on Tan C. It follows immediately from the Plücker formulas that

$$\deg \mathrm{Tan}\ C = 10.$$

(VII.1.4) Proposition: The map p has the following properties:

(i) $p^{-1}(C) = C_0$

(ii) $p^*(H) \sim C_0 + 5f_{\mathcal{O}}$.

Proof: Since the fibres of $\mathbb{P}(E)$ are mapped to lines by p it follows that the numerical equivalence class of H is given by

$$H \equiv C_0 + \lambda f_{\mathcal{O}}.$$

Since

$$H^2 = \deg \operatorname{Tan} C = 10 = 2\lambda$$

it follows that $\lambda = 5$.

Next we observe that each point of C lies on only one tangent as can be seen by projecting from this point. Hence $p^{-1}(C)$ is a section say

$$p^{-1}(C) = C' \equiv C_0 + \mu f_{\mathcal{O}}.$$

It follows from

$$\deg C = 5 = H.C' = 5 + \mu$$

that $\mu = 0$. This in fact implies assertion (i) namely

$$p^{-1}(C) = C_0.$$

Assertion (ii) now follows from the fact that

$$p^*\mathcal{O}(1)|C_0 = \mathcal{O}_{\mathbb{P}(E)}(5f_{\mathcal{O}})|C_0.$$

This concludes the proof.

2. The Horrocks-Mumford bundle

Here we want to recall briefly some basic facts about the Horrocks-Mumford bundle F which we shall use later.

(VII.2.1) In their paper [9] Horrocks and Mumford construct the bundle F as the cohomology of a monad

$$5\mathcal{O}_{\mathbb{P}_4}(2) \xrightarrow{a} 2\Lambda^2 T_{\mathbb{P}_4} \xrightarrow{b} 5\mathcal{O}_{\mathbb{P}_4}(3)$$

i.e. F is given by

$$F = {}^{\ker(a)}/{\operatorname{im}(b)}.$$

The difficult step in this construction is to find suitable maps a and b such that $b \circ a = 0$. Their construction is closely related to the Heisenberg group $H = H_5$. Details can be found in [9].

From the above description one concludes readily that the Chern classes of F are given by

$$c(F) = 1 + 5h + 10h^2.$$

This implies in particular that F is indecomposable.

(VII.2.2) Another way of looking at the Horrocks-Mumford bundle is the following. If $s \in \Gamma(F)$ is a general section then its zero-set

$$Z = \{s = 0\}$$

is an abelian surface of degree 10. The canonical bundle ω_Z of Z is trivial and F can be reconstructed from Z by means of the Serre-construction. Moreover, every abelian surface in $\mathbb{P}_4$ arises in this way (modulo projective transformations). It was, however, only recently that direct proofs for the existence of abelian surfaces in $\mathbb{P}_4$ were given. See [21], [22].

The bundle F is acted on by the Heisenberg group. This group action enables one to compute the cohomology groups of F. For future reference we want to quote the following result from [9]:

(VII.2.3) <u>Proposition</u>: <u>If</u> F <u>is the Horrocks-Mumford bundle then</u> $h^0(F(-1)) = 0$. <u>Moreover</u>

$$\dim \Gamma(F) = \dim \Gamma_H(F) = 4$$

<u>and the map</u>

$$\Lambda^2\Gamma(F) \to \Gamma(\Lambda^2 F) = \Gamma(\mathcal{O}_{\mathbb{P}_4}(5))$$

<u>defines an isomorphism</u>

$$\Lambda^2\Gamma(F) \cong \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)).$$

<u>In particular, the bundle</u> F <u>is generated by its global sections</u>

everywhere outside the 25 skew lines $L_{k\ell}$.

These will be the only properties of F which we shall need in what follows.

3. Another construction of the Horrocks-Mumford bundle

In this section we want to prove the main result of this chapter. Let $C \subseteq \mathbb{P}_4$ be an elliptic normal curve of degree 5 as in chapter I. Then its tangent surface Tan C has degree 10 and the main point is that the Horrocks-Mumford bundle can be reconstructed from Tan C by means of the Serre-construction. Here I should mention that there exist several proofs that Tan C is a locally complete intersection with trivial dualizing sheaf, i.e. that it gives rise to some rank 2 bundle on $\mathbb{P}_4$. Such proofs were given by Ellingsrud, Barth - Van de Ven and Ein. But then it remains to be seen that this bundle is in fact the Horrocks-Mumford bundle.

(VII.3.1) Theorem: There exists a section $s \in \Gamma(F)$ such that

$$\text{Tan } C = \{s = 0\}$$

as schemes, i.e. the Horrocks-Mumford bundle can be reconstructed from the tangent surface of an elliptic normal quintic via the Serre-construction.

Before we give a proof of this assertion we shall first prove an auxiliary result.

(VII.3.2) Lemma: There are at least 3 independent invariant quintics which contain the tangent surface Tan C, i.e.

$$\dim \Gamma_H(\mathcal{J}_{\text{Tan } C}(5)) \geq 3.$$

Proof: Let $P \in C$ be a 2-torsion point different from $\mathcal{O}$ and let T_P be the tangent line of C at P. We choose 3 different points $P_1, P_2, P_3 \in T_P$ not lying on C. Then the vector space

$$K := \{Q \in \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)) ; Q(P_i) = 0\}$$

has dimension

$$\dim K \geq 3.$$

It will be enough to prove that

$$K \subseteq \Gamma_H(\mathcal{J}_{\mathrm{Tan}\,C}(5)).$$

So let $Q \in K$ be an invariant quintic containing the points P_i. Since Q is also invariant under the involution ι it follows that Q also contains the points

$$\iota(P_i) \in T_P.$$

Hence Q intersects T_P in at least 6 points, i.e. it contains T_P. But then it also contains the tangent lines $T_{P+P_{k\ell}}$ at the 25 points $P+P_{k\ell}$ which form the orbit of P under the action of the Heisenberg group H_5. Now this implies that either Q contains Tan C or we have the following equality of divisors on $\tilde{X} = \mathbb{P}(E)$:

$$p^*(Q) = \sum_{(k,\ell)} f_{P+P_{k\ell}} + D$$

where D is an effective divisor whose class is

$$D \sim p^*(5H) - \sum_{(k,\ell)} f_{P+P_{k\ell}}$$

i.e.

$$D \sim 5C_0 + (f_{\mathcal{O}} - f_P).$$

But this is impossible since

$$h^0(\mathcal{O}_{\mathbb{P}(E)}(5C_0 + (f_{\mathcal{O}} - f_P)) = 0.$$

This proves the lemma.

We are now ready to give the

(VII.3.3) <u>Proof of the main theorem</u>: This can be done very much in the same way as in [9, p.77]. First note that by lemma (VII.4.1) we can choose a basis $s_1,\ldots,s_4 \in \Gamma(F)$ such that one of the following two possibilities holds:

(1) $s_1 \wedge s_2,\ s_3 \wedge s_4,\ s_1 \wedge s_3 - s_2 \wedge s_4 \in \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5))$

(2) $s_1 \wedge s_2,\ s_1 \wedge s_3$, 3rd independent element $\in \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5))$.

If $s \in \Gamma(F)$ is a section then we shall denote its pullback to $\tilde{X} = \mathbb{P}(E)$ by $\bar{s}$. If $s \wedge t \in \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5))$ and if $\bar{t} \neq 0$ then this implies that $\bar{s} = f \cdot \bar{t}$ for some rational function $f \in \mathbb{C}(\tilde{X})$.

To prove our result it will be sufficient to show that $\bar{s}_i = 0$ for some i. Then

$$\mathrm{Tan}\ C \subseteq \{s_i = 0\}.$$

On the other hand, since $h^0(F(-1)) = 0$ it follows that $\{s_i = 0\}$ is a surface of degree 10. Since this is also the degree of the tangent surface it follows immediately that the scheme $\{s_i = 0\}$ has Tan C as its support and that it is generically reduced. But since it is a locally complete intersection this implies in fact that it is reduced everywhere, i.e.

$$\mathrm{Tan}\ C = \{s_i = 0\}.$$

We shall give a proof by contradiction, i.e. we assume that $\bar{s}_i \neq 0$ for all i.

<u>Case 1</u>: We find that

$$\bar{s}_1 = f \cdot \bar{s}_2$$

$$\bar{s}_3 = g \cdot \bar{s}_4.$$

Next note that $\bar{s}_1 \wedge \bar{s}_3 \neq 0$ since F is generated by its global sections outside the lines $L_{k\ell}$. Since $\bar{s}_1 \wedge \bar{s}_3 = \bar{s}_2 \wedge \bar{s}_4$ we find that $fg = 1$, i.e. that

$$\bar{s}_1 = f \cdot \bar{s}_2$$

$$\bar{s}_4 = f \cdot \bar{s}_3.$$

Let D be the polar divisor of f and set $M := \mathcal{O}_{\mathbb{P}(E)}(D)$. Consider the map

$$\alpha : M \oplus M \to p^*F$$

$$(g_1, g_2) \mapsto g_1\bar{s}_2 + g_2\bar{s}_3 .$$

Since $\alpha(1,0) = \bar{s}_2$, $\alpha(0,1) = \bar{s}_3$, $\alpha(f,0) = \bar{s}_1$ and $\alpha(0,f) = \bar{s}_4$ it follows that α is an isomorphism whenever p^*F is generated by its global sections. On the other hand F is generated by its global sections outside the lines $L_{k\ell}$ which do not lie in the tangent surface Tan C. Since p is bijective this implies that α is an isomorphism outside a finite set of points and hence everywhere. But then it follows that

$$4(D^2) = (5H)^2 = 250$$

a contradiction.

<u>Case 2</u>: In this case we find

$$\bar{s}_2 = f \cdot \bar{s}_1$$

$$\bar{s}_3 = g \cdot \bar{s}_1$$

Let D be the minimal divisor such that $D \geq (f)_\infty$ and $D \geq (g)_\infty$ and define $M := \mathcal{O}_{\mathbb{P}(E)}(D)$. We can now look at the map

$$\alpha : M \oplus \mathcal{O} \to p^*F$$

$$(g_1, g_2) \mapsto g_1\bar{s}_1 + g_2\bar{s}_4 .$$

Exactly as before we see that α is an isomorphism. But this implies that

$$0 = c_2(M \oplus \mathcal{O}) = c_2(p^*F) = 100$$

which concludes the proof.

We want to conclude this section with two easy corollaries of our theorem.

(VII.3.4) Corollary: $\dim \Gamma(\mathcal{J}_{\mathrm{Tan}\ C}(5)) = \dim \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5)) = 3.$

Proof: This follows immediately from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_4} \xrightarrow{s} F \to \mathcal{J}_{\mathrm{Tan}\ C}(5) \to 0$$

and from the fact that $h^0(F) = 4$.

(VII.3.5) Corollary: The common intersection of all quintic hypersurfaces through the tangent surface Tan C consists of the union of Tan C and the 25 skew lines $L_{k\ell}$.

Proof: It is clear that the tangent surface and the lines $L_{k\ell}$ are contained in this intersection. Now let P be a point not on any of these varieties. Let $s_1,\dots,s_4 \in \Gamma(F)$ be a basis such that s_1 vanishes on Tan C. Note that $s_1(P) \neq 0$. We have to see that for some i one has $s_1 \wedge s_i(P) \neq 0$. But if this were not the case, the sections s_i would not generate the bundle F at P. This proves the corollary.

4. A lemma from linear algebra

The whole purpose of this section is to give a proof of the following easy lemma which was used in [9] and in a similar way in the proof of theorem (VII.3.1):

(VII.4.1) Lemma: Let W be a 4-dimensional complex vector space and let $K \subseteq \Lambda^2 W$ be a 3-dimensional subspace. Then there exists a basis $e_0,\dots,e_3$ of W such that either

i) $e_0\wedge e_1,\ e_2\wedge e_3,\ e_0\wedge e_2 - e_1\wedge e_3 \in K$ or

ii) $e_0\wedge e_1,\ e_0\wedge e_2$, 3rd independent element $\in K$.

Proof: Let

$$Q = Gr(1,3) \subseteq \mathbb{P}(\Lambda^2 W)$$

be the Plücker quadric of decomposable tensors. Since K has dimension

3 it follows that either $\mathbb{P}(K) \subseteq Q$ or that

$$C = \mathbb{P}(K) \cap Q$$

is a conic section. In any case we can choose linearly independent elements

$$u \wedge v, \quad w \wedge x \in K.$$

Case 1: $u \wedge v \wedge w \wedge x = 0.$

We can assume that $u, v, w \in K$ are linearly independent and rename these vectors as

$$e_0 := u, \quad e_1 := v, \quad e_2 := w.$$

Then

$$x = \sum_{i=0}^{2} \lambda_i e_i$$

and

$$e_2 \wedge x = e_2 \wedge (\lambda_0 e_0 + \lambda_1 e_1).$$

If $\lambda_0 = 0$ or $\lambda_1 = 0$ we are in case (ii). Otherwise we set

$$e_0' := \lambda_0 e_0 + \lambda_1 e_1$$

$$e_i' := e_i \qquad (i = 1, 2)$$

Then e_0', e_1' and e_2' are linearly independent and

$$e_0' \wedge e_1' = \lambda_0 e_0 \wedge e_1 \in K$$

$$e_0' \wedge e_2' = (\lambda_0 e_0 + \lambda_1 e_1) \wedge e_2 \in K$$

hence we are again in case (ii).

Case 2: $u \wedge v \wedge w \wedge x \neq 0.$

Then these vectors form a basis of W and we set

$$e_0 := u, \quad e_1 := v, \quad e_2 := w, \quad e_3 := x.$$

We can choose a basis of K which consists of vectors

$$e_0 \wedge e_1,\quad e_2 \wedge e_3,\quad e_0 \wedge e_2' - e_1 \wedge e_3' \in K$$

where

$$e_2', e_3' \in \langle e_2, e_3 \rangle .$$

First assume that e_2' and e_3' are linearly dependent say

$$e_2' = \lambda e_3'$$

Then we define

$$e_0'' := e_0 - \lambda e_1,\quad e_1'' := e_1,\quad e_2'' := e_2' .$$

These three vectors are linearly independent and we find that

$$e_0'' \wedge e_1'' = e_0 \wedge e_1 \in K$$

$$e_0'' \wedge e_2'' = e_0 \wedge e_2' - \lambda e_1 \wedge e_2' = e_0 \wedge e_2' - e_1 \wedge e_3' \in K$$

and we are once more back in (ii). Finally if e_2' and e_3' are linearly independent then we can choose e_0, e_1, e_2' and e_3' as the basis of W and are in case (i). This concludes the proof.

5. Further comments

(VII.5.1) There are other ways to construct the Horrocks-Mumford bundle F starting with an elliptic quintic curve $C = C_5 \subseteq \mathbb{P}_4$. Here we want to mention the perhaps most interesting one which, however, will not be used in what follows. This construction is originally due to Van de Ven and the present author and details can be found in [20].

Let us choose a 2-torsion point $P \in C$ different from $\mathcal{O}$. Then for each point $Q \in C$ we can consider the line

$$L(Q) := \overline{Q, P+Q},$$

The union of these lines

$$X := \bigcup_{Q \in C} L(Q)$$

is a smooth ruled surface of degree 5 in $\mathbb{P}_4$ whose base curve is the elliptic curve $C' := C/_{\langle P \rangle}$. In other words X is an <u>elliptic quintic scroll</u> in $\mathbb{P}_4$.

(VII.5.2) The main point is that X carries in a natural way a multiplicity 2 structure. Let $N_{X/\mathbb{P}_4}$ be the normal bundle of X in $\mathbb{P}_4$. For every ruling $L \subseteq X$ there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{L/X} & \to & N_{L/\mathbb{P}_4} & \to & N_{X/\mathbb{P}_4}|L & \to & 0 \\ & & \| & & \| & & & & \\ & & \mathcal{O}_L & & 3\mathcal{O}_L(1) & & & & \end{array}$$

Hence

$$N_{X/\mathbb{P}_4}|L = \mathcal{O}_L(1) \oplus \mathcal{O}_L(2).$$

If $\pi : X \to C'$ is the projection map we can define a line bundle $\mathcal{L} \subseteq N_{X/\mathbb{P}_4}$ by

$$\mathcal{L} := \pi^*(\pi_* N_{X/\mathbb{P}_4}(-2))(2)$$

Over each ruling $\mathcal{L}$ is just the uniquely determined rank 1 subbundle of degree 2.

This line bundle defines a multiplicity 2 structure in the following way. By construction we have a quotient map

$$N^*_{X/\mathbb{P}_4} \to \mathcal{L}^* \to 0$$

Let I be the ideal sheaf of X in $\mathbb{P}_4$. Then we can define an ideal sheaf $J \subseteq I$ by setting

$$J := \ker(w)$$

where

$$w : \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 = N^*_{X/\mathbb{P}_4} \to \mathcal{L}^* \to 0 .$$

We define

$$\mathcal{O}_{\tilde{X}} := \mathcal{O}_{\mathbb{P}_4}/\mathcal{J} .$$

In this way one finds a non-reduced structure $\tilde{X}$ on X which is of multiplicity 2. For details of this construction see [20].

(VII.5.3) It is easily checked that $\tilde{X}$ is a locally complete intersection. Moreover one can show [20, prop.4] that

$$\mathcal{L} \cong \omega^*_X .$$

From this one concludes [20, theorem 1] that

$$\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}} .$$

Hence the hypotheses for the Serre construction are fulfilled and there exists a rank 2 bundle F' on $\mathbb{P}_4$ together with a section $s \in \Gamma(F')$ such that $\tilde{X} = \{s = 0\}$. In fact one can show that $F' = F$, i.e. one has

(VII.5.4) Theorem: If F is the Horrocks-Mumford bundle then there exists a section $0 \neq s \in \Gamma(F)$ such that

$$\tilde{X} = \{s = 0\}.$$

In particular F can be reconstructed from $\tilde{X}$ by means of the Serre construction.

Proof: See [20].

VIII. The normal bundle of elliptic space curves of degree 5

In this chapter we want to apply our previous results to study the normal bundle of elliptic space curves of degree 5. Ellingsrud and Laksov [6] and also Eisenbud and Van de Ven (unpublished) were the first to work on this problem. In their paper [6] Ellingsrud and Laksov classified the normal bundles of elliptic quintics, thereby using a certain 1-parameter family of quintic hypersurfaces Y_M in $\mathbb{P}_4$. To describe and understand this family of quintic hypersurfaces was my original motive for this work.

In section 1 we shall first discuss the normal bundles of elliptic space curves of degree 5 with a node. We shall prove a slight strengthening of a result due to Ellingsrud and Laksov. In section 2 we shall then recall the theorem of Ellingsrud and Laksov, i.e. we shall construct the hypersurfaces Y_M and relate them to normal bundles of elliptic quintics. The only new results here are a proof that the Y_M form a linear family of hypersurfaces and an explicit equation for $Y_{\mathcal{O}} = \mathrm{Sec}\, C$. In section 3 we shall finally describe the linear family $\{Y_M\}$. After discussing the base locus of the linear system we shall prove that the space formed by the equations of these quintics equals

$$U = \Gamma_H(\mathcal{J}_{\mathrm{Tan}\, C}(5)) \cap \Gamma_H(\mathcal{J}_C^2(5)).$$

Then we shall explain the relation to the Horrocks-Mumford bundle F and finally we shall describe U as a subspace of $\Gamma_H(\mathcal{J}_C^2(5))$ using the basis which we have found in (VI.3) for this space.

1. The normal bundle of elliptic quintics with a node

Let $C \subseteq \mathbb{P}_4$ be an elliptic normal quintic. For every point $P \in \mathbb{P}_4$ we denote the projection of C from P by C_P. As long as P does not lie on the tangent surface of C the differential of the projection

$$\pi_P : C \to C_P$$

is nowhere 0. Hence the normal bundle of the map π_P in the sense of (V.1) exists and we shall denote it by

$$N_P := N_{\pi_P} .$$

If $P \notin \operatorname{Sec} C$ then $C_P \subseteq \mathbb{P}_3$ is smooth and N_P is the normal bundle in the usual sense, i.e.

$$N_P = N_{C_P/\mathbb{P}_3} = \mathcal{H}om_{\mathcal{O}_{C_P}}\left(\mathcal{J}_{C_P} / \mathcal{J}^2_{C_P} , \mathcal{O}_{C_P} \right)$$

Here we are interested in the case where C_P has a node. Although it is interesting in itself to study the normal bundle of such a curve we shall need this result mainly for applications in later sections of this chapter. The following proposition is a slight strengthening of a result of Ellingsrud and Laksov [6, p.17]. It says essentially that an elliptic quintic with a node behaves with respect to its normal bundle in exactly the same way as one would expect it from a smooth curve of the same degree and arithmetic genus [17].

(VIII.1.1) Proposition: For each point $P \in \operatorname{Sec} C - \operatorname{Tan} C$ the curve C_P lies on a unique quadric surface Q_P and has exactly one node. The following two cases occur

(i) If Q_P is smooth then N_P is indecomposable and there is a non-split exact sequence

$$0 \to \mathcal{O}_C \to N_P^*(2) \to \mathcal{O}_C \to 0.$$

(ii) If Q_P is a quadric cone with vertex E then E is a smooth point of C_P and

$$N_P^*(2) = \mathcal{O}_C(-E) \oplus \mathcal{O}_C(E) .$$

Proof: We have already seen in (IV.3) that C_P must lie on a unique quadric surface Q_P and that Q_P can be either smooth or a quadric cone. In either case this implies that the arithmetic genus of C_P is 2, hence C_P can only have one node. This can also be concluded from lemma (IV.1.1).

Case 1: Assume that Q_P is smooth. Then the equation of Q_P defines a subbundle

$$s : \mathcal{O}_{C_P} \to \mathcal{J}_{C_P}/\mathcal{J}^2_{C_P}(2)$$

which we can pull back to a subbundle

$$s : \mathcal{O}_C \to N^*_P(2).$$

It then follows from

$$\Lambda^2 N^*_P(2) = \mathcal{O}_C$$

that we have an exact sequence

$$0 \to \mathcal{O}_C \to N^*_P(2) \to \mathcal{O}_C \to 0.$$

We have to show that this sequence is non-split which is equivalent to

$$h^0(N^*_P(2)) = 1.$$

To see this, note that we have an exact sequence

$$0 \to N^*_P(1) \to N^*_C(1) \to V^*/V_P \otimes \mathcal{O}_C \to 0$$

where N_C denotes the normal bundle of C in $\mathbb{P}_4$ and

$$V^* = \Gamma(\mathcal{O}_{\mathbb{P}_4}(1)).$$

Moreover $V_P \subseteq V^*$ denotes as usual the space of linear forms vanishing on P. (For this sequence see also the next section). Tensoring with $\mathcal{O}_C(1)$ we get an exact sequence

$$0 \to N^*_P(2) \to N^*_C(2) \to V^*/V_P \otimes \mathcal{O}_C(1) \to 0.$$

On the other hand we have already seen that

$$\Gamma(\mathcal{J}_C(2)) \cong \Gamma(N^*_C(2)) .$$

Hence we have to see that the kernel of the map

$$\Gamma(\mathcal{J}_C(2)) \rightarrow V^*/V_P \otimes \Gamma(\mathcal{O}_C(1))$$

has dimension 1. For a quadric $\bar{Q}$ to be in this kernel means that if $P=(p_0,\dots,p_4)$ then

$$\sum_{i=0}^{4} \frac{\partial \bar{Q}_i}{\partial x_i} p_i = 0.$$

Since

$$\sum_{i=0}^{4} \frac{\partial \bar{Q}_i}{\partial x_i} p_i = \sum_{i=0}^{4} \frac{\partial \bar{Q}_i}{\partial x_i}(P) x_i$$

it follows that $\bar{Q}$ is a quadric through C which is singular at P. Since there is exactly one such quadric, namely the cone over Q_P, we are done.

Case 2: Now assume that Q_P is a quadric cone with vertex E. Projection from E defines a map

$$\rho : C_P \rightarrow C_0$$

of C_P onto a conic section C_0. Since

$$\deg C_P = 2\cdot\deg \rho + \mathrm{mult}_E C_P$$

it follows that E must be a smooth point of C_P.

As before the equation of the quadric Q_P defines a map

$$s : \mathcal{O}_C \rightarrow N_P^*(2)$$

which, however, has a zero at E. We claim that s vanishes of order 1 at E. To see this, we choose holomorphic coordinates t_1, t_2, t_3 near E such that

$$C_P = \{t_1 = t_2 = 0\}.$$

Then Q_P is given by an equation of the form

$$Q_P = \{f_1t_1 + f_2t_2 = 0\}.$$

Since Q_P is a cone over a non-degenerate conic it follows that at least one f_i must have a term ct_3 with $c \neq 0$. Hence s vanishes of order 1. But then it follows that there is an exact sequence

$$0 \to \mathcal{O}_C(E) \to N_P^*(2) \to \mathcal{O}_C(-E) \to 0$$

and since

$$h^1(\mathcal{O}_C(2E)) = h^0(\mathcal{O}_C(-2E)) = 0$$

this sequence splits. This concludes the proof.

2. The result of Ellingsrud and Laksov

(VIII.2.1) We first want to define a certain 1-parameter family of quintic hypersurfaces $Y_M \subseteq \mathbb{P}_4$. These hypersurfaces were first introduced by Ellingsrud and Laksov in [6] and are essential in the study of normal bundles of elliptic space curves of degree 5 in $\mathbb{P}_3$.

As always let $C \subseteq \mathbb{P}_4$ be an elliptic normal quintic with hyperplane section

$$L = \mathcal{O}_C(1) = \mathcal{O}_C(5\mathcal{O}).$$

Then

$$V^* = H^0(L) = H^0(\mathcal{O}_{\mathbb{P}_4}(1))$$

and by $V_P \subseteq V^*$ we denote the hyperplane of linear forms vanishing on P. Finally let

$$N = N_{C/\mathbb{P}_4}$$

be the normal bundle of C in $\mathbb{P}_4$.

For each point $P \in \mathbb{P}_4 - \operatorname{Tan} C$ we have the following commutative and exact diagram over C:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F_P \arrow[r] \arrow[d] & V_P \otimes \mathcal{O}_C \arrow[r] \arrow[d] & \mathcal{P}^1_C(L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N^* \otimes L \arrow[r] \arrow[d] & V^* \otimes \mathcal{O}_C \arrow[r] \arrow[d, "\nu_P"] & \mathcal{P}^1_C(L) \arrow[r] & 0 \\
 & \mathcal{O}_C \arrow[r, equal] \arrow[d] & \mathcal{O}_C \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{P}^1_C(L)$ is the bundle of first principal parts of L. Note that we can identify

$$F_P = N^*_P \otimes L .$$

Although we have started out with a point $P \in \mathbb{P}_4 - \operatorname{Tan} C$, the map ν_P and hence the map

$$N^* \otimes L \to \mathcal{O}_C$$

is also defined if $P \in \operatorname{Tan} C$. If $P \notin C$ then this map fails to be surjective exactly at those points of C which lie over the cusps of C_P. In any case we have still got an exact sequence

$$0 \to F_P \to N^* \otimes L \to \mathcal{O}_C$$

where the right hand map is surjective if and only if $P \notin \operatorname{Tan} C$.

In order to vary the point P we consider the product

$$\begin{tikzcd}
 & \mathbb{P}_4 \times C \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathbb{P}_4 & & C
\end{tikzcd}$$

This gives rise to a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & F \arrow[r] \arrow[d] & \Omega_{\mathbb{P}_4} \arrow[d] & \\
0 \arrow[r] & q^*(N^*\otimes L) \arrow[r] \arrow[d] & q^*(\overset{*}{V}\otimes\mathcal{O}_C) = p^*(\overset{*}{V}\otimes\mathcal{O}_{\mathbb{P}_4}) \arrow[r] \arrow[d] & \cdots \\
 & p^*(\mathcal{O}_{\mathbb{P}_4}(1)) \arrow[r, equal] \arrow[d] & p^*(\mathcal{O}_{\mathbb{P}_4}(1)) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

where the left-hand vertical row is exact over $(\mathbb{P}_4 - \text{Tan } C) \times C$. Tensoring this sequence with $q^*(L\otimes M)$ where $M \in \text{Pic}^0 C$ is a line bundle of degree 0, we get

$$(1_M) \quad 0 \to F \otimes q^*(L\otimes M) \to q^*(N^*\otimes L^2\otimes M) \xrightarrow{\phi_M} p^*\mathcal{O}_{\mathbb{P}_4}(1) \otimes q^*(L\otimes M).$$

Applying p_* leads to a morphism

$$\psi_M = p_*\phi_M : p_*q^*(N^*\otimes L^2\otimes M) \to \mathcal{O}_{\mathbb{P}_4}(1) \otimes p_*q^*(L\otimes M).$$

It follows from Riemann-Roch that

$$h^0(L\otimes M) = 5$$

for all $M \in \text{Pic}^0 C$. Similarly it follows from (V.2.2) that

$$h^0(N^*\otimes L^2\otimes M) = 5$$

for all $M \in \text{Pic}^0 C$. Hence the map

$$\psi_M : H^0(N^*\otimes L^2\otimes M) \otimes \mathcal{O}_{\mathbb{P}_4} \to \mathcal{O}_{\mathbb{P}_4}(1) \otimes H^0(L\otimes M)$$

can be viewed as a 5×5 matrix with entries linear forms. Following Ellingsrud and Laksov we define

$$Y_M := \{\det \psi_M = 0\}.$$

We now want to start collecting first properties of the hypersurfaces Y_M.

(VIII.2.2) Proposition: (i) For each M the variety $Y_M \subseteq \mathbb{P}_4$ is a hypersurface of degree 5.

(ii) The support of Y_M is given by

$$\operatorname{supp} Y_M = \{P; h^0(F_P \otimes L \otimes M) \neq 0\}.$$

(iii) $Y_{\mathcal{O}} = \operatorname{Sec} C$.

Proof: (i) All we have to see is that $Y_M \neq \mathbb{P}_4$. We first consider the case $M' \neq \mathcal{O}_C$. It follows from (VIII.1.1) case (i) that for general $P \in \operatorname{Sec} C - \operatorname{Tan} C$ one has

$$h^0(N_P^*(2) \otimes M') = 0.$$

Hence $Y_{M'} \neq \mathbb{P}_4$ for $M' \neq \mathcal{O}_C$ (cf. (ii)). Next assume $M = \mathcal{O}_C$. Again from (VIII.1.1) we can conclude that there exists a point $P \in Y_{M'} - \operatorname{Sec} C$. Then C_P is smooth and hence has semi-stable normal bundle ([10,p.61]). Hence $N_P^*(2)$ is given by an extension

$$0 \to M'^{-1} \to N_P^*(2) \to M' \to 0$$

from which one concludes that $Y_{\mathcal{O}} \neq \mathbb{P}_4$.

(ii) For each point P we have an exact sequence

$$0 \to F_P \otimes L \otimes M \to N^* \otimes L^2 \otimes M \to L \otimes M$$

and the assertion then follows from the definition of Y_M together with basechange.

(iii) We first note that Sec C is a hypersurface of degree 5. This follows since projection from a general line $L \subseteq \mathbb{P}_4$ maps C to a plane curve of degree 5 which by the genus formula must have 5 nodes. Hence it is enough to prove that

$$\operatorname{Sec} C \subseteq Y_{\mathcal{O}}\ .$$

But it follows from (VIII.1.1) that for each point $P \in \mathrm{Sec}\, C - \mathrm{Tan}\, C$ we have

$$h^0(F_P \otimes L) = h^0(N_P^*(2)) \neq 0$$

and this concludes the proof.

Our next result is that the quintics Y_M form a linear family of hypersurfaces.

(VIII.2.3) Proposition: The map

$$\det \phi : \mathrm{Pic}^0 C \to \mathbb{P} := \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}_4}(5)))$$

$$M \longmapsto Y_M$$

is not constant and admits a factorization

$$\begin{tikzcd} \mathrm{Pic}^0 C \arrow[rr, "\det \phi"] \arrow[dr] & & \mathbb{P} \\ & \mathrm{Pic}^0 C/\iota \cong \mathbb{P}_1 \arrow[ur, hook, "\text{linear}"'] & \end{tikzcd}$$

where ι denotes the involution on $\mathrm{Pic}^0 C$. In particular each point $P \notin \mathrm{Sec}\, C$ lies on a unique hypersurface $Y_M = Y_{M^{-1}}$.

Proof: We shall first prove that $\det \phi$ is not constant. To do this let $P \in \mathrm{Sec}\, C$ be a point such that C_P lies on a smooth quadric Q_P. Then it follows from (VIII.1.1) that

$$h^0(F_P \otimes L \otimes M) = h^0(N_P^*(2) \otimes M) = 0$$

for all $M \neq \mathcal{O}_C$. This implies that $Y_M \neq Y_{\mathcal{O}_C}$ if $M \neq \mathcal{O}_C$.

We can now turn to the second part of the proposition. Since we have chosen an origin $\mathcal{O}$ we can identify

$$C \times \mathrm{Pic}^{O} C = C \times C$$

and define the Poincaré-bundle E by

$$E := \mathcal{O}_{C \times C}(\Delta - C_O).$$

Here Δ is the diagonal and $C_O = \{\mathcal{O}\} \times C$. The bundle E has the following properties:

$$E \mid C \times \{M\} = M$$

$$E \mid \mathrm{Pic}^{O} C = \mathcal{O}_C(\mathcal{O}).$$

On the product $\mathbb{P}_4 \times C \times \mathrm{Pic}^{O} C$ we have a sequence

$$0 \to F \boxtimes L \boxtimes E \to N^* \boxtimes L^2 \boxtimes E \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}_4}(1) \boxtimes L \boxtimes E$$

where $\boxtimes$ denotes the tensor product of the pullbacks to $\mathbb{P}_4 \times C \times \mathrm{Pic}^{O} C$. The restriction of this sequence to $\mathbb{P}_4 \times C \times \{M\}$ is just (1_M). Let

$$f : \mathbb{P}_4 \times C \times \mathrm{Pic}^{O} C \to \mathbb{P}_4 \times \mathrm{Pic}^{O} C$$

be the projection map. It will then be enough to prove that

$$\Lambda^5 f_*(N^* \boxtimes L^2 \boxtimes E) = \mathcal{O}_{\mathbb{P}_4} \boxtimes \mathcal{O}_C(2\mathcal{O})$$

$$\Lambda^5 f_*(\mathcal{O}_{\mathbb{P}_4}(1) \boxtimes L \boxtimes E) = \mathcal{O}_{\mathbb{P}_4}(5) \boxtimes \mathcal{O}_C(4\mathcal{O})$$

since this implies that

$$\det \phi = \det f_* \phi \in \Gamma(\mathcal{O}_{\mathbb{P}_4}(5) \boxtimes \mathcal{O}_C(2\mathcal{O})) = \Gamma(\mathcal{O}_{\mathbb{P}_4}(5)) \boxtimes \Gamma(\mathcal{O}_C(2\mathcal{O})).$$

To prove the above assertion we shall use Grothendieck-Riemann-Roch (see e.g. [8, p.436]) which reads

$$\mathrm{ch}(f_! F) = f_*[\mathrm{ch}(F).\mathrm{Td}(T_f)].$$

Here T_f is the relative tangent sheaf of f and

$$\mathrm{ch}(f_! F) = \sum_{i\geq 0} (-1)^i R^i f_* F.$$

Since T_f is trivial and since

$$h^1(N^* \otimes L^2 \otimes M) = h^1(L \otimes M) = 0$$

for all $M \in \mathrm{Pic}^0 C$ we find that

$$\mathrm{ch}[f_*(N^* \otimes L^2 \otimes E)] = f_*[\mathrm{ch}(N^* \otimes L^2 \otimes E)]$$

$$\mathrm{ch}[f_*(L \otimes E)] = f_*[\mathrm{ch}(L \otimes E)].$$

It is easy to compute that

$$\mathrm{ch}[N^* \otimes L^2 \otimes E] = 3 + \mathbb{P}_4 \times (3\Delta + 2C_0) + 2\mathbb{P}_4 \times (\Delta . C_0).$$

By Grothendieck-Riemann-Roch this implies

$$\mathrm{ch}[f_*(N^* \otimes L^2 \otimes E)] = 5 + 2\mathbb{P}_4 \times \mathfrak{o}$$

and hence

$$\Lambda^5 f_*(N^* \otimes L^2 \otimes E) = \mathcal{O}_{\mathbb{P}_4} \otimes \mathcal{O}_C(2\mathfrak{o}).$$

Similarly one has

$$\mathrm{ch}[L \otimes E] = 1 + \mathbb{P}_4 \times (\Delta + 4C_0) + 4\mathbb{P}_4 \times (\Delta . C_0)$$

which implies in the same way as above that

$$\Lambda^5 f_*(L \otimes E) = \mathcal{O}_{\mathbb{P}_4} \otimes \mathcal{O}_C(4\mathfrak{o})$$

and hence

$$\Lambda^5 f_*(\mathcal{O}_{\mathbb{P}_4}(1) \otimes L \otimes E) = \mathcal{O}_{\mathbb{P}_4}(5) \otimes \mathcal{O}_C(4\mathfrak{o}).$$

This concludes the proof of our proposition.

(VIII.2.4) The next point we want to make is that it is easy to write down an explicit equation for the quintic hypersurface

$Y_{\mathfrak{G}} = \mathrm{Sec}\ C$. To do this recall that we have an isomorphism

$$H^0(\mathcal{J}_C(2)) \cong H^0(N^*(2))$$

and that a basis of $H^0(\mathcal{J}_C(2))$ is given by the quadrics

$$\bar{Q}_0 = x_0^2 + ax_2x_3 - \frac{1}{a}x_1x_4$$

$$\bar{Q}_1 = x_1^2 + ax_3x_4 - \frac{1}{a}x_2x_0$$

$$\bar{Q}_2 = x_2^2 + ax_4x_0 - \frac{1}{a}x_3x_1$$

$$\bar{Q}_3 = x_3^2 + ax_0x_1 - \frac{1}{a}x_4x_2$$

$$\bar{Q}_4 = x_4^2 + ax_1x_2 - \frac{1}{a}x_0x_3 .$$

We then have

(VIII.2.5) <u>Proposition</u>: <u>The quintic hypersurface</u> $Y_{\mathfrak{G}} = \mathrm{Sec}\ C$ <u>is given by</u>

$$Y_{\mathfrak{G}} = \mathrm{Sec}\ C = \{\det(\frac{\partial \bar{Q}_i}{\partial x_j}) = 0\}.$$

<u>Proof</u>: The map

$$\psi_{\mathfrak{G}} : H^0(N^* \otimes L^2) \otimes \mathcal{O}_{\mathbb{P}_4} \rightarrow \mathcal{O}_{\mathbb{P}_4}(1) \otimes H^0(L)$$

is given by

$$\psi_{\mathfrak{G}}(\bar{Q}_i) = \sum_{j=0}^{4} x_j \otimes \frac{\partial \bar{Q}_i}{\partial x_j} .$$

Since $\bar{Q}_i$ is a quadric it follows that

$$\sum_{j=0}^{4} x_j \otimes \frac{\partial \bar{Q}_i}{\partial x_j} = \sum_{j=0}^{4} \frac{\partial \bar{Q}_i}{\partial x_j} \otimes x_j$$

and the assertion follows immediately.

(VIII.2.6) Remark: We want to mention that one can at least in principle employ a similar method to compute the equation of a general Y_M - or at least reduce the problem to a computation involving certain theta-functions. To sketch this, note that the quadric

$$Q_0' := \bar{Q}_0 + a\bar{Q}_2 + a\bar{Q}_3$$

is singular at the origin $\mathcal{O}$. We can then choose elements $h_0 = \mathrm{id}$, $h_1,\dots,h_4 \in H_5$ (depending on a) such that the quadrics

$$Q_i' := h_i(Q_0')$$

form a basis of $H^0(\mathcal{J}_C(2))$. Next, note that the bundle

$$\tilde{E} = E \otimes \mathcal{O}_{C\times C}(C_0) = \mathcal{O}_{C\times C}(\Delta)$$

has a section $t \in H^0(\tilde{E})$ such that

$$0 \neq t(P) := t \,|\, C\times\{P\} \in H^0(\mathcal{O}_C(P)).$$

Since Q_0' is singular at the origin $\mathcal{O}$ we can define sections

$$v_i := Q_i' \cdot \frac{t(h_i(P))}{t(h_i(\mathcal{O}))} \in H^0(N^* \otimes L^2 \otimes M)$$

where $M = \mathcal{O}_C(P-\mathcal{O})$. At least for general P the v_i will be linearly independent and the map

$$\phi_M : H^0(N^* \otimes L^2 \otimes M) \otimes \mathcal{O}_{\mathbb{P}_4} \rightarrow \mathcal{O}_{\mathbb{P}_4}(1) \otimes H^0(L \otimes M)$$

is given by

$$\phi_M(v_i) = \sum_{j=0}^{4} x_j \otimes \frac{\partial Q_i'}{\partial x_j} \cdot \frac{t(h_i(P))}{t(h_i(\mathcal{O}))}$$

where

$$w_{ij} := \frac{\partial Q_i'}{\partial x_j} \cdot \frac{t(h_i(P))}{t(h_i(\mathcal{O}))} \in H^0(L \otimes M).$$

On the other hand we can define a basis $x_0',\dots,x_4' \in H^0(L \otimes M)$ by

$$x_i'(z) := x_i(z - \tfrac{P}{5})$$

and there is a linear relation

$$w_{ij} = \sum_{k=0}^{4} \lambda_{ij}^k x_k' .$$

Then Y_M is given by

$$Y_M = \{\det(\sum_{j=0}^{4} \lambda_{ij}^k x_j)_{ik} = 0\}.$$

Now we are ready to give the main result of this section.

(VIII.2.7) Theorem (Ellingsrud/Laksov): Each point $P \notin \mathrm{Sec}\, C$ lies on a unique quintic $Y_M = Y_{M^{-1}}$ and the following holds:

(i) If $M^2 \neq \mathcal{O}_C$ then the normal bundle N_P splits and

$$N_P^*(2) = M \oplus M^{-1} .$$

(ii) If $M^2 = \mathcal{O}_C$ then there is an exact sequence

$$0 \to M \to N_P^*(2) \to M \to 0$$

and there is a non-empty open set of points $P \in Y_M$ such that N_P is indecomposable.

Proof: We have already seen that each point P lies on a unique quintic $Y_M = Y_{M^{-1}}$. It then follows from (VIII.2.2) that there is a non-zero map

$$s : M \to N_P^*(2).$$

Since $P \notin \mathrm{Sec}\, C$ it follows that

$$h^0(N_P^*(2)) = 0$$

and hence s cannot have any zeroes, i.e., it defines an exact sequence

$$0 \to M \to N_P^*(2) \to M^{-1} \to 0 .$$

If $M^2 \neq \mathcal{O}_C$ this sequence must necessarily split since

$$h^1(M^2) = 0.$$

Now assume that $M^2 = \mathcal{O}_C$. Then the above sequence is non-split if and only if

$$h^0(N_P^*(2) \otimes M) = 1.$$

By semi-continuity it will be enough to find at least one point for which this holds. To do this we choose a point $P' \in \operatorname{Sec} C$ such that $C_{P'}$ is a nodal curve on a quadric cone $Q_{P'}$. We can do this because of (IV.3). It then follows from (VIII.1.1) that

$$N_{P'}^*(2) = \mathcal{O}_C(-E) \oplus \mathcal{O}_C(E)$$

and this concludes the proof.

(VIII.2.8) Remarks: (i) One consequence of the above theorem is that the normal bundle of a smooth elliptic space curve of degree 5 is always semi-stable. This is no longer true for higher degree as was shown in [10].

(ii) I have heard that Ellingsrud has constructed for each theta-characteristic $M \neq \mathcal{O}_C$ examples of smooth curves $C_P \subseteq \mathbb{P}_3$ such that

$$N_P^*(2) = M \oplus M.$$

This means that every semi-stable rank 2 bundle on C with determinant $\mathcal{O}_C$ occurs as the normal bundle of a smooth elliptic curve of degree 5 with the exception of the trivial bundle.

3. The quintic hypersurfaces Y_M

The purpose of this section is to describe the linear family $\{Y_M\}$. To do this we shall first determine the base locus of this linear system. The next step is to prove that the equations of the Y_M are

H_5-invariant. We shall then prove that the Y_M are exactly those quintics which contain the tangent surface of C and are singular along C and relate this to the Horrocks-Mumford bundle. Finally we want to determine the subspace $U \subseteq \Gamma_H(\mathcal{J}_C^2(5))$ which belongs to the family $\{Y_M\}$ in terms of the basis $\Gamma_H(\mathcal{J}_C^2(5))$ which we have found earlier.

(VIII.3.1) We shall first have to recall a few basic facts from earlier chapters. We showed in chapter IV that there is a 1-dimensional family of rank 3 quadrics through C whose singular lines form a ruled surface $F \subseteq \mathbb{P}_4$ of degree 15. Moreover, we gave an explicit description of a birational map

$$\phi : S^2C = \mathbb{P}(E_0) \to F$$

Here E_0 is the unique indecomposable rank 2 bundles over C with determinant $\mathcal{O}_C(\mathfrak{o})$. We had also seen that

$$D = \phi^{-1}(C) \sim C_0 + 12f_{\mathfrak{o}}$$

$$\phi^*(H) \sim C_0 + 7f_{\mathfrak{o}}$$

where $C_0 \subseteq \mathbb{P}(E_0)$ is the unique section with

$$\mathcal{O}_{C_0}(C_0) = \mathcal{O}_{C_0}(\mathfrak{o}) .$$

In (VII.1) we studied the desingularization

$$p : \tilde{X} = \mathbb{P}(E_1) \to \mathrm{Tan}\, C$$

of the tangent developable of C. Recall that E_1 was given by the non-split extension

$$0 \to \mathcal{O}_C \to E_1 \to \mathcal{O}_C \to 0.$$

Let $C_1 \subseteq \mathbb{P}(E_1)$ be the unique section with $C_1^2 = 0$. Then we had seen that

$$p^{-1}(C) = C_1$$

$$p^*(H) \sim C_1 + 5f_{\mathcal{O}}.$$

It follows directly from our explicit description of the map ϕ that the intersection of the tangent surface of C with F consists of two curves, namely

$$\mathrm{Tan}\, C \cap F = C \cup E$$

where

$$E = \phi(\bar{\Delta})$$

and $\bar{\Delta} \subseteq S^2C$ is the image of the diagonal $\Delta \subseteq C \times C$ under the canonical projection. Since

$$\bar{\Delta} \sim 4C_0 - 2f_{\mathcal{O}}$$

we can compute the degree of E to be

$$\deg E = \bar{\Delta}.H = (4C_0 - 2f_{\mathcal{O}}) \cdot (C_0 + 7f_{\mathcal{O}}) = 30.$$

Moreover, since

$$D \cap \bar{\Delta} = \{ \overline{(P,P)} \in S^2C \,;\, 5P \sim 5\mathcal{O}\}$$

it follows that

$$C \cap E = \{P \in C,\ 5P \sim 5\mathcal{O}\}$$

i.e. C and E intersect exactly in the 5-torsion points of C.

The last remark we want to make concerns the pre-image of the curve E on the desingularization of the tangent surface. Since E intersects each tangent only once, $p^{-1}(E)$ is a section of $\mathbb{P}(E_1)$ and since C and E intersect exactly in the 5-torsion points of C it follows easily that

$$p^{-1}(E) \sim C_1 + 25f_{\mathcal{O}}.$$

We shall now turn to the common intersection of the family Y_M:

(VIII.3.2) Proposition: The base locus of the linear system $\{Y_M\}$ consists of the union of the tangent surface Tan C with the surface F, i.e.

$$\bigcap_M Y_M = \mathrm{Tan}\ C \cup F.$$

Proof: Since $Y_{\mathcal{O}} = \mathrm{Sec}\ C$ it follows that the common intersection of the Y_M is contained in the secant variety of C. If $P \in \mathrm{Sec}\ C - \mathrm{Tan}\ C$ then it follows from (VIII.1.1) that P is in the common intersection of the Y_M if and only if C_P lies on a quadric cone Q_P, i.e., if and only if $P \in F$. So it remains to see that the tangent surface Tan C is contained in each of the hypersurfaces Y_M. This follows easily from degree considerations but we also want to give a direct proof which follows [6].

To do this we fix some point $P \in \mathrm{Tan}\ C - C$. Again C_P lies on a unique quadric Q_P. By arguments which we have already used in the proof of (VIII.1.1) it follows that C_P has no other singularities but a simple cusp over some point R. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & F_P & \to & \pi_P^*(\Omega_{\mathbb{P}_3}(1)) & \to & \Omega_C(1) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N^*(1) & \to & \Omega_{\mathbb{P}_4}(1)|C & \to & \Omega_C(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & V^*/_{V_P} \otimes \mathcal{O}_C & = & V^*/_{V_P} \otimes \mathcal{O}_C & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Over the point R the map

$$\pi_P^*\Omega_{\mathbb{P}_3} \to \Omega_C$$

has a simple zero, i.e. we have an exact sequence

$$0 \to F_P \to \pi_P^*(\Omega_{\mathbb{P}_3}(1)) \to L(-R) \to 0$$

from which one concludes that

$$\Lambda^2 F_P(1) = \mathcal{O}_C(R).$$

Now assume that Q_P is smooth. Then similarly as in (VIII.1.1) this quadric defines a subbundle $\mathcal{O}_C \subseteq F_P(1)$ and hence an exact sequence

$$0 \to \mathcal{O}_C \to F_P(1) \to \mathcal{O}_C(R) \to 0$$

which proves that

$$h^0(F_P(1) \otimes M) \neq 0$$

for all $M \in \mathrm{Pic}^0 C$, i.e.

$$P \in \bigcap_M Y_M .$$

It remains to consider the case where Q_P is a quadric cone. Then the vertex E of Q_P must be a smooth point of C_P and we get an exact sequence

$$0 \to \mathcal{O}_C(E) \to F_P(1) \to \mathcal{O}_C(R-E) \to 0$$

which must split because of

$$h^1(\mathcal{O}_C(2E-R)) = 0 .$$

This concludes the proof.

The following consequence was also known to Ellingsrud and Laksov although their argument is different.

(VIII.3.3) Corollary: The quintic hypersurfaces Y_M are irreducible.

Proof: First note that the secant variety Sec C is irreducible. Assume that some quintic Y_M is reducible. Then it follows from the above proposition that Y_M must be the union of a quadric Q_2 and a cubic hypersurface Q_3. Moreover Q_2 must intersect Sec C in the tangent surface Tan C and Q_3 meets Sec C in F. By reasons of degree Q_2 cannot contain the surface F as well. Hence

$$\phi^*(Q_2) = D + \bar{\Delta} + D'$$

where D' is some effective divisor on S^2C. But this is impossible since

$$D + \bar{\Delta} \sim 5C_0 + 10f_{\mathfrak{o}}$$

and

$$\phi^*(2H) \sim 2C_0 + 14f_{\mathfrak{o}} .$$

The next step is to prove that the equations of the quintic hypersurfaces Y_M are invariant under the Heisenberg group H_5.

(VIII.3.4) Proposition: The equations of the quintic hypersurfaces Y_M are H_5-invariant, i.e. if U is the affine 2-dimensional space of quintic forms which belongs to the family Y_M then

$$U \subseteq \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5)).$$

In particular the Y_M are linear combinations of the fundamental pentahedra.

Proof 1: We have already seen that the Y_M contain the tangent surface Tan C. On the other hand we have seen in (VII.3.4) that

$$\Gamma(\mathcal{J}_{\text{Tan } C}(5)) = \Gamma_H(\mathcal{J}_{\text{Tan } C}(5))$$

and this implies our assertion immediately.

Proof 2: Since the above proof makes use of the whole set-up of the Horrocks-Mumford bundle it may also be interesting to give a second independent proof. To do this we first claim that the Y_M are invariant under H_5 as hypersurfaces, i.e. that H_5 operates on U by scalar multiplication. This is clear for $Y_{\mathcal{O}} \doteq$ Sec C. Let Y_M be some other quintic. Moreover let $h \in H_5$ be an element of the Heisenberg group which operates on C by translation with a point $P = P_h$. Let

$Q \in Y_M$ be any point not on Sec C and set

$$Q' := h(Q).$$

By π_Q and $\pi_{Q'}$ we denote the projections from Q and Q' respectively. The relation between these two maps from C to 3-space is

$$\pi_{Q'} = \pi_Q \circ T_{-P}$$

where T_{-P} denotes translation on C by -P. Hence

$$N_{Q'} = T^*_{-P} N_Q$$

and since T^*_{-P} operates trivially on $\mathrm{Pic}^0 C$ it follows that

$$N_{Q'} = N_Q$$

which implies that $Q' \in Y_M$, i.e. the hypersurface Y_M is H_5-invariant.

It now suffices to show that the equation of at least one hypersurface Y_M is H_5-invariant. Recall from (VIII.2.5) that

$$Y_{\mathcal{O}} = \{\det\left(\frac{\partial \bar{Q}_i}{\partial x_j}\right) = 0\}.$$

It is convenient to write

$$\bar{Q}_0 = \sum_{i,j} q_{ij} x_i x_j$$

where

$$q_{00} = 1, \quad q_{23} = q_{32} = \frac{a}{2}, \quad q_{14} = q_{41} = -\frac{1}{2a}$$

and $q_{ij} = 0$ otherwise. One finds that

$$\frac{1}{2}\left(\frac{\partial \bar{Q}_i}{\partial x_j}\right) = \begin{pmatrix} q_{00}x_0 & q_{14}x_4 & q_{23}x_3 & q_{32}x_2 & q_{41}x_1 \\ q_{41}x_2 & q_{00}x_1 & q_{14}x_0 & q_{23}x_4 & q_{32}x_3 \\ q_{32}x_4 & q_{41}x_3 & q_{00}x_2 & q_{14}x_1 & q_{23}x_0 \\ q_{23}x_1 & q_{32}x_0 & q_{41}x_4 & q_{00}x_3 & q_{14}x_2 \\ q_{14}x_3 & q_{23}x_2 & q_{32}x_1 & q_{41}x_0 & q_{00}x_4 \end{pmatrix}$$

which we can also write as

$$\frac{1}{2}\left(\frac{\partial \bar{Q}_i}{\partial x_j}\right) = (q_{j-i,i-j}x_{2i-j})_{ij} \ .$$

Hence we have to prove H_5-invariance of the determinant

$$D = \sum_{\lambda \in S_5} \prod_i q_{\lambda(i)-i,i-\lambda(i)}x_{2i-\lambda(i)}$$

which can be done as follows:

(i) $$\tau(D) = \sum_\lambda \prod_i \varepsilon^{-2i+\lambda(i)} q_{\lambda(i)-i,i-\lambda(i)}x_{2i-\lambda(i)}$$

Since

$$\sum_i (-2i+\lambda(i)) \equiv 0 \bmod 5$$

it follows immediately that $\tau(D) = D$.

(ii) $$\begin{aligned} \sigma(D) &= \sum_\lambda \prod_i q_{\lambda(i)-i,i-\lambda(i)}x_{2i-\lambda(i)-1} \\ &= \sum_\lambda \prod_i q_{\lambda(i)-i+1,i-\lambda(i)-1}x_{2(i-1)-\lambda(i)} \\ &= \sum_\lambda \prod_i q_{\lambda(i-1)-(i-1),(i-1)-\lambda(i-1)}x_{2(i-1)-\lambda(i-1)} \\ &= \sigma(D) . \end{aligned}$$

This concludes the proof.

Although the fact that

$$U \subseteq \Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$$

provides us with some information about the family $\{Y_M\}$, since we know the space $\Gamma_H(\mathcal{O}_{\mathbb{P}_4}(5))$ of invariant quintic forms quite well, the situation is not yet quite satisfactory. One would like to have an accessible necessary and sufficient condition for an invariant quintic to define one of the Y_M. Of course one could say that the linear family $\{Y_M\}$ is determined by its base locus $\mathrm{Tan}\ C \cup F$. But we want to replace the condition that the Y_M contain the ruled surface F by another condition which will then enable us to relate this description to the Horrocks-Mumford bundle.

(VIII.3.5) <u>Theorem</u>: <u>The space</u> U <u>of quintic forms which belong to the family</u> $\{Y_M\}$ <u>is given by</u>

$$U = \Gamma_H(\mathcal{J}_{\mathrm{Tan}\ C}(5)) \cap \Gamma_H(\mathcal{J}_C^2(5))$$

<u>i.e. an invariant quintic form defines one of the hypersurfaces</u> Y_M <u>if and only if it vanishes on the tangent surface</u> $\mathrm{Tan}\ C$ <u>and is singular along</u> C.

<u>Proof</u>: It will clearly be sufficient to prove that

$$\Gamma_H(\mathcal{J}_F(5)) = \Gamma_H(\mathcal{J}_C^2(5)) \quad .$$

We first want to show that any quintic Q through F must be singular along C. To see this recall that there are exactly 4 lines belonging to F through each point P of C. These 4 lines correspond to the 4 quadric cones through the curve C_P. They span $\mathbb{P}_4$ and this implies that every hypersurface containing F must be singular along C.

In order to prove the other inclusion assume that $Q \in \Gamma_H(\mathcal{J}_C^2(5))$ defines a quintic hypersurface which does not contain F. Since Q is invariant it must go through the 25 lines $L_{k\ell}$. Moreover since Q is singular along C it follows that the pullback of Q to S^2C is of the form

$$\phi^*(Q) = 2D + \sum_{5P\sim 5\mathcal{O}} f_P + D'$$

for some effective divisor $D' \subseteq S^2C$. But this gives a contradiction, since

$$2D + \sum_{5P\sim 5\mathcal{O}} f_P \sim 2C_O + 49f_{\mathcal{O}}$$

whereas

$$\phi^*(5H) \sim 5C_O + 35f_{\mathcal{O}}.$$

This proves the theorem.

Remark: We saw in (VI.2.4) that

$$\dim \Gamma_H(\mathcal{J}_C^2(5)) = 3 .$$

There we also constructed an explicit basis of this vector space.

We now want to rephrase this result in terms of the Horrocks-Mumford bundle F. Recall that

$$h^0(F) = 4$$

and that all the sections of F are H_5-invariant, i.e.

$$\Gamma(F) = \Gamma_H(F).$$

(VIII.3.6) Lemma: The vector space

$$W := \{f \in \Gamma(F) \; ; \; f(\mathcal{O}) = 0\}$$

has dimension 3.

Proof: This is an immediate consequence of proposition (VI.2.4) which says that the scheme-theoretic intersection of the H_5-invariant quintics $f \wedge f'$ consists exactly of the 25 skew lines $L_{k\ell}$. Hence all sections are linearly dependent over the point $\mathcal{O} \in L_{00}$ but there must be at least one section which does not vanish at $\mathcal{O}$ since the intersection of these quintics could not be smooth at this point otherwise.

Note that the sections $f \in W$ must vanish at all the 5-torsion points of C since they are invariant under the Heisenberg group H_5. By our results from chapter VII there exists a unique section $s \in W$ such that

$$\operatorname{Tan} C = \{s = 0\}.$$

We can now formulate the following

(VIII.3.7) Theorem: The space U is given by

$$U = s \wedge W = \{s \wedge f \; ; \; f \in W\}.$$

Proof: Clearly any quintic $s \wedge f$ contains the tangent surface Tan C. Since $s(\mathcal{O}) = f(\mathcal{O}) = 0$ any such quintic must be singular at the origin $\mathcal{O}$ and hence at all the 5-torsion points of C. But any quintic hypersurface which is singular at 25 points of the degree 5 curve C must be singular along the whole curve. The assertion now follows from our theorem.

(VIII.3.8) We want to conclude this section with yet one more description of U, i.e., we want to determine the 2-dimensional subspace $U \subseteq \Gamma_H(\mathcal{J}_C^2(5))$ in terms of the basis of the 3-dimensional space $\Gamma_H(\mathcal{J}_C^2(5))$ which we have found in (VI.2.4). Unfortunately, the numerical data involved do not seem to be very illuminating.

We shall first have to say a word about the tangent $T_{\mathcal{O}}$ of C at the origin $\mathcal{O}$. Using the quadric equations $\bar{Q}_i$ of C it is easy to see that $T_{\mathcal{O}}$ is given by the linear equations

$$-x_1 - ax_2 + ax_3 + x_4 = 0$$

$$-\frac{1}{a}x_0 - 2ax_1 + a^2x_3 - ax_4 = 0$$

$$a^2x_0 + \frac{1}{a}x_1 + 2x_2 + x_3 = 0$$

The tangent $T_{\mathcal{O}}$ is invariant under the involution ι (as a line) and contains exactly two fixed points of ι, namely the origin

$$\mathcal{O} = (0 : -a : 1 : -1 : a)$$

and the point

$$S = (10a^3 : -3a - a^6 : 1-3a^5 : 1-3a^5 : -3a - a^6).$$

(VIII.3.9) Lemma: $U = \{Q \in \Gamma_H(\mathcal{J}_C^2(5)) \,;\, Q(S) = 0\}$

Proof: We have already seen that every quintic $Q \in \Gamma_H(\mathcal{J}_C^2(5))$ contains the surface F. Let us assume that Q does not contain the tangent surface Tan C. Then we have to show that $Q(S) \neq 0$. Since Q is singular along C it follows that the pullback of Q to the desingularization $\tilde{X} = \mathbb{P}(E_1)$ contains the section C_1 at least 4-fold plus the curve $p^{-1}(E)$. Since

$$4C_1 + p^{-1}(E) \sim 5C_1 + 25f_{\mathcal{O}} \sim 5H$$

it follows that

$$Q \cap \text{Tan } C = C \cup E .$$

Since S is neither on C nor on E this concludes the proof.

Now recall from (VI.3.2) that the following quintic forms are a basis of $\Gamma_H(\mathcal{J}_C^2(5))$

$$Q_0 := x_0\bar{Q}_0^2 + x_1\bar{Q}_1^2 + x_2\bar{Q}_2^2 + x_3\bar{Q}_3^2 + x_4\bar{Q}_4^2$$

$$Q_1 := x_0\bar{Q}_2\bar{Q}_3 + x_1\bar{Q}_3\bar{Q}_4 + x_2\bar{Q}_4\bar{Q}_0 + x_3\bar{Q}_0\bar{Q}_1 + x_4\bar{Q}_1\bar{Q}_2$$

$$Q_2 := x_0\bar{Q}_1\bar{Q}_4 + x_1\bar{Q}_2\bar{Q}_0 + x_2\bar{Q}_3\bar{Q}_1 + x_3\bar{Q}_4\bar{Q}_2 + x_4\bar{Q}_0\bar{Q}_3 .$$

What remains to be done now, is to compute the value of the quintics Q_i at the point S. Using the computer of Brown University, J.Lubin found the following result:

$$\lambda_0 := Q_0(S) = 32a^2(a^{10} + 11a^5 - 1)^2(1 + 14a^5 - a^{10})$$

$$\lambda_1 := Q_1(S) = 160a^5(a^{10} + 11a^5 - 1)^2(2 - a^5)$$

$$\lambda_2 := Q_2(S) = 160a^4(a^{10} + 11a^5 - 1)^2(1 + 2a^5)$$

Since we are not interested in common factors of the λ_i we set

$$\bar{\lambda}_0 := 1 + 14a^5 - a^{10}$$

$$\bar{\lambda}_1 := 5a^3(2 - a^5)$$

$$\bar{\lambda}_2 := 5a^2(1 + 2a^5) \ .$$

This finally leads us to

(VIII.3.10) <u>Proposition</u>: <u>The space</u> U <u>which belongs to the linear family</u> $\{Y_M\}$ <u>is given by</u>

$$U = \{ \sum_{i=0}^{2} c_i Q_i \ ; c_0\bar{\lambda}_0 + c_1\bar{\lambda}_1 + c_2\bar{\lambda}_2 = 0 \}.$$

IX. Elliptic quintics and special surfaces of small degree

In this chapter we want to say a few more words about the normal bundle N_P of a smooth elliptic quintic $C_P \subseteq \mathbb{P}_3$ from a geometric point of view. In the previous chapter we have seen that there always exists a subbundle $M \subseteq N_P^*(2)$ of degree 0 which in general splits off. Note that 0 is the maximal degree of any subbundle of $N_P^*(2)$. In this chapter we want to describe, how a given subbundle M of maximal degree can be realized geometrically. Recall that a smooth elliptic quintic $C_P \subseteq \mathbb{P}_3$ never lies on a quadric. On the other hand every such curve lies e.g. on five independent cubic surfaces and every such surface defines a map

$$s : \mathcal{O}_C \to N_P^*(3) = \mathcal{J}_{C_P}(3) / \mathcal{J}^2_{C_P}(3) .$$

The map s has a zero whenever C passes through a singularity of S. Hence, if S has 5 singularities along C (properly counted), then S represents a subbundle of maximal degree. Similarly a quartic surface must have 10 singularities along C to give rise to a subbundle $M \subseteq N_P^*(2)$ of degree 0. The problem is, whether it is always possible to find suitable surfaces (of low degree) with sufficiently many singularities along C. The aim of this chapter is, to prove, that a given subbundle $M \subseteq N_P^*(2)$ of (maximal) degree 0 can always be represented by a surface S which is the projection of a complete intersection of two quadric hypersurfaces in $\mathbb{P}_4$. We shall then see that in special cases (which are a degeneration of the situation described above), a maximal subbundle M can be represented by certain ruled cubic surfaces.

1. The general case

(IX.1.1) As always let $C \subseteq \mathbb{P}_4$ be an elliptic normal quintic. Moreover, let

$$L = \{ \lambda_0 Q_0 + \lambda_1 Q_1 \ ; \ (\lambda_0 : \lambda_1) \in \mathbb{P}_1 \}$$

be a pencil of quadrics through C whose base locus is

$$S = Q_0 \cap Q_1 .$$

Then the pencil L gives rise to a map

$$\bar{\phi} : 2\mathcal{O}_C \xrightarrow{(Q_0,Q_1)} N_C^*(2) = \mathcal{J}_C(2)/\mathcal{J}_C^2(2) \ .$$

Next let $P \notin \mathrm{Sec}\, C$ be a fixed centre of projection. Then we have a commutative and exact diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & N_P^*(2) \\
 & & \downarrow \\
2\mathcal{O}_C & \xrightarrow{\bar{\phi}} & N_C^*(2) \\
\downarrow{\scriptstyle (Q_0',Q_1')} & & \downarrow \\
\mathcal{O}_C(1) & = & \mathcal{O}_C(1) \\
 & & \downarrow \\
 & & 0
\end{array}$$

The pencil L, therefore, gives rise to a map

$$\phi : \mathcal{O}_C(-1) \to N_P^*(2)$$

where

$$\phi := \bar{\phi} \circ \begin{pmatrix} -Q_1' \\ Q_0' \end{pmatrix} .$$

Note that the map ϕ (up to a non-zero scalar) does not depend on the choice of Q_0 and Q_1, but only on the pencil L. Hence we shall from now on write ϕ_L instead of ϕ.

(IX.1.2) The map ϕ_L has the following <u>geometric interpretation</u>: First assume for the sake of simplicity that $P \notin S$ and that S is irreducible. Then the projection from P maps S onto an irreducible quartic surface S_P which contains the curve C_P. The surface S_P is singular along a double conic C_0 which arises in the following way. Since $P \notin S$ there exists a unique quadric $Q \in L$ which contains P. Moreover, since $P \notin \mathrm{Sec}\, C$ it follows that Q is smooth at P. Let H_P be the tangent hyperplane of Q at P. Then the degree 4 curve

$$D := H_P \cap S$$

is the branch curve of the projection π_P and will be mapped 2:1 onto the double conic C_O. The quintic C_P intersects the conic C_O in 5 points, namely the images of the hyperplane section $H_P \cap C$.

Since S_P has degree 4 it defines a map

$$s : \mathcal{O}_C \to N_C^*(4)$$

Note that (up to a twist) the kernel s^t is the normal bundle N_{C_P/S_P} of C_P in S_P - at least outside the singularities of S_P. The map s has a zero whenever C_P goes through a singular point of S_P, hence in particular at those points where C_P intersects the double conic C_O. Therefore, s can be viewed as a morphism

$$s : \mathcal{O}_C \to N_C^*(3)$$

and this map is just $\phi_L(1)$. We can also say this as follows: Let $R \in C$ be a smooth point of S and let E be the tangent plane of S at R. Then the projection from P maps E onto the tangent plane E_P of S_P at $\pi_P(R)$ unless E contains P in which case we get a pinch point of S_P. But the plane E_P just represents the image of the morphism ϕ_L in $N_P^*(2) \subseteq V_P \otimes \mathcal{O}_C(1)$ over the point R. This discussion also shows that the map ϕ_L has a zero at R if and only if S is either singular at R or if R lies over a pinch point of S_P (see also lemma (IX.1.6)).

A similar interpretation holds if $P \in S$ or if S is reducible. First note that if S contains P then it is smooth at P since there is no quadric through C which is singular in P. Then the projection from P maps the blow up $\tilde{S}$ of S at P to $\mathbb{P}_3$. The image of $\tilde{S}$ under this map then plays the same role as the surface S_P did in the above case, and we shall again denote it by S_P. We finally want to remark that it can in fact happen that the surface S is reducible. Since C does not lie on a quadric surface this can only be the case if S is the union of a plane $\mathbb{P}_2$ and a cubic surface F which then must contain C. We shall come back to this case in section 2 of this chapter.

Motivated by this discussion we make the following

(IX.1.3) <u>Definition</u>: Let $M \subseteq N_P^*(2)$ be a line subbundle. Then we say that the surface S (resp. its projection S_P) <u>represents the subbundle</u> M if and only if the map ϕ_L is non-zero and factors through M, i.e. if and only if there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_C(-1) \arrow[r, "\phi_L \neq 0"] \arrow[dr, dashed, "\phi_L'"'] & N_P^*(2) \\ & M \arrow[u, phantom, "\cup\!\!\mid"] \end{tikzcd}$$

(IX.1.4) We have seen in (IV.2) that

$$h^0(\mathcal{J}_C(2)) = 5 .$$

Hence let

$$\mathbb{P}_4' := \mathbb{P}_4(\Gamma(\mathcal{J}_C(2)))$$

be the projectively 4-dimensional space of quadrics through C. By

$$G = Gr(1,4)$$

we denote the Grassmannian of lines in $\mathbb{P}_4'$, i.e. the variety of pencils L of quadrics which contain C.

For the rest of this section we want to fix once and for all a centre of projection $P \notin \mathrm{Sec}\, C$. Let $R \in C$ be some point on the elliptic normal quintic C. Then the quadrics $Q \in \mathbb{P}_4'$ which are singular at R define a line

$$L_0 \subseteq \mathbb{P}_4'$$

The line L_0 is given by the pencil of quadrics which cuts out the elliptic quartic C_R. Next we define a hyperplane $\mathbb{P}_3 \subseteq \mathbb{P}_4'$ by

$$\mathbb{P}_3 := \{Q \in \mathbb{P}_4' ; \Big(\sum_{i=0}^{4} \frac{\partial Q}{\partial x_i}(R) x_i \Big)(P) = 0\} .$$

I.e. $\mathbb{P}_3$ consists of all those quadrics which are either singular at R or whose tangent hyperplane at R goes through P. That this defines a hyperplane $\mathbb{P}_3 \subseteq \mathbb{P}_4'$ follows from the fact that the condition

$$\left(\sum_{i=0}^{4} \frac{\partial Q}{\partial x_i}(R)x_i \right)(P) = 0$$

is linear and non-empty. The latter holds, since C is the scheme-theoretic intersection of the quadrics containing it.

We now define varieties

$$S_R^1 := \{L \in G \; ; \; L \cap L_0 \neq \phi\}$$

$$S_R^2 := \{L \in G \; ; \; L \subseteq \mathbb{P}_3\}.$$

Note that S_R^1 is just the variety of pencils L for which the base locus S is singular at R. Similarly S_Q^2 consists of those pencils for which the tangent plane of S at R (if it exists) contains P. Since $P \neq R$ this just means that $\pi_P(R)$ is a pinch point of S_P.

(IX.1.5) Observation: Note that both varieties S_R^1 and S_R^2 are defined by Schubert conditions. More precisely, in the cohomology ring $H^*(G)$ of G one has

$$S_R^1 \sim \sigma_{20}$$

$$S_R^2 \sim \sigma_{11}$$

where σ_{20} and σ_{11} are the respective Schubert cycles. For this notation see [7, p.139 ff.].

Now let

$$L = \{\lambda_0 Q_0 + \lambda_1 Q_1 \; ; \; (\lambda_0 : \lambda_1) \in \mathbb{P}_1\}$$

be a pencil of quadrics through C and let

$$\phi_L : \mathcal{O}_C(-1) \to N_C^*(2)$$

be the corresponding map as defined in (IX.1.1).

(IX.1.6) Lemma: The map ϕ_L has a zero at $R \in C$ if and only if

$$L \in S_R := S_R^1 \cup S_R^2 .$$

Proof: Assume that $L \in S_R^1$. We can then assume that Q_0 is singular at R. This implies that the map

$$Q_0 : \mathcal{O}_C \to N_C^*(2)$$

has a zero at R and the same holds for the map

$$Q_0' : \mathcal{O}_C \to \mathcal{O}_C(1).$$

But this implies that the map

$$\phi_L = -Q_0Q_1' + Q_1Q_0'$$

also has a zero at R. Next suppose that $L \in S_R^2$. Then Q_0' and Q_1' have zeroes at R and so has ϕ.

Finally assume that $L \notin S_R^1 \cup S_R^2$. We can then assume that Q_0 and Q_1 are smooth at R and that only the tangent hyperplane of Q_0 at R contains P. This means that of all the maps involved, only Q_0' has a zero at R and hence we are done.

The next proposition plays an important role in the proof of the main result of this chapter.

(IX.1.7) Proposition: For each pencil $L \in G$ of quadrics through C the associated map

$$\phi_L : \mathcal{O}_C(-1) \to N_C^*(2)$$

is non-zero.

Proof: Let

$$L = \{\lambda_0 Q_0 + \lambda_1 Q_1 \ ; \ (\lambda_0 : \lambda_1) \in \mathbb{P}_1\}$$

be a pencil of quadrics through C and let

$$S = Q_0 \cap Q_1$$

be its base locus. We have to show that a general point $R \in C$ is a smooth point of S which is not mapped to a pinch point of S_P. To do this we consider the maps

$$Q_0, Q_1 : \mathcal{O}_C \to N_C^*(2) .$$

Every quadric Q through C is in at most one point R of C singular. If Q has an isolated singularity at R one sees exactly as in the proof of (VIII.1.1) that the map Q has a simple zero at R. The only other possibility is that Q is singular along a line L_0 which meets C in R. But then it follows from the fact that L_0 is not tangent to C in R that Q again has a simple zero at R. Now assume that S is singular along the whole curve C. This can only be the case, if the two maps

$$Q_0, Q_1 : \mathcal{O}_C \to N_C^*(2)$$

define the same subbundle $M \subseteq N_C^*(2)$. By what we have said above, M has degree 0 or 1. In any case Q_0 and Q_1 define the same section (up to a scalar) in $N_C^*(2)$. But this is a contradiction to the fact that

$$H^0(\mathcal{J}_C(2)) \cong H^0(N_C^*(2)) .$$

So it remains to see that a general point of C is not mapped to a pinch point of S_P. We can assume that the centre of projection P lies on Q_0 and is a smooth point of Q_0 since $P \notin \operatorname{Sec} C$. Let H_P be the tangent hyperplane of Q_0 at P. Now let $R \in C$ be a point which does not lie on H_P. Then R cannot lie over a pinch point. Otherwise the line L_1 through P and R would be tangent to Q_0 at R and hence would be contained in Q_0. But this would imply that $L_1 \subseteq H_P$ which we have excluded.

This proves our proposition.

We are now ready to formulate and prove the main result of this section.

(IX.1.8) <u>Theorem</u>: <u>For each subbundle</u> $M \subseteq N_P^*(2)$ <u>of degree</u> 0 <u>there</u>

<u>exists a pencil</u> L <u>of quadrics such that the associated surface</u> S <u>represents the subbundle</u> M.

<u>Proof</u>: It follows from theorem (VIII.2.7) that

$$h^0(N_P^*(3)) = 10.$$

We set

$$\mathbb{P}_9 := \mathbb{P}(H^0(N_P^*(3))).$$

From what we have said above and from proposition (IX.1.7) it follows that we have a map

$$\begin{aligned} \phi : G &\to \mathbb{P}_9 \\ L &\mapsto \phi_L. \end{aligned}$$

Now let $M \subseteq N_P^*(2)$ be a fixed line subbundle of degree 0. Since

$$h^0(M(1)) = 5$$

it follows that the maps

$$s : \mathcal{O}_C(-1) \to N_P^*(2)$$

which factor through M form a (projectively) 4-dimensional space

$$\mathbb{P}_4 \subseteq \mathbb{P}_9 .$$

It will be enough to prove that

$$\phi(G) \cap \mathbb{P}_4 \neq \phi.$$

But this will follow, if we can show that

$$\dim \phi(G) \geq 5.$$

Since the Grassmannian G has dimension 6, it will be enough to show that for general $L \in G$ one has

$$\dim \phi^{-1}(\phi(L)) \leq 1.$$

In order to see this, we fix a point $R_1 \in C$ and consider the variety

$$S_{R_1} = S^1_{R_1} \cup S^2_{R_2} \quad .$$

Recall that the map ϕ_L has a zero at R_1 if and only if $\phi_L \in S_{R_1}$. Moreover, we had seen that S_{R_1} has dimension 4 and that in $H^*(G)$ one has

$$S_{R_1} \sim \sigma_{20} + \sigma_{11} \ .$$

Note that it follows from the Schubert calculus on $Gr(1,4)$ that S_{R_1} intersects each cycle of dimension at least 2.

It is a consequence of proposition (IX.1.7) that for each $L \in S_{R_1}$ the associated surface S is smooth at a general point R_2 of C and that its tangent plane at R_2 does not contain P. Hence

$$S_{R_1} \cap S_{R_2} \neq \phi$$

and for each component V_i of this intersection we have

$$2 \leq \dim V_i \leq 3.$$

As before we can choose a point $R_3 \in C$ such that

$$S_{R_1} \cap S_{R_2} \cap S_{R_3} \neq \phi$$

and such that each component W_i of this intersection has dimension

$$0 \leq \dim W_i \leq 2 \ .$$

If all these components are at most 1-dimensional then we are done, because for each

$$L \in S_{R_1} \cap S_{R_2} \cap S_{R_3}$$

we have

$$\phi^{-1}(\phi(L)) \subseteq S_{R_1} \cap S_{R_2} \cap S_{R_3}$$

since these are the only pencils for which the associated map ϕ_L has zeroes at R_1, R_2 and R_3. If there is a component W_i of dimension 2 then we can choose yet another point P_4 such that

$$S_{R_1} \cap S_{R_2} \cap S_{R_3} \cap S_{R_4} \neq \phi$$

and where each component of this intersection has dimension at most 1. Then we are done by the same reasoning as above.

This proves the theorem.

2. A special case

In this section we want to study a special case of the above situation, namely the case where the quartic surface S degenerates into the union of a ruled cubic surface F and a plane $\mathbb{P}_2$. We shall, however, approach the problem from a somewhat different angle.

(IX.2.1) Again let $C \subseteq \mathbb{P}_4$ be an elliptic normal quintic with origin $\mathcal{O}$. For each point $P_0 \in C$ we can define an involution

$$\begin{aligned} \kappa : C &\to C \\ z &\mapsto -z + P_0 \; . \end{aligned}$$

If $P_0 = \mathcal{O}$ then this is just the involution ι which we have considered earlier. Note that κ lifts to a linear map of $\mathbb{P}_4$ if and only if P_0 is a 5-torsion point of C. We can associate a ruled surface F to the involution κ by taking the union of the lines spanned by z and $\kappa(z)$, resp. the tangents of C at z if $z = \kappa(z)$, i.e.

$$F = \bigcup_{z \in C} \overline{(z, \kappa(z))} \; .$$

Clearly F is a rational ruled surface. It was already studied by C.Segre in [16] who also knew the following result.

(IX.2.2) Proposition: The surface F is a smooth ruled cubic surface, i.e. it is isomorphic to the rational ruled surface $\Sigma_1 = \mathbf{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1))$ and the embedding is given by the full linear system $|C_0 + 2f|$ where C_0 is the unique section of Σ_1 with $C_0^2 = -1$.

Moreover, the divisor class of C is $2C_0 + 3f$.

Proof: We first want to prove that F has degree 3. Let $P_1, P_2, P_3 \in C$ be different points with

$$(1) \qquad \sum_{i=1}^{3} P_i \simeq -P_0 .$$

Let E be the plane spanned by P_1, P_2 and P_3. Then for each point $z \in C$ there is a unique hyperplane H through E which contains z. Because of (1) it follows that

$$H.C = P_1 + P_2 + P_3 + z + \kappa(z).$$

Now choose planes E' and E'' as above such that the degree 3 divisors which are cut out by E, E' and E'' on C span the linear system $|4\mathcal{O} - P_0|$. Let H, H' and H'' be the pencils of hyperplanes through E, E' and E'' respectively. These pencils can be parametrized by

$$C/\kappa = \mathbb{P}_1 .$$

Then for each $\lambda \in \mathbb{P}_1$ the hyperplanes $H(\lambda)$, $H'(\lambda)$ and $H''(\lambda)$ intersect in a line which is a ruling of F. Hence F can be constructed by a Steiner construction, i.e.

$$F = \bigcup_{\lambda \in \mathbb{P}_1} (H(\lambda) \cap H'(\lambda) \cap H''(\lambda))$$

and from this description it follows immediately (cf. [7, p.530 ff.]) that

$$\deg F = 3.$$

Now we have to show that F is isomorphic to Σ_1. Clearly F is a rational ruled surface and we have a map

$$\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-n)) \to F$$

for some integer $n \geq 0$. This map is bijective, since no two secants of C intersect outside C. Let $C_0 \subseteq \Sigma_n$ be a section with $C_0^2 = -n$. This section is determined uniquely unless $n = 0$. Let H be the hyperplane section. Then

$$H \sim C_0 + \lambda f \ .$$

Since

$$3 = \deg F = H^2 = -n + 2\lambda$$

it follows that n is odd. Moreover, since H is base point free it follows that $\lambda \geq n$ and hence $n \leq 3$. It remains to exclude the case $n = 3$. To do this note that

$$H.C_0 = (C_0 + \lambda f).C_0 = -n + \lambda \ .$$

If $n = 3$ then $\lambda = 3$ and hence

$$H.C_0 = 0$$

which is impossible. It follows that $n = 1$ and hence

$$H \sim C_0 + 2f.$$

The surface F spans $\mathbb{P}_4$ and since

$$\dim | C_0 + 2f | = 4$$

it follows that the map from Σ_1 to F is given by the full linear system $|C_0 + 2f|$ and this is well known to define an embedding.

It remains to determine the divisor class of C. Since C intersects each fibre in 2 points, we have

$$C \sim 2C_0 + \mu f.$$

But then it follows from the fact that

$$5 = \deg C = C.H = 2 + \mu$$

that

$$C \sim 2C_0 + 3f$$

which concludes our proof.

(IX.2.3) <u>Remarks</u>: (i) Note that one can construct the cubic ruled surface F also by taking the projection of the Veronese surface $V \subseteq \mathbb{P}_5$ from a point on V itself.

(ii) The linear system $|C_0 + f|$ has projective dimension

$$\dim |C_0 + f| = 2.$$

Since

$$H.(C_0 + f) = (C_0 + 2f).(C_0 + f) = 2$$

it follows that each divisor $\tilde{C} \in |C_0 + f|$ is mapped to a degree 2 curve on F. If $\tilde{C}$ is a section then this curve will be a conic section, otherwise it will be the union of two lines. In any case, every element $\tilde{C} \in |C_0 + f|$ determines a plane $\mathbb{P}_2$ and the union of F and $\mathbb{P}_2$ is the complete intersection of two quadrics in $\mathbb{P}_4$ (see [2, p.58]), i.e.

$$F \cup \mathbb{P}_2 = Q_0 \cap Q_1.$$

In this sense we are dealing with a special case of the situation treated in section 1 of this chapter.

(iii) Note that by our construction of F each fibre f cuts out a divisor of class $\mathcal{O} + P_0$ on C, i.e. we write

$$C.f \sim \mathcal{O} + P_0 .$$

Since

$$H.C \sim 5\mathcal{O}$$

one can conclude that

$$C_0.C \sim 3\mathcal{O} - 2P_0 .$$

This implies that

$$C.(C_0 + f) \sim 4\mathcal{O} - P_0$$

and restriction to C defines an isomorphism

$$\Gamma(\mathcal{O}_F(C_0 + f)) \cong \Gamma(\mathcal{O}_C(4\mathfrak{o} - P_0)).$$

(iv) Note that the unique section $C_0 \subseteq \Sigma_1$ with $C_0^2 = -1$ is mapped to a line L in $\mathbb{P}_4$ which intersects C in the point $3\mathfrak{o} - 2P_0$. Clearly L depends on the choice of P_0. If one projects from a point L the cubic surface F is mapped to a quadric cone in $\mathbb{P}_3$. Hence L is the singular line of a rank 3 quadric through C. Indeed, by varying P_0 one gets every vertex of a rank 3 quadric through C in this way. It is interesting to note that L is one of the 25 skew lines $L_{k\ell}$ which are part of the <u>configuration</u> which we studied in chapter II if and only if P_0 is one of the 5-torsion points of C (see also C.Segre's paper [16]). In fact if $P_0 = \mathfrak{o}$ this follows immediately from the fact that L meets C in $\mathfrak{o}$ and that ι leaves L pointwise fixed. A similar argument applies also to the other 5-torsion points.

(IX.2.4) <u>Lemma</u>: $N_{C/F} = \mathcal{O}_C(9\mathfrak{o} - P_0)$.

<u>Proof</u>: By what we have said in (iii) it follows that

$$\begin{aligned} N_{C/F} &= \mathcal{O}_C(C) \\ &= \mathcal{O}_C(2C_0 + 2f) \\ &= \mathcal{O}_C(9\mathfrak{o} - P_0) \,. \end{aligned}$$

(IX.2.5) Now let $P \notin \mathrm{Sec}\, C$ be a centre of projection. Then we have the following maps between normal bundles

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & \mathcal{O}_C(-1) \arrow[d] \\
0 \arrow[r] & N_{C/F}(-2) \arrow[r] \arrow[dr, "\psi"] & N_C(-2) \arrow[d] \\
 & & N_P(-2) \arrow[d] \\
 & & 0
\end{tikzcd}$$

The arguments of section 1 also show that $\psi \not\equiv 0$. Hence its image defines a subbundle

$$\tilde{M} \subseteq N_P(-2)$$

of degree ≥ -2.

The geometric situation is the following. The point P lies on a unique plane $\mathbb{P}_2$ spanned by some conic $\tilde{C} \in |C_0 + f|$. Projection from P maps F onto a cubic surface $F_P \subseteq \mathbb{P}_3$ which has a double line, namely the image of the conic $\tilde{C}$. Since

$$C.\tilde{C} = 3$$

the curve C_P meets this line in 3 points. Hence the subbundle of $N_P(-2)$ which is defined by the surface F_P has degree ≥ -2. Note that the surface F (resp. F_P) represents the subbundle

$$M := \ker(N_P^*(2) \to \tilde{M}^*)$$

precisely in the sense of section 1 of this chapter. Moreover

$$\deg M = \deg \tilde{M} \geq -2 .$$

(IX.2.6) Since we are really interested in subbundles of $N_P(-2)$ of degree 0 we shall have to choose special centres of projection. Therefore, let $\tilde{C} \in |C_0 + f|$ be a smooth conic section on F which intersects C in three different points P_1, P_2 and P_3. Let P be the point of intersection of two of the tangents of $\tilde{C}$ at the points P_i, say P_1 and P_2. First note that $P \notin \mathrm{Sec}\, C$, since the only double points with respect to π_P lie on $\tilde{C}$ (cf. the discussion in (IX.1.2)). On the other hand the map

$$\psi : N_{C/F}(-2) \to N_P(-2)$$

has zeroes at P_1 and P_2. Hence the resulting subbundle

$$\tilde{M} \subseteq N_P(-2)$$

has degree ≥ 0. Since there are no subbundles of degree greater than 0 it follows that

$$\deg \widetilde{M} = 0 \ .$$

We can summarize this as follows:

(IX.2.7) <u>Proposition</u>: <u>The smooth elliptic quintic</u> C_P <u>lies on the ruled cubic surface</u> F_P <u>and meets the double line of</u> F_P <u>in three points, two of which are pinch points of</u> F_P. <u>The subbundle of</u> N_P <u>defined by</u> F_P <u>has degree</u> 0.

(IX.2.8) <u>Remarks</u>: (i) We want to point out that this is a special situation and that not every degree 0 subbundle of every elliptic quintic C_P can be represented in this way. This follows from an easy dimension count. In the above construction one can vary the point P_0 and for fixed P_0 there exists a 2-dimensional family of conics $\widetilde{C} \in |C_0 + f|$. This gives only 3 parameters altogether.

(ii) The last point we want to make is that this "variety of special projection centres" is not contained in one of the hypersurfaces Y_M , but intersects them all. The reason is that the line bundle $\widetilde{M}$ is given by

$$\begin{aligned} \widetilde{M} &= \mathcal{O}_C(2P_1 + 2P_2 + P_3 - 5\mathcal{O}) \\ &= \mathcal{O}_C(P_1 + P_2 - \mathcal{O} - P_0) . \end{aligned}$$

So even for fixed P_0 we can always arrange $P_1 + P_2$ so that $\widetilde{M}$ is any given line bundle of degree 0. Note that $\widetilde{M} = \mathcal{O}_C$ if the conic $\widetilde{C}$ degenerates into two lines C_0 and f in which case we can project from a general point of the secant f which intersects C in

$$P_1 + P_2 \sim P + \mathcal{O}.$$

Then F is mapped to a smooth quadric surface containing the nodal curve C_P.

Klaus HULEK
Mathematisches Institut
Bismarckstrasse 1 1/2
D-8520 ERLANGEN
R.F.A.

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RÉSUMÉ

Le sujet de cette monographie est la géométrie projective des courbes elliptiques. Les principaux aspects développés sont les suivants:

1. En utilisant les fonctions théta spéciales, les symétries des courbes elliptiques normales et l'action du groupe de Heisenberg peuvent être explicitées.

2. Le fibré de Horrocks-Mumford est relié aux courbes elliptiques de plusieurs façons ; en particulier, il peut être reconstitué à partir du "rouleau" tangent des surfaces quintiques normales.

3. La géométrie du fibré de Horrocks-Mumford est aussi intimement liée au fibré normal des courbes elliptiques gauches de degré 5.

Le but principal de ce texte est d'étudier les relations entre ces différents aspects.