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THE INTEGRAL GEOMETRY OF LINE COMPLEXES AND A THEOREM OF GELFAND-GRAEV

BY

Victor GUILLEMIN

1. Introduction

Let $P = \mathbb{CP}^3$ be the complex three-dimensional projective space and let $G = \mathbb{CG}(2, 4)$ be the Grassmannian of complex two-dimensional subspaces of $\mathbb{C}^4$. To each point $p \in G$ corresponds a complex line l_p in P . Given a smooth function, f , on P we will show in § 2 how to define properly the line integral,

$$(1.1) \quad \int_{l_p} f(\lambda) d\lambda d\bar{\lambda} = \hat{f}(p).$$

A complex hypersurface, S , in G is called *admissible* if there exists no smooth function, f , which is not identically zero but for which the line integrals, (1.1) are zero for all $p \in S$. In other words if S is admissible, then, in principle, f can be determined by its integrals over the lines, l_p , $p \in S$. In the 60's GELFAND and GRAEV settled the problem of characterizing which subvarieties, S , of G have this property. We will describe their result (and, in fact, sketch a rough proof of it) in § 3. At first glance their result is rather puzzling : Admissibility turns out *not* to be a generic property of varieties. In fact very few S 's possess this property.

The purpose of this paper is to describe how this result can be used as the rationale for a method of constructing multi-branched analytic solutions of the wave equation on compactified Minkowski space with prescribed singularities. We will describe this method in § 4 and illustrate it with

examples in §§ 5-6. Finally in § 7 we will describe an analogue of the Gelfand-Graev theorem for compactified Minkowski space.

2. The Gelfand line transform

Let $f : f(z, \bar{z})$ be a smooth function on $\mathbf{C}^2 - 0$ which is bihomogeneous of bidegree $(-2, -2)$; i.e.

$$(2.1) \quad f(\lambda z, \bar{\lambda} \bar{z}) = |\lambda|^{-4} f(z, \bar{z})$$

for all $\lambda \in \mathbf{C}^*$. Let $dz = dz_1 \wedge dz_2$. Since f is not in $\mathcal{L}^1$ the integral

$$(2.2) \quad \int f(z, \bar{z}) dz d\bar{z}$$

diverges; however, one can still make sense of (2.2) as follows. Let

$$\Xi = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

and let ω be the form of type 1-1:

$$(2.3) \quad \omega = \iota(\Xi)\iota(\bar{\Xi})f dz \wedge d\bar{z}.$$

This form has nice properties with respect to the principle fibration : $\mathbf{C}^2 - 0 \xrightarrow{\pi} \mathbf{CP}^1$. Namely it vanishes when restricted to fibers; and, by (2.1), it is invariant under the action of the structure group, $\mathbf{C}^*$. Thus there exists a form, μ , of type 1-1 on $\mathbf{CP}_1$ such that

$$\omega = \pi^* \mu.$$

We define (2.2) to be the integral

$$(2.4) \quad \int_{\mathbf{CP}^1} \mu.$$

It is clear that we can formulate the definition, (2.4), in a coordinate-free way. If V is a complex vector space of dimension 2, f a smooth bihomogeneous function on $V - 0$ of bidegree $(-2, -2)$ and Ω an element of $\wedge^{2,2}(V^*)$ then the integral

$$(2.5) \quad \int_V f \Omega$$

is well-defined (independent of coordinates).

Consider now a bihomogeneous function, f , on $\mathbf{C}^4 - 0$ of bidegree $(-2, -2)$. Given a point $p \in G$, let V be the complex 2-dimensional subspaces of $\mathbf{C}^4$ represented by p . We will define the *line transform*, $\hat{f}$, of f at p as follows. By definition it will be an element of the space

$$(2.6) \quad \Lambda^{2,2}(V^*)^*.$$

Notice that an element of $(2,6)$ is defined by describing how it pairs with an element, Ω , of $\Lambda^{2,2}(V^*)$. For $\hat{f}(p)$ the answer is given by the integral (2,5); i.e. by definition :

$$(2.7) \quad \langle \hat{f}(p), \Omega \rangle = \int_V f \Omega.$$

Functions on $\mathbf{C}^4 - 0$ which are bihomogeneous of bidegree $(-2, -2)$ can be regarded as sections of a line bundle, $\mathcal{L} \rightarrow P$. We will denote by $\mathcal{M}$ the line bundle on G whose fiber at p is $(2,6)$. With this notation we can regard the line transform described above as an integral operator

$$(2.8) \quad R : \Gamma(\mathcal{L}) \rightarrow \Gamma(\mathcal{M}), \quad Rf = \hat{f}.$$

It is not hard to show that R is injective and to describe its range using the representation theory of $SL(4, \mathbf{C})$. We prefer here to give a more elementary description of its range. Let U_0 be the open subset of G consisting of all points $p \in G$ for which the restriction of $dz_1 \wedge dz_2$ to V is non-zero. (As above V is the 2-dimensional subspace of $\mathbf{C}^4$ represented by p). Then, V can be described by linear equations of the form

$$\begin{aligned} z_3 &= az_1 + bz_2, \\ z_4 &= cz_1 + dz_2, \end{aligned}$$

where a, b, c and d depend on V . In fact a, b, c and d are coordinate functions on U_0 , and $dz_1 \wedge dz_2$ provides one with a trivialization of $\mathcal{M}$ over U_0 ; so for $p \in U_0$

$$(2.9) \quad \begin{aligned} \hat{f}(p) &= \hat{f}(a, b, c, d) \\ &= \int f(z_1, z_2, az_1 + bz_2, cz_1 + dz_2) dz_1 dz_2 d\bar{z}_1 d\bar{z}_2. \end{aligned}$$

Differentiating under the integral sign one obtains

$$(2.10)_0 \quad \Delta_0 f = \left(\frac{\partial}{\partial a} \frac{\partial}{\partial d} - \frac{\partial}{\partial b} \frac{\partial}{\partial c} \right) f = 0.$$

Similarly

$$(2.11)_0 \quad \overline{\Delta}_0 f = \left(\frac{\partial}{\partial \bar{a}} \frac{\partial}{\partial \bar{d}} - \frac{\partial}{\partial \bar{b}} \frac{\partial}{\partial \bar{c}} \right) f = 0.$$

More generally given a decomposable element ν , of $\Lambda^2(\mathbf{C}^4)^*$ let U_ν be the open subset of G consisting of all points, p , for which the restriction of ν to V is non-zero. Then ν defines a trivialization of $\mathcal{M}$ over U_ν ; and, with respect to this trivialization, there exists second order differential operators, Δ_ν and $\overline{\Delta}_\nu$, analogous to (2.10)₀ and (2.11)₀, such that

$$(2.10)_\nu \quad \Delta_\nu f = 0$$

and

$$(2.11)_\nu \quad \overline{\Delta}_\nu f = 0$$

on U_ν . Let $\mathcal{M}_1$ be the line bundle over G whose fiber at p is $\Lambda^{2,2}(V^*) \otimes \Lambda^2(V^*)^* \otimes \Lambda^2(C^4/V)^*$, and let $\overline{\mathcal{M}}_1$ be its complex conjugate. Patching together the Δ_ν 's and $\overline{\Delta}_\nu$'s one gets intrinsically defined second order differential operators

$$(2.12) \quad \Delta : \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M}_1)$$

and

$$(2.13) \quad \overline{\Delta} : \Gamma(\mathcal{M}) \rightarrow \Gamma(\overline{\mathcal{M}}_1)$$

such that $\Delta \hat{f} = \overline{\Delta} \hat{f} = 0$. This proves one half of the following proposition.

PROPOSITION. — *A section $g \in \Gamma(\mathcal{M})$ satisfies the equations*

$$(2.14) \quad \Delta g = \overline{\Delta} g = 0$$

if and only if $g = \hat{f}$ for some section, f , of $\mathcal{L}$.

We recall next that if $p \in G$ the cotangent space to G at p can be identified with

$$(2.15) \quad \text{Hom}(\mathbf{C}^4/V, V).$$

Let Σ_p be the set of rank one elements in this space. Since $\mathbf{C}^4/V$ and V are two-dimensional the set, Σ_p , is a quadratic cone inside T_p^* . This shows that G is equipped with an intrinsic (complex) conformal structure such that Σ_p

is the cone of “light-like” rays at p . We will say more about this conformal structure in § 4.

Let Σ be the fiber bundle over G whose fiber at p is Σ_p . We claim that Σ is the characteristic variety of the system of partial differential equations (2.14). In fact let a, b, c and d be the coordinate functions on U_0 described above and let α, β, γ and δ be the dual cotangent coordinates. Then for $p \in U_0$

$$\Sigma_p = \{(\alpha, \beta, \gamma, \delta), \alpha\delta - \beta\gamma = 0\},$$

whereas

$$\sigma(\Delta_0)(\alpha, \beta, \gamma, \delta) = \alpha\delta - \beta\gamma$$

by (2.10).

3. Admissibility

Let S be a complex hypersurface in G . One calls S admissible if the integral transform

$$\Gamma(\mathcal{L}) \rightarrow \Gamma(M1S), \quad f \rightarrow \hat{f}1S$$

is injective. In [2], Gelfand et al. show that the following S 's are admissible :

Example 1. — Let W be a non-singular curve in P and let S be the set of all points $p \in G$ such that W and l_p intersect.

Example 2. — Let W be a non-singular surface in P and let S be the set of all points $p \in G$ such that l_p has at least one point of tangency with W .

Their main result is the following converse statement :

THEOREM. — *If S is admissible then near a generic point S is locally as in example one or as in example two.*

We will sketch a proof of this below. We first claim :

LEMMA. — *For S to be admissible it has to be characteristic with respect to the differential operator, Δ .*

“Proof”. — If S were non-characteristic then the Cauchy problem

$$(2.1) \quad \Delta g = \overline{\Delta}g = 0, \quad g = 0 \text{ on } S$$

would be well-posed. But if g is a non-trivial solution of (2.1) then, by the proposition in § 2, $g = \hat{f}$ and $\hat{f}1S = 0$. Contradiction.

Unfortunately, if S is non-characteristic at a point, p , the Cauchy problem (3.1) is well-posed only in a small neighborhood of p ; whereas, to get a

contradiction, we need to find a non-trivial *global* solution of (3.1). Therefore, this “proof” is not completely convincing. There is a convincing proof involving (3.1); but we won’t attempt to describe it here.

We next require some facts about the characteristic variety, Σ , of the differential operator, Δ . Since Σ is a co-isotropic subvariety of T^*G , it is equipped with a canonical null-foliation. We will show that this null-foliation is *fibrating* with $\mathbb{CP}^1$ ’s as fibers and $T^*P - 0$ as base.

Proof. — A typical element of $T^*P - 0$ is of the form, $\xi \otimes x$, with $x \in \mathbb{C}^4 - 0$, $\xi \in (\mathbb{C}^4)^* - 0$, and $\langle x, \xi \rangle = 0$. Given a point, p , in G let V be the two-dimensional subspace of $\mathbb{C}^4$ represented by p . We will say that p belongs to $\gamma_{x,\xi}$ if

$$(3.2) \quad x \in V \text{ and } \xi \in V^0.$$

The set, $\gamma_{x,\xi}$, defined by (3.2) is a complex line in G ; and it is easy to see that the $\gamma_{x,\xi}$ ’s are exactly the light rays on G associated with the canonical conformal structure. Q.E.D.

We will denote by

$$(3.3) \quad \pi : \Sigma \rightarrow T^*P - 0$$

the null-fibration. Now let S be a hypersurface in G which is characteristic with respect to Δ . Then its conormal vector at each point is “light-like”; so the conormal bundle

$$\Lambda = N^*S - 0$$

is contained in Σ . Since Λ is Lagrangian this implies that *for every point in Λ the leaf of the null-foliation passing through this point is also in Λ* . Therefore Λ has to be of the form $\pi^{-1}(\Lambda_1)$ where Λ_1 is a Lagrangian submanifold of $T^*P - 0$. At “generic” points Λ_1 is locally of the form

$$\Lambda_1 = N^*W - 0,$$

where W , the projection of Λ_1 into P , is a submanifold of P . Therefore we have proved that

$$(3.4) \quad N^*S = \pi^{-1}(N^*W)$$

at “generic” points of N^*S . From (3.4) it is easy to deduce Gelfand’s theorem in the following form

THEOREM. — *The hypersurface, S , consists of all points, $p \in G$, such that l_p intersects W non-transversally.*

4. The Penrose transform

The Penrose transform is the holomorphic analogue of line transform described in § 2. It was used by PENROSE and his collaborators to construct solutions of the wave equation on compactified Minkowski space. (See [3] and [4].) Before describing it we will review some facts about the geometry of compactified Minkowski space. A good reference for the material below is the survey article of WELLS, [5].

We have already observed that G is equipped with a canonical (complex) conformal structure. It has three real forms, on which the induced (real) conformal structures are of type $(++++)$, $(++--)$ and $(+++-)$ respectively, and these are S^4 , $\mathbf{RG}(2, 4)$ and compactified Minkowski space, which we will denote by M . A good way to view M as a submanifold of G is as follows. Consider on $\mathbf{C}^4$ the Hermitian form

$$(4.1) \quad H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2.$$

For each $p \in G$ let V_p be the two-dimensional subspace of $\mathbf{C}^4$ represented by p . Then

$$(4.2) \quad M = \{p \in G, H = 0 \text{ on } V_p\}.$$

From this description of M one sees easily that the group $SU(2, 2)$ acts as conformality transformations on M .

We will now show how one can take a holomorphic function defined on an appropriate open subset of P and convert it via the *Penrose transform* into a solution of the conformal wave equation on M . Incidentally the version of the Penrose transform which we will describe below is very close to the version which one finds in PENROSE's earlier papers. (See [4].) Later EASTWOOD, PENROSE and WELLS found a more elegant and general definition, involving sheaf cohomology, which we won't attempt to describe here. (See [1].)

To start with, let f be a meromorphic function on $\mathbf{C}^2 - 0$ which is homogeneous of degree -2 , i.e. satisfies

$$f(\lambda z) = \lambda^{-2} f(z)$$

for all $\lambda \in \mathbf{C}^*$. Let Ξ be the vector field, $z_1 \partial/(\partial z_1) + z_2 \partial/(\partial z_2)$, and let ω be the one form, $\iota(\Xi) f dz_1 \wedge dz_2$. As in § 2 ω is of the form $\omega = \pi^* \mu$, where μ is a meromorphic one-form on $\mathbf{CP}^1$ and $\pi : \mathbf{C}^2 - 0 \rightarrow \mathbf{CP}^1$ is the canonical projection. Given a contour, γ , on $\mathbf{CP}^1$ not intersecting the poles of μ , we will denote by $\text{Res}_\gamma f dz_1 dz_2$ the integral

$$(4.3) \quad \text{Res}_\gamma f dz_1 dz_2 = \int_\gamma \mu.$$

It is clear that this definition is independent of the choice of coordinates, i.e. if V is a two-dimensional complex vector space, f a homogeneous meromorphic function on $V - 0$ of degree -2 and Ω an element of $\Lambda^{2,0}(V^*)$, then for an appropriate contour, γ , on the projective space $\mathbf{P}V$, the residue

$$(4.4) \quad \text{Res}_\gamma f \Omega$$

is well-defined.

Now let f be a meromorphic function on $\mathbf{C}^4 - 0$ which is homogeneous of degree -2 and let W be the set of rays (in P) on which f is singular. W is an algebraic subvariety of P , but it need not be non-singular; so we will denote by W_0 the non-singular points of W and by W_1 the curve of singular points. Let S be the set of all points, $P \in G$, such that the line, l_p , either intersects W_1 or has a common point of tangency with W_0 . Let p be a point not on S and let $V = V_p$ the subspace of $\mathbf{C}^4$ represented by p . We will define $\hat{f}(p) \in \Lambda^{2,0}(V^*)^*$ by the formula

$$(4.5) \quad \langle \hat{f}(p), \Omega \rangle = \text{Res}_\gamma f \Omega$$

for $\Omega \in \Lambda^{2,0}(v^*)$, γ being a contour on the line $l_p = \mathbf{P}V$ avoiding points of $W \cap l_p$. Let $\mathcal{M}$ be the line bundle on G with fiber

$$\Lambda^{2,0}(V^*)^* = \Lambda^{2,0}(V)$$

at p . If one varies the contour, γ , continuously with respect to p , one gets from (4.5) a multi-branched holomorphic section of $\mathcal{M}$ over $G - S$ which satisfies the holomorphic analogue of the wave equation discussed in § 2. By the theorem of Gelfand discussed in § 3, S is *characteristic* with respect to the wave equation; so the Penrose transform can be regarded as a tool for constructing multi-branched holomorphic solutions of the wave equation on G with singularities along a prescribed characteristic hypersurface. Restricted to M these solutions often become single-valued with singularities along a prescribed *real* characteristic hypersurface. (See § 7). We won't attempt here to give a systematic description of these solutions; but, in the next couple of sections, we will illustrate this method by means of examples.

5. Characteristic hypersurfaces of the first kind

We saw in § 3 that there are two kinds of characteristic hypersurfaces in G . The first kind consists of all lines which pass through a fixed curve, and the second kind consists of all lines which have a common point of tangency with a fixed surface. In this section we will describe how to construct single-valued holomorphic solutions of the wave equation with singularities along characteristic hypersurfaces of the first kind.

Let W be the algebraic curve in $\mathbf{CP}^3$ defined by the equations

$$Q_1(z) = Q_2(z) = 0,$$

where Q_1 and Q_2 are homogeneous polynomials in (z_1, z_2, z_3, z_4) with no common factor. Let the function, f , in (4.5) be of the form

$$(5.1) \quad f = Q_3 / Q_1^{m_1} Q_2^{m_2},$$

where $\deg Q_3 = m_1 \deg Q_1 + m_2 \deg Q_2 - 2$, and choose the contour, γ , in (4.5) so that it surrounds all the zeroes of Q_1 on the projective line, $l = \mathbf{PV}$, but none of the zeroes of Q_2 . Then the expression (4.5) is well-defined providing no point on l is simultaneously a zero of Q_1 and Q_2 ; i.e. providing the line, l , doesn't intersect the curve, W . In other words, let S be the characteristic hypersurface of the first kind consisting of all points, $p \in G$, for which the line, l_p , intersects W . Then, to each function of the form (5.1), there corresponds a holomorphic solution of the wave equation with singularities on S . Notice that this correspondence is *not* injective. If either m_1 or m_2 were equal to zero in (5.1), then the contour, γ , would surround *all* zeroes of $Q_1^{m_1} Q_2^{m_2}$; so the expression (4.5) would be identically zero. The most satisfactory way to describe this correspondence is in sheaf-theoretic terms: Let $\mathcal{L}_{\text{can}}$ be the canonical line bundle of the projective space P and let $\mathcal{L} = \mathcal{L}_{\text{can}}^2$. Let U_1 and U_2 be the subsets of P on which Q_1 and Q_2 are non-zero. Functions of the form (5.1) are identical with sections of $\mathcal{L}$ over $U_1 \cap U_2$ and functions of the form (5.1) with $m_2 = 0$ (respectively, $m_1 = 0$) are just sections of $\mathcal{L}$ over U_1 (respectively U_2). By MAYER-VICTORIS:

$$(5.2) \quad \Gamma(U_1, \mathcal{L}) \oplus \Gamma(U_2, \mathcal{L}) \xrightarrow{\rho} \Gamma(U_1 \cap U_2, \mathcal{L}) \rightarrow H^1(U_1 \cup U_2, \mathcal{L}) \rightarrow 0,$$

and the image of ρ is contained in the kernel of the Penrose transform; so the Penrose transform is actually a map of $H^1(P - W, \mathcal{L})$ into the space of holomorphic solutions of the wave equation with singularities on S . This is the way the Penrose transform is described in [1] (where it is shown, in addition, that it is bijective).

Example. — Let $Q_1 = z_1, Q_2 = z_2$ and $f = (z_1 z_2)^{-1}$.

In this example S is the characteristic cone consisting of all points, $p \in G$, for which the line l_p intersects the fixed line, l , defined by the equations $z_1 = z_2 = 0$. The apex of this cone is the point, p_0 , represented by the line, l , itself.

Let $U = G - S$. Notice that U consists of all points $p \in G$ with the property that the two-form $dz_1 \wedge dz_2$ doesn't vanish on the space $V = V_p$. Hence there is a natural trivialization of the line bundle, $\mathcal{M}$, over U ; and, with respect to this trivialization, the solution of the wave equation associated with f takes at p the value

$$(5.3) \quad \int_{\gamma} \mu,$$

μ being the one form

$$\mu = \iota \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) \frac{dz_1 dz_2}{z_1 z_2} = \frac{dz}{z},$$

where $z = z_2/z_1$ and γ is a contour on the line, l_p , surrounding the point $z = 0$. However, it is clear that this integral is $2\pi i$ for all p ; i.e. the Penrose transform, $\hat{f}$, of f is the constant function $\hat{f} = 2\pi i$.

Next let U' be the set of points $p \in G$ for which the two-form $dz_3 \wedge dz_4$, restricted to V_p , doesn't vanish. If $p \in U \cap U'$ the subspace, V_p , of $\mathbf{C}^4$ can be described by a pair of equations of the form

$$\begin{aligned} z_1 &= az_3 + bz_4, \\ z_2 &= cz_3 + dz_4, \end{aligned}$$

and, restricted to V_p ,

$$(5.4) \quad dz_1 \wedge dz_2 = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} dz_3 \wedge dz_4.$$

The fact that neither of these restrictions is zero says that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

As in § 2, a, b, c and d can be employed as coordinate functions on $U \cap U'$, and with respect to these coordinates, the transition function relating the trivializations of $\mathcal{M}$ given by $dz_3 \wedge dz_4$ and by $dz_1 \wedge dz_2$ is just the function

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by (5.4). Therefore, in terms of the trivialization given by $dz_3 \wedge dz_4$, $\widehat{f}$ is equal to

$$\widehat{f}(a, b, c, d) = 2\pi i / (ad - bc)$$

on U' ; i.e. $\widehat{f}$ is the so-called *elementary solution* of the wave equation :

$$\frac{\partial}{\partial a} \frac{\partial}{\partial d} - \frac{\partial}{\partial b} \frac{\partial}{\partial c}.$$

6. Characteristic hypersurfaces of the second kind : an example

Let Q be an arbitrary non-degenerate quadratic form on $\mathbf{C}^4$. After making an appropriate change of coordinates we can assume that

$$(6.1) \quad Q(z) = z_1 z_2 + z_3 z_4.$$

Let W be the quadratic surface in P defined by $Q = 0$, and let S be the characteristic hypersurface in G associated with W . In other words $p \in S$ if and only if l_p is tangent to W . Notice that for $p \in S$ either l_p intersects W in a single point or l_p is entirely contained in W . Let S_1 be the set of points for which the second alternative holds. It is easy to see that S_1 is the singular locus of S and is the disjoint union of two $\mathbf{CP}^1$'s (corresponding to the two rulings of W).

In this section we will compute the Penrose transform of the function

$$(6.2) \quad f = 1/Q.$$

Before we do so, however, let's consider a somewhat simpler problem. Let $q = q(z_1, z_2)$ be a non-degenerate quadratic form on $\mathbf{C}^2 - 0$, and let's compute the residue

$$\text{Res}_\gamma (dz_1 dz_2 / q),$$

where γ is a contour on $\mathbf{CP}^1$ surrounding one of the zeroes of q . We can make a linear change of coordinates

$$(6.3) \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

so that $q(z_1, z_2) = w_1 w_2$. Moreover, if

$$q(z) = k_{11} z_1^2 + k_{12} z_1 z_2 + k_{21} z_2 z_1 + k_{22} z_2^2,$$

with $k_{12} = k_{21}$, and J and K are the matrices

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

then

$$(6.4) \quad BJB^t = K.$$

With this change of coordinates we get

$$\text{Res}_\gamma (dz_1 dz_2/q) = (\det B)^{-1} \text{Res}_\gamma (dw_1 dw_2/w_1 w_2).$$

In § 5 we showed that the residue on the right was just $2\pi\sqrt{-1}$; so we get the formula

$$(6.5) \quad \text{Res}_\gamma (dz_1 dz_2/q) = 2\pi(\det K)^{-1/2}.$$

since $-(\det B)^2 = \det K$ by (6.4).

Let's come back now to the problem of computing the Penrose transform of (6.2). Let U be the subset of G consisting of all points, p , for which the restriction of $dz_1 \wedge dz_2$ to V_p is non-zero. If $p \in U$ the equations of V_p are

$$\begin{aligned} z_3 &= az_1 + bz_2, \\ z_4 &= cz_1 + dz_2, \end{aligned}$$

and a, b, c and d can be employed as coordinate functions on U . Let A be the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the quadratic form, $z_1 z_2 + z_3 z_4$, restricted to V_p , is of the form

$$(z_1, z_2)(J + AJA^t)(z_1, z_2)^t,$$

and, therefore, by (6.5), the Penrose transform of (6.2) is the function

$$(6.6) \quad \hat{f} = \hat{f}(a, b, c, d) = 2\pi \det(J + AJA^t)^{1/2}.$$

Affine Minkowski space sits inside of U as the set of matrices

$$A = \begin{pmatrix} u & w \\ \bar{u} & v \end{pmatrix}$$

with w complex and u and v real; so the restriction of (6.6) to affine Minkowski space is

$$(6.7) \quad 2\pi[(\det A + 1)^2 + 4|w|^2]^{-1/2},$$

or, in terms of the more familiar space-time coordinates,

$$(6.8) \quad \begin{aligned} u &= (1/\sqrt{2})(x_0 + x_1), & v &= (1/\sqrt{2})(x_0 - x_1), & w &= (1/\sqrt{2})(x_2 + ix_3), \\ \hat{f} &= 2\pi h(x)^{-1/2}, \end{aligned}$$

where

$$(6.9) \quad h(x) = \left[\frac{1}{2}(x_0^2 - x_1^2 - x_2^2 - x_3^2) + 1 \right]^2 + 4(x_2^2 + x_3^2).$$

Notice that (6.9) is non-negative, so $\hat{f}$ is *single-valued* on Minkowski space with singularities on the hyperbola

$$x_0^2 - x_1^2 = 2, \quad x_2 = x_3 = 0.$$

Remark. — One gets a very similar formula for the Penrose transformation of the function

$$P/Q^k,$$

P being a homogeneous polynomial in z of degree $2k - 2$.

7. Characteristic hypersurfaces in compact Minkowski space

There is an interesting analogue of the theorem proved in § 3 for compact Minkowski space. For $p \in M$ let Σ_p be the light cone in $T_p^* - 0$ and let Σ be the fiber bundle over M whose fiber at p is Σ_p . Since Σ is of codimension 1 as a submanifold of $T^*M - 0$, it is co-isotropic and hence is equipped with a null-foliation. We will show that this null-foliation is fibrating with $\mathbf{RP}^1$'s as fibers and an extremely interesting symplectic manifold as base. Recall that in § 4 we introduced the Hermitian form

$$(7.1) \quad H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2$$

on $\mathbf{C}^4$. Let P^+ (respectively N) be the set of points in P where H is positive (respectively zero). It is not hard to show that P is a non-degenerate complex domain in the sense that at each point, p , on its boundary, N , the Levi form at p is non-degenerate. Moreover, at each point, p , of N there is an “inward

pointing" real covector $\xi \in T_p^*N - 0$ annihilated by the tangential Cauchy-Riemann vectors at p . Let τ be the fiber bundle over N whose fiber at p is the ray

$$(7.2) \quad \tau_p = \{\lambda\xi, \lambda > 0\}.$$

Since the Levi form is non-degenerate τ is a symplectic submanifold of $T^*N - 0$.

THEOREM. — *The null-foliation of Σ is fibrating with $\mathbf{RP}^1$'s as fibers and τ as base.*

Proof. — Given $x \in N$ let γ_x be the set of all points $p \in M$ such that the complex line, l_p , in N , associated with p , contains the point x . It is fairly obvious that γ_x is an $\mathbf{RP}^1$. On the other hand it is not hard to see that γ_x is a light ray with respect to the conformal Lorentzian structure on M and that all light rays are of this form. (See for instance § 3 of [5].) Q.E.D.

Let $\pi : \Sigma \rightarrow \tau$ be the null-fibration. Given a characteristic hypersurface, S , in M let Λ be its conormal bundle. To say that S is characteristic is equivalent to saying that Λ is contained in Σ ; so Λ must necessarily be of the form

$$(7.3) \quad \Lambda = \pi^{-1}(\Lambda_1),$$

where Λ_1 is a conic Lagrangian submanifold of τ . Since the fiber, τ_p , of τ above $p \in N$ consists of the single ray (7.2), Λ_1 is completely determined by its projection, W , on N . Moreover, since N is the projectivization of a conic symplectic manifold it has an intrinsic contact structure, and Λ_1 is Lagrangian in τ if and only if W is Legendrian in N . We leave for the reader to show that (7.3) translates into the following statement :

THEOREM. — *Let S be a characteristic (light-like) hypersurface in M . Then there exists a unique Legendrian submanifold, W , in N such that*

$$(7.4) \quad S = \{p \in M, l_p \text{ intersects } W\}.$$

Conversely if W is a Legendrian submanifold of N the set (7.4) is a characteristic hypersurface in M .

REFERENCES

- [1] EASTWOOD (M.), PENROSE (R.) and WELLS, JR. (R.O.). — Cohomology and massless fields, *Communications in Math. Phys.*, t. **78**, 1981, p. 305–351.
- [2] GELFAND (I.M.), GRAEV (M.I.) and VILENKIN (N.Ya.). — *Generalized Functions*, vol. 5. — New York, Academic Press, 1966.
- [3] HUGHSTON (L.P.) and WARD (R.S.). — *Advances in Twistor Theory*. — London, Pitman, 1979.
- [4] PENROSE (R.) and MACCALLUM (M.A.H.). — Twistor theory : an approach to the quantization of fields and space-time, *Phys. Rep.*, t. **6C**, 1974, p. 241–316.
- [5] WELLS (R.O.). — Complex manifolds and mathematical physics, *Bull. A.M.S.*, t. **1,2**, 1981, p. 296–336.

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