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ISOMETRIC IMMERSIONS OF RIEMANNIAN MANIFOLDS

BY

M. GROMOV

Denote by $\mathcal{G}_r$ the space of C^r -smooth quadratic differential forms g on a smooth manifold V (that are C^r -sections of the symmetric square of the cotangent bundle of V) and let $\mathcal{F}_r^q$ be the space of C^r -maps $f : V \rightarrow \mathbf{R}^q$. Denote by $\mathcal{D} : \mathcal{F}_r^q \rightarrow \mathcal{G}_{r-1}$, for $r \geq 1$, the (first order non-linear differential) operator which assigns to each f the *induced* quadratic form g on V , that is

$$(1) \quad g(\partial, \partial') = \langle D_f \partial, D_f \partial' \rangle,$$

for all pairs of tangent vectors ∂ and ∂' in $T_v(V)$, $v \in V$, where

$$D_f : T(V) \rightarrow T(\mathbf{R}^q)$$

stands for the differential of f and where $\langle \ , \ \rangle$ denotes the Euclidean scalar product in $T_w(\mathbf{R}^q) = \mathbf{R}^q$ for all $w = f(v) \in \mathbf{R}^q$. The relation (1) can be expressed in local coordinates $u_1, \dots, u_n$ on V , for $n = \dim V$, by $p = n(n+1)/2$ equations in the partial derivatives

$$\partial_i f = \frac{\partial}{\partial u_i} f, \quad i = 1, \dots, n,$$

$$(2) \quad g(\partial_i, \partial_j) = \langle \partial_i f, \partial_j f \rangle, \quad 1 \leq i \leq j \leq n.$$

Observe that the induced form g is always positive semi-definite : it is positive definite if and only if f is an *immersion*, (i.e. the differential D_f is injective on $T_v(V)$ for all $v \in V$). Thus $\mathcal{D}$ restricts to an operator from immersions to positive forms, called

$$\mathcal{D}_+ : \text{Im}_r \rightarrow \mathcal{G}_{r-1}^+.$$

The study of $\mathcal{D}_+$ was originally motivated by the *isometric immersion problem* asking for a solution f to the equation $\mathcal{D}_+f = g$ for a given Riemannian metric $g \in \mathcal{G}^+$. This question was raised by SCHLÄFLI in 1873 and one probably believed at that time that the existence of an isometric immersion $f : (V, g) \rightarrow \mathbf{R}^q$ might be helpful in the study of the intrinsic geometry of (V, g) . Although this belief has not materialized so far, the operator $\mathcal{D}_+$ has turned out to be an amusing non-linear specimen worth of a study in its own right. An essential feature of $\mathcal{D}$ is the abundance of *characteristic directions*. In fact, *all directions* are characteristic for $\mathcal{D}$ if $\dim V \geq 2$. Namely, for no hypersurface $V_0 \subset V$ the initial value problem

$$(3) \quad \mathcal{D}f = g, \quad f|_{V_0} = f_0$$

can be solved unless the initial map $f_0 : V_0 \rightarrow \mathbf{R}^q$ satisfies certain differential equations of its own. Indeed, if the map $f : (V, g) \rightarrow \mathbf{R}^q$ is isometric, then f_0 is isometric for the restricted metric $g_0 = g|_{V_0}$ on V_0 . Now if f_0 is isometric, then the system (3) can be locally solved in the real analytic case by applying the Cauchy-Kovalevskia theorem to an auxiliary second order system obtained by differentiating (1) *two* times and then by eliminating the third derivatives with an appropriate anti-symmetrization. To see how it works, we assume, for the sake of simplicity, the manifold V to be the metric product,

$$(V, g) = (V_0 \times \mathbf{R}, g_0 + dt^2)$$

and we write the equations (2) as follows :

$$(4) \quad \begin{aligned} \langle \partial_i f, \partial_j f \rangle &= g_{ij} = g(\partial_i, \partial_j), \quad 1 \leq i, j \leq n-1, \\ \langle \partial_i f, \partial_t f \rangle &= 0, \quad \langle \partial_t f, \partial_t f \rangle = 1, \end{aligned}$$

where ∂_t stands for $\partial/\partial t$, and where the functions g_{ij} on $V = V_0 \times \mathbf{R}$ are constant in t . Hence

$$0 = \partial_t \langle \partial_i f, \partial_j f \rangle = \langle \partial_{it} f, \partial_j f \rangle + \langle \partial_i f, \partial_{jt} f \rangle.$$

Next,

$$0 = \partial_j \langle \partial_i f, \partial_t f \rangle = \langle \partial_{ij} f, \partial_t f \rangle + \langle \partial_i f, \partial_{jt} f \rangle.$$

Now, by alternating i and j we obtain with the above,

$$(5) \quad \langle \partial_{ij} f, \partial_t f \rangle = 0.$$

which implies by differentiating in t ,

$$(5') \quad \langle \partial_{ij} f, \partial_{tt} f \rangle + \langle \partial_t f, \partial_{ijt} f \rangle = 0.$$

On the other hand

$$0 = \partial_{ij} \langle \partial_t f, \partial_t f \rangle = 2(\langle \partial_t f, \partial_{ijt} f \rangle + \langle \partial_{it} f, \partial_{jt} f \rangle).$$

Therefore,

$$(6) \quad \langle \partial_{tt} f, \partial_{ij} f \rangle = \langle \partial_{ij} f, \partial_{jt} f \rangle.$$

Finally, we differentiate in t the last two equations in (4) and obtain

$$(7) \quad \begin{aligned} \langle \partial_{tt} f, \partial_i f \rangle &= -\langle \partial_t f, \partial_{it} f \rangle, \\ \langle \partial_{tt} f, \partial_t f \rangle &= 0. \end{aligned}$$

LEMMA. — *A C^3 -map $f : V_0 \times \mathbf{R} \rightarrow \mathbf{R}^q$ satisfies the equation $\mathcal{D}f = g = g_0 + dt^2$ if and only if it satisfies the system (6) + (7) (which consists of $p = n(n+1)/2$ equations of the second order) as well as the following initial value conditions on $V_0 = V_0 \times 0$:*

$$(8) \quad \begin{aligned} \langle \partial_i f, \partial_j f \rangle &= g_0(\partial_i, \partial_j), & \langle \partial_t f, \partial_t f \rangle &= 1, \\ \langle \partial_t f, \partial_i f \rangle &= 0, & \langle \partial_t f, \partial_{ij} f \rangle &= 0. \end{aligned}$$

Proof. — The “only if” claim (i.e. the implication (2) $\Rightarrow$ (6) + (7) + (8)) has been already established. The “if” part follows by reversing the above computation.

COROLLARY. — *Let $f_0 : (V_0, g_0) \rightarrow \mathbf{R}^q$ be a real analytic isometric immersion whose derivatives $\partial_i f_0, \partial_{ij} f_0, 1 \leq i, j \leq n-1$, are linearly independent at every point $v_0 \in V$. Then, if V_0 is a contractible manifold and if $q \geq p = n(n+1)/2$, there is a neighborhood $U \subset V_0 \times \mathbf{R}$ of $V_0 = V_0 \times 0$ which admits a real analytic isometric immersion $f : (U, g) \rightarrow \mathbf{R}^q$, such that $f|_{V_0} = f_0$.*

Proof. — The immersed manifold $f(V_0) \subset \mathbf{R}^q$ admits, under our assumptions, a real analytic unit vector field $X_0 : V_0 \rightarrow T(\mathbf{R}^q) | V_0$ which is normal to the vectors $\partial_i f_0$ and $\partial_{ij} f_0, 1 \leq i, j \leq n-1$ at every point $v_0 \in V_0$. Then the initial data $f|_{V_0} = f_0$ and $\partial_t f|_{V_0} = X_0$ satisfy the assumption of the lemma. Furthermore, the independence of $\partial_i f_0, \partial_{ij} f_0$ and X_0 at all points $v_0 \in V$, allows one to resolve the system (7) in $\partial_{tt} f$, and then to apply the Cauchy-Kovalevskaya theorem (see 3.1.2. in [G] for details).

These considerations are due to JANET (see [J], [Bu]) who applied them to an arbitrary metric g and thus proved by induction in n the following.

THEOREM [JANET, 1926]. — *If a metric g on V is real analytic, then a small neighborhood $U \subset V$ of any given point $v_0 \in V$ admits a real analytic isometric immersion $(U, g) \rightarrow \mathbf{R}^p$ for $p = n(n+1)/2$.*

(This result was also proven by É. CARTAN by a somewhat different method; see [C] and [B]).

Observe that a *generic* real analytic manifold (V, g) admits no analytic (not even C^∞) immersion into $\mathbf{R}^q$ for $q < p = n(n+1)/2$. This is seen by viewing the (infinite dimensional!) space $\mathcal{F}^q$ of maps $V \rightarrow \mathbf{R}^q$ as a q -dimensional variety while the space g of metrics on V is assigned dimension $p = n(n+1)/2$ (as metrics are sections of a p -dimensional bundle over V). Infact a simple application of Sard's theorem to finite dimensional jet spaces shows (see [G-R]) the image $\mathcal{D}(\mathcal{F}_\infty^q) \subset \mathcal{G}_\infty$ to be a meager subset in $\mathcal{G}_\infty$.

A similar consideration suggests the following *rigidity* of generic immersions $V \rightarrow \mathbf{R}^q$ for $q < p$. To state this we divide $\mathcal{F}^q$ by the group Is of isometries of $\mathbf{R}^q$ and observe $\mathcal{D}$ to admit a factorization to an operator

$$\overline{\mathcal{D}} : \mathcal{F}_\infty^q / \text{Is} \rightarrow \mathcal{G}_\infty.$$

CONJECTURE. — *If $q < p = n(n+1)/2$ then the operator $\overline{\mathcal{D}}$ is one-to-one on an open dense subset in $\mathcal{F}_\infty^q / \text{Is}$. (See [B] for the recent progress in the rigidity problem).*

The above conjecture claims the double points of the map (operator) $\overline{\mathcal{D}}$ to be nowhere dense in the C^∞ -topology.

Yet the subset of double points is expected to be quite substantial for $p < 2q$. The following result (see § 3.3.4. in [G]) shows this subset to be C^0 -dense for $p \ll 2q$.

THEOREM. — *If $q \geq p/2 + 2n + 2$ then arbitrary continuous maps f_1 and f_2 of V into $\mathbf{R}^q$ admit C^0 -approximations by real analytic immersions, say by f'_1 , and by f'_2 respectively, such that $\mathcal{D}f'_1 = \mathcal{D}f'_2$.*

Now we turn to the global isometric immersion problem for $q \geq p$. If $q = p$, the structure of $\mathcal{D}^+$ (and, in particular, of the image $\mathcal{D}^+(\text{Im}_\infty) \subset \mathcal{G}_\infty^+$) appears formidably complicated. Yet, for $q > p$, one expects, the operator $\mathcal{D}^+$ to behave like a reasonable smooth map of a q -dimensional variety to a p -dimensional one. In particular, one expects the following dual to the rigidity conjecture.

CONJECTURE. — *If $q > p$, then there is an open dense subset Ω in Im_∞ on which the operator $\mathcal{D}^+$ is a submersion (in particular an open map) with infinite dimensional fibers.*

The truth of this conjecture for $q \geq p + 2n$ was established by J. NASH (see [N_2]) in 1956 in the course of his solution of the isometric immersion

problem. Namely, NASH considers *free* maps $f : V \rightarrow \mathbf{R}^q$ whose osculating spaces (generated by the derivatives $\partial_i f$ and $\partial_i \partial_j f$) have dimension $n+p$ at all points $v \in V$. Then he proves the above conjecture for $\Omega =$ the subspace of free maps in Im_∞ . Moreover, the techniques in § 3.1. of [G] show *the map* $\mathcal{D}^+ : \Omega \rightarrow \mathcal{G}_\infty^+$ *to be a Serre fibration* for $q \geq p + 2n + 3$. This immediately implies the following

ISOMETRIC IMMERSION THEOREM (see § 3.1.7. in [G]). — *Every C^∞ -smooth riemannian manifold V admits an isometric C^∞ -immersion into $\mathbf{R}^q$ for $q = p + 2n + 3$. (One does not know what happens for $p \leq q < p + 2n + 3$).*

The above immersion theorem remains valid for real analytic manifolds and maps but the image

$$D_+(\text{Im}_r) \subset \mathcal{G}_{r-1}^+$$

is poorly understood for $1 \leq r < \infty$. However, the isometric immersion problem for $r = 0$ admits the following solution (see $[N_1]$, $[K]$).

THEOREM [NASH-KUIPER]. — *If V admits some immersion into $\mathbf{R}^q$, for $q > n \approx \sqrt{2}p$, then there also exists an isometric C^1 -immersion $(V, g) \rightarrow \mathbf{R}^q$ for an arbitrary C^0 -metric g on V . Moreover, the operator $\mathcal{D}^+ : \text{Im}_1 \rightarrow G_0^+$ is a Serre fibration.*

See § 2.4.9. in [G] for a conceptual proof of this remarkable theorem.

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