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BROWNIAN MOTION ON A SMALL GEODESIC BALL

by

Mark A. Pinsky

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1. Introduction

Let $\{X_t, t \geq 0\}$ be the Brownian motion process of a Riemannian manifold (M, g) . The exit time from the geodesic ball centered at $m \in M$ is defined by

$$T_\varepsilon = \inf\{t > 0: d(X_t, m) = \varepsilon\}$$

where $d(\cdot, \cdot)$ is the distance function defined by g .

In a previous paper [4] we studied the mean exit time $E_m(T_\varepsilon)$ and obtained three non-zero terms of the asymptotic expansion when $\varepsilon \downarrow 0$. This was used to prove the following stochastic characterization of the Euclidean space (R^n, g_0) : If for each $m \in M$, $E_m(T_\varepsilon) = \varepsilon^2/2n + O(\varepsilon^8)$ when $\varepsilon \downarrow 0$, then (M, g) is locally isometric to (R^n, g_0) provided $n < 6$. In case $n = 6$, we provided an example of a non-flat symmetric Riemannian manifold whose asymptotic expansion is $\varepsilon^2/2n + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$.

In this paper we shall extend our analysis to the second moment $E_m(T_\varepsilon^2)$, $m \in M$, $\varepsilon \downarrow 0$. By combining the previous techniques with the "stochastic Taylor formula" we obtain a three-term asymptotic expansion for the second moment, given at the end of section 4. As a

by-product we have the following characterization of Euclidean space $(\mathbb{R}^n, g_0)$ valid in any dimension $n < \infty$: If for each $m \in M$, $E_m(T_\varepsilon) = \text{const. } \varepsilon^2 + O(\varepsilon^8)$ and $E_m(T_\varepsilon^2) = \text{const. } \varepsilon^4 + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$, then (M, g) is locally isometric to $(\mathbb{R}^n, g_0)$. Similar characterizations are obtained for any space of constant curvature.

The present work, which could be formulated in non-stochastic terms, may be viewed as complementary to the general theory of semi-martingales on manifolds as formulated by Laurent Schwartz [5]. In particular our stochastic Taylor formula (proposition 2.1 below) is a consequence of the martingale formulation of diffusion processes.

2. Notations and Definitions

Let (M, g) be an n -dimensional Riemannian manifold. We use the following notations.

- $\bar{M}_m$ is the tangent space at $m \in M$.
- $B_m(\varepsilon)$ is the ball of radius ε in M with center at $m \in M$.
- $\bar{B}_m(\varepsilon)$ is the ball of radius ε in $\bar{M}_m$ with center at $0 \in \bar{M}_m$.
- $\exp_m$ is the exponential mapping (which is defined on all of $\bar{M}_m$ in case M is complete; otherwise it is a mapping) from $\bar{B}_m(\varepsilon)$ to $B_m(\varepsilon)$ for sufficiently small $\varepsilon > 0$.
- Φ_ε is the mapping on functions defined by

$$(\Phi_\varepsilon f)(\exp_m \varepsilon x) = f(x);$$

Φ_ε maps from $C^\infty(\bar{B}_m(1))$ to $C^\infty(B_m(\varepsilon))$ for sufficiently small $\varepsilon > 0$.

Δ is the Laplace-Beltrami operator of the Riemannian manifold:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right) \text{ where } g^{ij} = (g^{-1})^{ij}, \quad g = \det(g_{ij}).$$

The following result, which will be used repeatedly, was proved in [4].

Proposition 2.0: There exist second order differential operators $(\Delta_{-2}, \Delta_0, \Delta_1, \dots)$ on $C^\infty(\bar{M}_m)$ such that for each $N \geq 0$ and each $f \in C^\infty(\bar{M}_m)$

$$(2.1) \quad \Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon f = \varepsilon^{-2} \Delta_{-2} f + \sum_{j=0}^N \varepsilon^j \Delta_j f + O(\varepsilon^{N+1}) \quad (\varepsilon \downarrow 0).$$

Δ_j maps polynomials of degree k to polynomials of degree $k + j$. In any normal coordinate chart $(x_1, \dots, x_n)$ we have

$$(2.2) \quad \Delta_{-2} f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta_0 f = (1/3) \sum_{i,a,j,b=1}^n R_{iajb} x_a x_b \frac{\partial^2 f}{\partial x_i \partial x_j} - (2/3) \sum_{i,a=1}^n \rho_{ia} x_a \frac{\partial f}{\partial x_i}.$$

Here R_{iajb} is the Riemann tensor and $\rho_{ij} = \sum_{a=1}^n R_{iaja}$ is the Ricci tensor at $m \in M$.

Let (X_t, P_x) be the Brownian motion process with infinitesimal generator Δ . For each $m \in M$ let T_ϵ be the exit time from the geodesic ball $B_m(\epsilon)$. To study the moments of T_ϵ we invoke the following "stochastic Taylor formula."

Proposition 2.1 [1,2]: Let (X_t, P_x) be a Feller-Markov process with infinitesimal generator A . Let T be a stopping time with $E_x(T^{N+1})$ finite and let f be a function in the domain of A^{N+1} . Then

$$f(x) - E_x f(X_T) = \sum_{k=1}^N \frac{(-1)^k}{k!} E_x \left\{ T^k A^k f(X_T) \right\} + \frac{(-1)^{N+1}}{N!} E_x \left\{ \int_0^T u^N A^{N+1} f(X_u) du \right\}$$

(If $N=0$ the sum is empty and we have the Dynkin formula $E_x f(X_T) - f(x) = E_x \left\{ \int_0^T A f(X_u) du \right\}$.)

Corollary 2.2: Let T_ϵ be the exit time from the geodesic ball $B_m(\epsilon)$ and let $u_0 = 1$, $u_k(x) = (1/k!) E_x(T_\epsilon^k)$ for $k \geq 1$. Then in the interior of $B_m(\epsilon)$ we have $\Delta u_k = -u_{k-1}$ ($k=1,2,\dots$) and on the boundary we have $u_k = 0$ ($k=1,2,\dots$). In particular $\Delta^k u_N = (-1)^k u_{N-k}$, $0 \leq k \leq N$, $N \geq 1$.

Proof: Let $\bar{u}_0 = 1$ and let $\bar{u}_k$ be the classical solution of the elliptic problem $\Delta \bar{u}_k = -\bar{u}_{k-1}$ with $\bar{u}_k = 0$ on the boundary of $B_m(\epsilon)$. Taking $T = \min(R, T_\epsilon)$ and $f = \bar{u}_{N+1}$ in the proposition 2.1 we have

$$\bar{u}_{N+1}(x) - E_x \bar{u}_{N+1}(X_T) = \sum_{k=1}^N \frac{1}{k!} E_x \left\{ T^k \bar{u}_{N-k+1}(X_T) \right\} + \frac{1}{(N+1)!} E_x (T^{N+1})$$

Thus

$$\frac{1}{(N+1)!} E_x (T^{N+1}) \leq 2 \|\bar{u}_{N+1}\|_\infty + \sum_{k=1}^N \frac{\|\bar{u}_{N-k+1}\|_\infty}{k!} E_x (T^k)$$

Letting $R \rightarrow \infty$ in this inequality and using induction we see that $E_x(T_\epsilon^{N+1})$ is finite. Taking $T = T_\epsilon$ above yields

$$\bar{u}_{N+1}(x) - 0 = \frac{1}{(N+1)!} E_x(T_\varepsilon^{N+1}) = u_{N+1}(x)$$

This completes the necessary identification.

The exact solution u_2 is not available for a general Riemannian manifold. Therefore, following [4] we shall construct an approximate solution v_2 in the form

$$(2.3) \quad v_2 = \phi_\varepsilon (\varepsilon^4 g_0 + \varepsilon^6 g_2 + \varepsilon^7 g_3 + \varepsilon^8 g_4)$$

where g_0, g_2, g_3, g_4 are functions on $\bar{B}_m(1)$ satisfying

$$(2.4) \quad \Delta_{-2} g_0 = -f_0 \quad g_0|_{\partial \bar{B}_m(1)} = 0$$

$$(2.5) \quad \Delta_{-2} g_2 + \Delta_0 g_0 = -f_2 \quad g_2|_{\partial \bar{B}_m(1)} = 0$$

$$(2.6) \quad \Delta_{-2} g_3 + \Delta_1 g_0 = -f_3 \quad g_3|_{\partial \bar{B}_m(1)} = 0$$

$$(2.7) \quad \Delta_{-2} g_4 + \Delta_0 g_2 + \Delta_2 g_0 = -f_4 \quad g_4|_{\partial \bar{B}_m(1)} = 0$$

The functions f_0, f_2, f_3, f_4 are solutions of the following set of equations:

$$(2.8) \quad \Delta_{-2} f_0 = -1 \quad f_0|_{\partial \bar{B}_m(1)} = 0$$

$$(2.9) \quad \Delta_{-2} f_2 + \Delta_0 f_0 = 0 \quad f_2|_{\partial \bar{B}_m(1)} = 0$$

$$(2.10) \quad \Delta_{-2} f_3 + \Delta_1 f_0 = 0 \quad f_3|_{\partial \bar{B}_m(1)} = 0$$

$$(2.11) \quad \Delta_{-2} f_4 + \Delta_0 f_2 + \Delta_2 f_0 = 0 \quad f_4|_{\partial \bar{B}_m(1)} = 0$$

Letting $v_1 = \phi_\varepsilon (\varepsilon^2 f_0 + \varepsilon^4 f_2 + \varepsilon^5 f_3 + \varepsilon^6 f_4)$ we have $\Delta v_2 = -v_1 + O(\varepsilon^8)$, $\Delta^2 v_2 = 1 + O(\varepsilon^6)$. Applying proposition 2.1 with $N=1$, $f=v_2$ we have $v_2(p) = (\frac{1}{2})E_p(T^2(1+O(\varepsilon^6))) = (\frac{1}{2})E_p(T^2) + O(\varepsilon^{10})$. To summarize, we have the following:

Proposition 2.3: The function v_2 defined by (2.3) - (2.7) satisfies
 $v_2|_{\partial B_m(\varepsilon)} = 0 = \Delta v_2|_{\partial B_m(\varepsilon)}$, $\Delta v_2 = -v_1 + O(\varepsilon^8)$, $\Delta^2 v_2 = 1 + O(\varepsilon^6)$, and
 $v_2(m) = \frac{1}{2}E_m(T_\varepsilon^2) + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$.

3. Determination of g_0, g_2

In this section we shall prove

Proposition 3.1. We have

$$g_0 = (1/2n)^2 (1 - r^2) - (1/8n(n+2)) (1 - r^4)$$

$$g_2 = \left(\rho - \frac{\tau r^2}{n} \right) \left[\frac{n+2}{6n^2(n+4)^2} (1 - r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)} (1 - r^4) \right]$$

$$+ \tau \left[\frac{1 - r^2}{24n^3(n+2)} + \frac{1 - r^4}{24n^3(n+2)} - \frac{1 - r^6}{24n^2(n+2)(n+4)} \right]$$

where $\rho = \sum_{i,j=1}^n \rho_{ij} x_i x_j$ is the Ricci tensor, $r^2 = \sum_{i=1}^n x_i^2$ and
 $\tau = \sum_{i=1}^n \rho_{ii}$ is the scalar curvature.

Proof: Recall from the previous work [4]

$$f_0 = (1/2n) (1 - r^2)$$

$$f_2 = \left(\rho - \frac{\tau r^2}{n} \right) \frac{1 - r^2}{6n(n+4)} + \tau \frac{1 - r^4}{12n^2(n+2)}$$

$$\Delta_{-2}(r^2) = 2n, \Delta_{-2}(r^4) = 4(n+2)r^2, \Delta_{-2}(r^6) = 6(n+4)r^4$$

$$\Delta_0(r^2) = -\frac{2}{3}\rho, \Delta_0(r^4) = -\frac{4}{3}\rho r^2, \Delta_0(r^6) = -2\rho r^4$$

$$\Delta_{-2}(\rho) = 2\tau, \Delta_{-2}(r^2\rho) = 2\tau r^2 + 2(n+4)\rho, \Delta_{-2}(r^4\rho) = 2\tau r^4 + 4(n+6)\rho r^2.$$

$$\Delta_0(\rho) = \frac{2}{3}(\rho \# R - 2\rho \circ \rho), \Delta_0(r^2\rho) = \frac{2r^2}{3}(\rho \# R - 2\rho \circ \rho) - \frac{2}{3}\rho^2,$$

$$\Delta_0(r^4\rho) = \frac{2r^4}{3}(\rho \# R - 2\rho \circ \rho) - \frac{4}{3}\rho^2 r^2,$$

where in the last two formulas we have used the fact that $\Delta_0(fg) = f\Delta_0 g + g\Delta_0 f$ if $f=f(r)$ is a radial function and g is arbitrary. A lengthy but straightforward computation then shows that $\Delta_{-2}g_0 = -f_0$, $\Delta_{-2}g_2 = -f_2 - \Delta_0 g_0$, as required. Clearly both g_0, g_2 satisfy the required boundary conditions.

4. Determination of $g_4(0)$

We introduce the Green's operator:

$$P: C^\infty(\bar{B}_m(1)) \longrightarrow C^\infty(\bar{B}_m(1))$$

defined uniquely by the properties that for all $f \in C^\infty(\bar{B}_m(1))$

$$\begin{aligned} \Delta_{-2}(Pf) + f &= 0 & \text{in } \bar{B}_m(1) \\ Pf &= 0 & \text{on } \partial\bar{B}_m(1) . \end{aligned}$$

With this notation we have from (2.8) - (2.11)

$$\begin{aligned} f_0 &= P1 \\ f_2 &= P\Delta_0 f_0 \\ f_3 &= P\Delta_1 f_0 \\ f_4 &= P\Delta_0 f_2 + P\Delta_2 f_0 \end{aligned}$$

Similarly equations (2.4) - (2.7) can be written in the form

$$\begin{aligned} g_0 &= Pf_0 \\ g_2 &= Pf_2 + P\Delta_0 g_0 \\ g_3 &= Pf_3 + P\Delta_1 g_0 \\ g_4 &= Pf_4 + P\Delta_0 g_2 + P\Delta_2 g_0 \\ &= P^2\Delta_0 f_2 + P^2\Delta_2 f_0 + P\Delta_0 g_2 + P\Delta_2 g_0 . \end{aligned}$$

Therefore to compute g_4 we must first compute $\Delta_0 f_2$, $\Delta_2 f_0$, $\Delta_0 g_2$, $\Delta_2 g_0$. To handle the terms $P\Delta_0 g_2$ and $P\Delta_2 g_0$ we may use lemma 6.3 of [4]. To handle the terms $P^2\Delta_0 f_2$ and $P^2\Delta_2 f_0$ we invoke the following lemma, where the integrals are normalized so that $\int_{S^{n-1}} d\theta = 1$

Lemma 4.1. Let j be the solution of the biharmonic Poisson equation $\Delta_{-2}^2 j = r^k g(\theta)$ in the unit ball $\bar{B}_m(1)$ and satisfying the boundary conditions $j = 0$ and $\Delta_{-2} j = 0$ on the boundary $\partial\bar{B}_m(1) = S^{n-1}$. Then

$$j(0) = \frac{n+k+4}{2(k+4)n(n+k)(n+k+2)} \int_{S^{n-1}} g(\theta) d\theta$$

Proof. Let $G(x,y)$ be the Green's function for the biharmonic equation $\Delta_{-2}^2 G = \delta$ with the same boundary conditions. Then

$$j(x) = \int_{\bar{B}_m(1)} G(x,y) |y|^k g(y/|y|) dy. \text{ Let } \bar{g} = \int_{S^{n-1}} g(\theta) d\theta \text{ be the mean value}$$

of g on the unit sphere. Then

$$j(0) = \int_{\bar{B}_m(1)} G(0,y) |y|^k [g(y/|y|) - \bar{g}] + \int_{\bar{B}_m(1)} G(0,y) |y|^k dy.$$

The first integral is zero, since $G(0,y) = G(|y|)$, a radial function. The second integral is the solution of the problem $\Delta_{-2}^2 j = r^k \bar{g}$, which is directly computed as

$$j(r) = \frac{\bar{g}}{(k+2)(n+k)} \left[\frac{1-r^2}{2n} - \frac{1-r^{k+4}}{(k+4)(n+k+2)} \right].$$

Thus

$$j(0) = \frac{\bar{g}}{(k+2)(n+k)} \left[\frac{1}{2n} - \frac{1}{(k+4)(n+k+2)} \right]$$

which is of the required form.

For small values of k , we have for example

$$k=0: \quad j(0) = \frac{(n+4)}{8n^2(n+2)} \bar{g}$$

$$k=2: \quad j(0) = \frac{(n+6)}{12n(n+2)(n+4)} \bar{g}$$

$$k=4: \quad j(0) = \frac{(n+8)}{16n(n+4)(n+6)} \bar{g}.$$

We also recall the following integral formulas which were used in [4] where integration is with respect to the normalized uniform surface measure on S^{n-1} .

Lemma 4.2

$$\int_{S^{n-1}} \left(\rho - \frac{\tau r^2}{n} \right) = \frac{2}{n(n+2)} \left(\|\rho\|^2 - \frac{\tau^2}{n} \right)$$

$$\int_{S^{n-1}} \rho \# R = \frac{\|\rho\|^2}{n}$$

$$\int_{S^{n-1}} \rho \circ \rho = \frac{\|\rho\|^2}{n}$$

$$\int_{S^{n-1}} R \# R = \frac{1}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2} \|R\|^2 \right)$$

$$\int_{S^{n-1}} \nabla^2 \rho = \frac{2}{n(n+2)} \Delta \tau$$

It is easily checked that this implies that $\int_{S^{n-1}} \Delta_0 \left(\rho - \frac{\tau r^2}{n} \right) = -(2/3n) \left(\|\rho\|^2 - \frac{\tau^2}{n} \right)$.

Computation of $P^2 \Delta_2 f_0$: We have

$$\Delta_2 f_0 = (1/90n) (9 \nabla^2 \rho + 2 R \# R)$$

Both of these terms are homogeneous with $k=4$. Applying the above lemmas 4.1 and 4.2 we have

$$(P^2 \Delta_2 f_0)(0) = \frac{n+8}{90 \cdot 16 n^2 (n+4)(n+6)} \left[\frac{18}{n(n+2)} \Delta \tau + \frac{2}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2} \|R\|^2 \right) \right]$$

Computation of $P \Delta_2 g_0$: We have

$$\Delta_2 g_0 = \frac{1}{90} (9 \nabla^2 \rho + 2 R \# R) \left(\frac{1}{2n^2} - \frac{r^2}{2n(n+2)} \right)$$

which is a combination of terms with $k=4$ and $k=6$. Applying lemma 6.3 of [4] and lemma 4.2 above, we have

$$(P \Delta_2 g_0)(0) = \frac{n^2 + 20n + 48}{90 \cdot 48 n^2 (n+2)(n+4)(n+6)} \left[\frac{18}{n(n+2)} \Delta \tau + \frac{2}{n(n+2)} \left(\|\rho\|^2 + \frac{3}{2} \|R\|^2 \right) \right]$$

Computation of $P^2 \Delta_0 f_2$: We have

$$\Delta_0 f_2 = \left(\rho - \frac{\tau r^2}{n} \right) \frac{\rho}{9n(n+4)} + \frac{(1-r^2)}{6n(n+4)} \left[\frac{2}{3} (\rho \# R - 2 \rho \circ \rho) + \frac{2 \tau \rho}{3n} \right] + \frac{\tau \rho r^2}{9n^2 (n+2)}$$

which is a combination of terms with $k=2$ and $k=4$. Applying lemmas 4.1 and 4.2 we have

$$(P^2 \Delta_0 f_2)(0) = - \frac{n^2 + 12n + 48}{432 n^3 (n+2)(n+4)^2 (n+6)} \left(\|\rho\|^2 - \frac{\tau^2}{n} \right) + \frac{n+8}{144 n^4 (n+2)(n+4)(n+6)} \tau^2$$

Computation of $P\Delta_0 g_2$: We have

$$\begin{aligned}
 \Delta_0 g_2 &= \left(\rho - \frac{\tau r^2}{n} \right) \Delta_0 \left[\frac{n+2}{6n^2(n+4)^2} (1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)} (1-r^4) \right] \\
 &+ \left[\frac{n+2}{6n^2(n+4)^2} (1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)} (1-r^4) \right] \Delta_0 \left(\rho - \frac{\tau r^2}{n} \right) \\
 &+ \tau \Delta_0 \left[\frac{1-r^2}{24n^3(n+2)} + \frac{1-r^4}{24n^3(n+2)} - \frac{1-r^6}{24n^2(n+2)(n+4)} \right] \\
 &= \left(\rho - \frac{\tau r^2}{n} \right) \left[\frac{\rho(n+2)}{9n^2(n+4)^2} - \frac{\rho r^2(n+3)}{9n(n+2)(n+4)(n+6)} \right] \\
 &+ \left[\frac{n+2}{6n^2(n+4)^2} (1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)} (1-r^4) \right] \left[\frac{2}{3} (\rho \# R - 2\rho \circ \rho) + \frac{2\tau\rho}{3n} \right] \\
 &+ \tau \left[\frac{\rho}{36n^3(n+2)} + \frac{\rho r^2}{18n^3(n+2)} - \frac{\rho r^4}{12n^2(n+2)(n+4)} \right]
 \end{aligned}$$

which is a combination of terms with $k=4$ and $k=6$. Applying lemma 4.2 above and lemma 6.3 of [4] we have after some lengthy algebra

$$\begin{aligned}
 (P\Delta_0 g_2)(0) &= - \frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3(n+2)^2(n+4)(n+6)} \left(\|\rho\|^2 - \frac{\tau^2}{n} \right) \\
 &+ \frac{5n^2 + 106n + 240}{864n^4(n+2)^2(n+4)(n+6)} \tau^2
 \end{aligned}$$

These results are recorded in the table in the Appendix. We summarize the result in the following form.

Theorem 4.3. For small $\varepsilon > 0$

$$\frac{1}{2} E_m(T_\varepsilon^2) = c_0 \varepsilon^4 + c_1 \varepsilon^6 \tau_m + \varepsilon^8 \left[c_2 \Delta \tau + c_3 \tau^2 + c_4 \|\rho\|^2 + c_5 \|R\|^2 \right]_m + o(\varepsilon^{10})$$

where the constants $c_0, c_1, c_2, c_3, c_4, c_5$ depend on the dimension n . In fact $c_0 = g_0(0)$ and $c_1 = g_2(0)$ given by proposition 3.1; c_2, c_3, c_4, c_5

are given in the appendix. Here $\tau = \sum_{i=1}^n \rho_{ii}$ is the scalar curvature

and $\Delta\tau = \sum_{i=1}^n \nabla_{ii}^2 \tau$ is the Laplacian of the scalar curvature. Also $\|R\| = \left\{ \sum R_{ijkl}^2 \right\}^{\frac{1}{2}}$ and $\|\rho\| = \left\{ \sum \rho_{ij}^2 \right\}^{\frac{1}{2}}$ are the lengths of the curvature tensor and the Ricci curvature.

5. Converse theorems

Theorem 5.1. Let (M, g) be a Riemannian manifold such that for all $m \in M$ we have $E_m(T_\epsilon) = \text{const. } \epsilon^2 + O(\epsilon^8)$ and $E_m(T_\epsilon^2) = \text{const. } \epsilon^4 + O(\epsilon^{10})$ when $\epsilon \downarrow 0$. Then (M, g) is locally isometric to (R^n, g_0) .

Proof. From the first hypothesis and theorem 1.1 of [4] we have that for all $m \in M$, $\tau_m = 0$ and $\|R\|_m = \|\rho\|_m$. From the second hypothesis and theorem 4.3 above we have in addition that $c_4 \|\rho\|_m^2 + c_5 \|R\|_m^2 = 0$. This is possible for $\|R\|_m \neq 0$ if and only if $c_4 + c_5 = 0$. From the table of values in the Appendix this entails the equality

$$18(n+4)^2(n+6)(2n^2 + 25n + 48) = 33n^5 + 792n^4 + 8292n^3 + 38208n^2 + 69120n$$

Multiplying out the left side it is seen that the left side is strictly greater than the right side for every $n \geq 1$. Therefore $c_4 + c_5 \neq 0$ and we must have $\|R\|_m = 0 = \|\rho\|_m$ and (M, g) is locally isometric to (R^n, g_0) .

Theorem 5.2. Let (M, g) be a Riemannian manifold such that for all $m \in M$ we have $E_m^{(M, g)}(T_\epsilon) - E_m^{(M_\lambda, g_\lambda)}(T_\epsilon) = O(\epsilon^8)$ and $E_m^{(M, g)}(T_\epsilon^2) - E_m^{(M_\lambda, g_\lambda)}(T_\epsilon^2) = O(\epsilon^{10})$ when $\epsilon \downarrow 0$ where (M_λ, g_λ) is a space of constant sectional curvature λ . Then (M, g) is locally isometric to (M_λ, g_λ) .

Proof. From the first hypothesis and theorem 1.1 of [4] we have that for all $m \in M$

$$\tau_m = \tau(\lambda)$$

$$\|R\|_m^2 - \|\rho\|_m^2 = \|R(\lambda)\|^2 - \|\rho(\lambda)\|^2$$

where $\tau(\lambda)$, $R(\lambda)$, $\rho(\lambda)$ are the values for a space of constant sectional curvature. From the second hypothesis and theorem 4.3 above, we have further

$$c_4 \|\rho\|_m^2 + c_5 \|R\|_m^2 = c_4 \|\rho(\lambda)\|^2 + c_5 \|R(\lambda)\|^2$$

The proof of theorem 5.1 above shows that $c_4 + c_5 \neq 0$. Therefore the above equations uniquely determine the values $\|R\|_m^2 = \|R(\lambda)\|^2$, $\|\rho\|_m^2 = \|\rho(\lambda)\|^2$. It is well known that this implies that (M, g) has constant sectional curvature.

6. Appendix. Table of the coefficients of $g_4(0) = c_2 \Delta\tau + c_3 \tau^2 + c_4 \|\rho\|^2 + c_5 \|R\|^2$

coefficient of

in	$\Delta\tau$	$\ \rho\ ^2$	$\ R\ ^2$
$P^2 \Delta_0 f_2$	0	$-\frac{n^2 + 12n + 48}{432n^3 (n+2)^2 (n+6)}$	0
$P^2 \Delta_2 f_0$	$\frac{n+8}{80n^3 (n+2) (n+4) (n+6)}$	$\frac{n+8}{720n^3 (n+2) (n+4) (n+6)}$	$\frac{n+8}{480n^3 (n+2) (n+4) (n+6)}$
$P \Delta_0 g_2$	0	$-\frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3 (n+2)^2 (n+4)^3 (n+6)^2}$	0
$P \Delta_2 g_0$	$\frac{n^2 + 20n + 48}{240n^3 (n+2)^2 (n+4) (n+6)}$	$\frac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4) (n+6)}$	$\frac{n^2 + 20n + 48}{1440n^3 (n+2)^2 (n+4) (n+6)}$
TOTAL	$c_2 = \frac{2n^2 + 25n + 48}{120n^3 (n+2)^2 (n+4) (n+6)}$	$c_4 = -\frac{33n^5 + 792n^4 + 8292n^3 + 38208n^2 + 83520n + 69120}{12960n^3 (n+2)^2 (n+4)^3 (n+6)^2}$	$c_5 = \frac{2n^2 + 25n + 48}{720n^3 (n+2)^2 (n+4) (n+6)}$

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