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LONG RANGE FORCES AND TOPOLOGY OF INSTANTON MODULI SPACES†

Clifford Henry Taubes‡

Cet article décrit le théorème suivant de l'auteur : les espaces de modules pour les connexions "auto-duales" sur S^4 sont connexes.

I. MODULI SPACES

There is a distinguished set of connections on a principal $SU(2)$ bundle, $P \to S^4$. These are the connections whose curvatures are self-dual with respect to the Hodge $*$ from the standard metric on T^*S^4. This set is Aut P invariant, and the quotient,

$$\mathfrak{M}(P) = (P_s \times \{\text{self-dual connections on } P\})/\text{Aut } P \tag{1.1}$$

is naturally a smooth manifold. Here, P_s is the fibre of P at the south pole, $s \in S^4$. If the degree of P, k (minus the second Chern class of $P \times_{SU(2)} \mathbb{C}^2$), is nonnegative, then $\mathfrak{M}(P)$ is non-empty and has dimension $8k$. The space $\mathfrak{M}(P)$ is called the moduli space of self-dual connections on P. Little is known about the topology of $\mathfrak{M}(P)$; but from work by Atiyah and Jones [3] and the author [23], one conjectures that (1) $\pi_0(\mathfrak{M}(P)) = 1$, and (2) the pointed homotopy groups $\pi_n(\mathfrak{M}(P))$ and $\pi_{n+3}(S^3)$ are isomorphic if $0 < n \leqslant 2k-2$.

Similar conjectures exist for the moduli spaces $\mathfrak{M}(P)$ from principal bundles $P \to S^4$ with rank 2 or larger compact, simple Lie group as structure group. A small step towards proving these conjectures was recently made by the author [22] with the following theorem:

THEOREM 1.1. <u>Let</u> $P \to S^4$ <u>be a principal</u> $G = SU(2)$ <u>or</u> $SU(3)$ <u>bundle</u>. <u>If the moduli space</u> $\mathfrak{M}(P)$ <u>is not empty</u>, <u>then it is path connected</u>.

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The SU(2) and SU(3) cases are similar and treated with full detail in [22]. Here, and henceforth, only the SU(2) case will be discussed.

An SU(2) bundle $P \to S^4$ with degree $k=0$ is isomorphic to $S^4 \times SU(2)$. In this case, $\mathfrak{M}_0 =$ point $=$ the Aut P orbit of the product connection on $P = S^4 \times SU(2)$. For an SU(2) bundle with $k=1$, $\mathfrak{M}_1 \simeq 5$-Ball $\times SO(3)$ [2], so it is connected. For an SU(2) bundle with $k=2$, Hartshorne has shown that $\mathfrak{M}_2$ is connected using algebraic geometry via the Ward correspondence and twistor theory [12]. Prior to Theorem 1.1, the connectivity of $\mathfrak{M}(P)$ on SU(2) bundles with degree >2 was a conjecture.

It is the purpose of this note to describe the primary ideas involved in the proof of Theorem 1.1.

The theorem is obtained with the help of the first Morse inequality, the "mountain pass lemma", for the Yang-Mills functional on the infinite dimensional space of Aut P orbits of all connections on P:

$$\mathcal{B}(P) = (P_s \times \{\text{smooth connections on } P\})/\text{Aut } P \ . \tag{1.2}$$

Before describing this situation in greater detail, it is worthwhile examining the following simpler application of the first Morse inequality. Consider a subset $X \subset S^2$ which is $f^{-1}(0)$ for a C^3 function $f: S^2 \to [0,1]$ having no critical points outside of X with Morse index less than 2. Under these circumstances, the first Morse inequality implies that X is connected. In this simple case, the idea behind this application of Morse theory is easy to describe. Given two points, $p_0, p_1 \in X$, consider the space of paths in S^2 connecting p_0 to p_1:

$$\Theta = \{\phi \in C^0([0,1];S^2): \phi(0) = p_0 \text{ and } \phi(1) = p_1\} \ .$$

Associated to Θ is the number

$$f_\infty = \inf_{\phi \in \Theta} \sup_{t \in [0,1]} f(\phi(t)) \ . \tag{1.3}$$

This number is the altitude, as defined by f, of the lowest mountain pass between p_0 and p_1. Whatever the value of f_∞, a mini-max argument (Ljusternik-Šnirelman theory [13]) will provide a critical point, p_*, of f where the Morse index of f is zero or one. Hence, $p_* \in X$. But the continuity of f insures that $f(p_*) = f_\infty$, and so

$f_\infty = 0$. But if $f_\infty = 0$, then p_0 and p_1 are in the same connected component of X.

The proof of Theorem 1.1 requires an infinite dimensional analog of this example. Consider the Yang-Mills functional on $\mathcal{B}(P)$ [5], [1]. At a connection A on P, the value of the Yang-Mills functional is 1/2 times the L^2-norm of the curvature of A, F_A:

$$\mathcal{YM}(A) = \frac{1}{2} \int_{S^4} |F_A|^2(x)\, dvol(x) \quad . \tag{1.4}$$

In Equation (1.4), the pointwise norm on F_A is the Aut P invariant norm on the vector bundle $\mathrm{AdP} \otimes \overset{2}{\wedge} T^*$ that is induced from the Killing form on $\mathfrak{su}(2) = \mathrm{Lie\ alg\ SU}(2)$ and from the standard metric on T^*S^4.

A smooth connection A which is a critical point of $\mathcal{YM}$ satisfies the Yang-Mills equations,

$$D_A * F_A = 0 \quad . \tag{1.5}$$

Here, $*$ is the standard Hodge dual on $\overset{2}{\wedge} T^*S^4$, and D_A is the exterior, covariant derivative. A self-dual connection is a connection whose curvature satisfies

$$F_A = * F_A \quad . \tag{1.6}$$

(An anti-self dual connection has $F_A = -*F_A$. These can be found on bundles with negative degree and they are obtainable from self-dual connections by reversing the orientation of S^4.) The Bianchi identities force a self-dual connection to satisfy the Yang-Mills equations.

As $\mathcal{YM}(\cdot)$ is Aut P invariant, it descends to a positive functional on $\mathcal{B}$. A critical point of $\mathcal{YM}$ on $\mathcal{B}$ is the Aut P orbit of a pair $(h,A) = $ (point in P_s, solution to the Yang-Mills equations on P).

For those P with non-negative degree, $k \geqslant 0$, the Yang-Mills functional restricts to $\mathcal{M}$ with the constant value k; this is the infinum of $\mathcal{YM}$ on $\mathcal{B}$. In fact a connection A on such a principal bundle is self-dual if and only if $\mathcal{YM}(A) = k$.

Bourguignon, Lawson and Simons [6] have shown that any local minimum of $\mathcal{YM}$ on $\mathcal{B}(P)$ is self-dual if degree(P) is non-negative, or it is anti-self-dual if degree(P) is nonpositive. Presently, more information about the index of the hessian of $\mathcal{YM}$ is available.

The following theorem summarizes what is known:

THEOREM 1.2 (C.H. Taubes [23]). *Let* $P \to S^4$ *be a principal* SU(2) *bundle with degree* k. *If* $b \in \mathcal{B}(P)$ *is a critical point of* $\mathcal{YM}$ *which is neither self-dual, nor anti-self-dual, then the index of the hessian of* $\mathcal{YM}$ *at* b *is at least* $2|k|+2$.

If the Yang-Mills functional were to satisfy the Palais-Smale condition [4] on $\mathcal{B}(P)$, then Theorem 1.1 would follow readily from Theorem 1.2 with an argument which is by now standard [15]. Indeed, the conjecture concerning $\pi_n(\mathcal{M}(P))$ from the opening paragraph would follow with $n \leq 2k$, given the Palais-Smale condition. However, this condition is not satisfied, a fact which the noncompactness of $\mathcal{M}_1$ demonstrates.

The difficulties in the proof of Theorem 1.1 arise when circumventing the pathology of the Yang-Mills functional. Following the teachings of Sacks and Uhlenbeck [17], one must analyze the mechanisms by which the Palais-Smale condition fails. This analysis, together with Theorem 1.2, leads to

THEOREM 1.3. *Let* $P \to S^4$ *be a principal* SU(2) *bundle with index* $k \geq 0$. *Let* $p_0, p_1 \in \mathcal{M}(P)$. *Suppose there exists a continuous path,* $\phi \in C^0([0,1];\mathcal{B})$, *with* $\phi(0) = p_0$, $\phi(1) = p_1$ *for which*

$$\sup_{t\in[0,1]} \mathcal{YM}(\phi(t)) < k+2 \quad . \tag{1.7}$$

Then p_0 *is connected to* p_1 *by a path in* $\mathcal{M}$.

Theorem 1.3 motivates the investigation into the behavior of paths with endpoints in $\mathcal{M}(P)$. This investigation produced the following theorem:

THEOREM 1.4. *Let* $P \to S^4$ *be a principal* SU(2) *bundle with index* $k \geq 0$. *Suppose that for every principal* SU(2) *bundle* $P' \to S^4$ *with index* $0 \leq k' < k$, $\mathcal{M}(P')$ *is path connected. Then any two* $p_0, p_1 \in \mathcal{M}(P)$ *are connected by a continuous curve which satisfies Eq. (1.7).*

Armed with Theorems 1.3 and 1.4 and the knowledge that $\mathcal{M}(P(k=0)=$ point, one obtains Theorem 1.1 by the obvious induction argument.

The threshold behavior that is exhibited by Theorem 1.3 appears to be a common phenomenon in elliptic variational problems which are borderline for the Palais-Smale condition. For the Yang-Mills-Higgs functional on $\mathbb{R}^3$, a similar theorem was proved in order to establish the existence, using Ljusternik-Šnirelman theory, of a non-minimal critical point [21]. Brèzis and Nirenberg require a morally analogous theorem in their study of the inhomogeneous Yamabe problem [9]. Brèzis and Coron derive analogous theorems studying Rellich's conjecture [7], and harmonic maps of the 2-disc into S^2 [8].

In the remainder of this article, I will outline the ideas which enter the proofs of Theorems 1.3 and 1.4. Section 2 contains the discussion of Theorem 1.3. This is a discourse on Ljusternik-Šnirelman theory as applied to $\mathfrak{YM}$ on the space $\mathcal{B}(P)$. Section 3 contains the discussion of Theorem 1.4. The principal problem there is the following one: There is a natural connected sum operation on pairs $\{(P,b)\}$ where $P \to S^4$ is a principal $SU(2)$ bundle and $b \in \mathcal{B}(P)$. Given $(P,b) + (P',b') = (P'',b'')$, when is

$$\mathfrak{YM}(b'') < \mathfrak{YM}(b) + \mathfrak{YM}(b') \quad ?$$

II. LJUSTERNIK-ŠNIRELMAN THEORY FOR $\mathfrak{YM}$

If $P \to S^4$ is a principal $SU(2)$ bundle, then the homotopy groups of $\mathcal{B}(P)$ can be studied via the fibration

$$1 \to \mathrm{Aut}\, P \to P_s \times \{\text{smooth connections on } P\} \to \mathcal{B}(P) \to 1 \quad . \tag{2.1}$$

From Eq. (2.1), one concludes that $\pi_0(\mathcal{B}) = 1$ and $\pi_\ell(\mathcal{B}) \simeq \pi_{\ell+3}(S^3)$ for $\ell > 0$. The homotopy groups of $\mathfrak{M} \to \mathcal{B}$ can be found using the exact sequence

$$\to \pi_{\ell+1}(\mathcal{B};\mathfrak{M}) \to \pi_\ell(\mathfrak{M}) \to \pi_\ell(\mathcal{B}) \to \pi_\ell(\mathcal{B};\mathfrak{M}) \to \quad . \tag{2.2}$$

In order to calculate $\pi_\ell(\mathfrak{M})$ from Eq. (2.2) it is necessary to know which path components of $C^0((D^\ell,S^{\ell-1});(\mathcal{B},\mathfrak{M}))$ have maps that lie entirely in $\mathfrak{M}$. Here $D^\ell \subset \mathbb{R}^\ell$ is the unit ℓ-disc, $\partial D^\ell = S^{\ell-1}$.

Suppose that $\Theta \subseteq C^0((D^\ell;S^{\ell-1}),(\mathcal{B};\mathfrak{M}))$ is a path component. The preceding considerations give import to the number

$$A(\Theta) = \inf_{\phi\in\Theta}\ \sup_{y\in D^\ell}\ (\mathfrak{YM}(\phi(y)) - |k|). \tag{2.3}$$

If $A(\Theta) > 0$, then the Ljusternik-Šnirelman procedure presumes to associate to Θ (via mini-max arguments) a critical point of $\mathfrak{YM}$ on $\mathcal{B}$ which is not in $\mathfrak{M}$. The procedure is partially successful as it yields the two theorems below.

THEOREM 2.1. <u>Let</u> $P \to S^4$ <u>have non-negative degree</u> k. <u>There exists</u> $\varepsilon > 0$ <u>with the property that</u> $\mathcal{B}_\varepsilon(P) = \mathcal{YM}^{-1}([k,k+\varepsilon])$ <u>is a strong deformation retract of</u> $\mathfrak{M}$.

THEOREM 2.2. <u>Let</u> $P \to S^4$ <u>have non-negative degree</u> k. <u>For</u> $\ell > 0$, <u>let</u> $\Theta \subseteq C^0((D^\ell;S^{\ell-1}),(\mathcal{B};\mathfrak{M}))$ <u>be a path component</u>. <u>If</u> $\mathcal{A}(\Theta) \notin 2\mathbb{Z}$, <u>then there exists a principal</u> $SU(2)$ <u>bundle</u> $P' \to S^4$ <u>and a smooth critical point of</u> $\mathcal{YM}$ <u>in</u> $\mathcal{B}(P')$. <u>This critical point is nonminimal</u>, <u>and there</u>, <u>the index of the hessian of</u> $\mathcal{YM}$ <u>is less than</u> $\ell+1$.

The theorems above and Theorem 1.2 have Theorem 1.3 as a corollary. Indeed, consider any two path components $\mathfrak{M}_0, \mathfrak{M}_1 \subset \mathfrak{M}$. Let Θ be the component of $C^0((D^1,\partial D^1),(\mathcal{B};\mathfrak{M}))$ consisting of those curves ϕ with $\phi(0) \in \mathfrak{M}_0$ and $\phi(1) \in \mathfrak{M}_1$. Because of Eq. (1.7), $\mathcal{A}(\Theta) < 2$. If $\mathcal{A}(\Theta)$ were nonzero then Theorems 1.2 and 2.2 would be in contradiction. Therefore, $\mathcal{A}(\Theta) = 0$, in which case the deformation retract from Theorem 2.1 provides a path in $\mathfrak{M}$ which connects $\mathfrak{M}_0$ and $\mathfrak{M}_1$.

Theorem 2.1 is a natural extension of the gap phenomena discovered by Bourguignon, Lawson and Simons [6], see also [5]. It is also the correct interpretation for the implicit function theorem in [24].

Theorem 2.1 is proved from the following observations. Let $P_- = \frac{1}{2}(1-*)$ be the metric-orthogonal projection of AdP valued 2-forms onto AdP valued, anti-self-dual 2-forms. There exists $\varepsilon > 0$ with the property that if $b = [h,A] \in \mathcal{B}_\varepsilon(P)$, then the second order, elliptic operator $\Delta_A \equiv P_- D_A * D_A$ on AdP valued, anti-self-dual 2-forms is uniformly positive [6], [24]. After giving $\mathcal{B}$ one of the standard L^p_k-Sobolev topologies (with $kp > 4$) [11],[16], a smooth vector field on $\mathcal{B}_\varepsilon$ is defined by the projection onto $T\mathcal{B}_\varepsilon$ of $a = -*D_A\Delta_A^{-1}P_-F_A$. This vector field induces a smooth flow, $\Psi:[0,\infty) \times \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ by integration. The flow satisfies $\Psi(0,b) = b$,

$$\mathcal{YM}(\Psi(t,b)) = k + (\mathcal{YM}(b) - k)\exp(-2t) \quad ,$$

and $\Psi(t,b)$ converges strongly as $t \to \infty$ to $\Psi_\infty(b) \in \mathfrak{M}(P)$. The flow provides the required retract of $\mathcal{B}_\varepsilon$ onto $\mathfrak{M}$.

Theorem 2.2 is obtained by generalizing the direct minimization technique that Sedlacek [18] derived with K. Uhlenbeck's compactness theorems [25],[26]. The theorem is brethren to the Collar theorem of Donaldson [10] and Uhlenbeck [11]. Its grandparents are the harmonic map existence results of Sacks and Uhlenbeck [17] and Meeks and Yau [14].

Before describing the proof, a digression concerning some vector bundles over $\mathcal{B}$ is required. Let $\bar{\mathfrak{C}} = P_s \times \{\text{smooth connections on } P\}$. Due to Eq. (2.1), the tangent bundle, $T\mathcal{B} \to \mathcal{B}$ is obtained from the exact sequence

$$0 \to \bar{\mathfrak{C}} \times_{\text{Aut } P} \Gamma(\text{adP}) \to \bar{\mathfrak{C}} \times_{\text{Aut } P} (\mathfrak{su}(2) \times \Gamma(\text{AdP} \otimes T^*)) \to T\mathcal{B} \to 0 \quad .$$

Here, $\Gamma(\cdot)$ is the space of smooth sections of $(\cdot)$ over S^4.

By pull-back, and then restriction, the gradient of $\mathfrak{YM}$ defines a linear functional, $\nabla\mathfrak{YM}$, on

$$\mathfrak{v} = \bar{\mathfrak{C}} \times_{\text{Aut } P} (\{0\} \times \Gamma(\text{AdP} \otimes T^*)) \to \mathcal{B} \ .$$

There is an alternative way to define $\nabla\mathfrak{YM} \in \mathfrak{v}^*$: The affine structure of the space of connections on P provides a map

$$\mathfrak{f} : \mathfrak{v} \to \mathcal{B} \quad . \tag{2.4}$$

The map $\mathfrak{f}$ sends $v = [h,A,\hat{v}] \in \mathfrak{v}_b$ in the fibre over $b = [h,A] \in \mathcal{B}$ to the point $\mathfrak{f}(v) = [h,A+\hat{v}] \in \mathcal{B}$. Then

$$\nabla\mathfrak{YM}_b(v) = \frac{d}{dt}\,\mathfrak{YM}(\mathfrak{f}(tv))\Big|_{t=0} \quad .$$

With $\mathfrak{f}$, one can define a "covariant" hessian of $\mathfrak{YM}$. This section, $\mathcal{H}$, of $(\text{Sym}_2\mathfrak{v})^* \to \mathcal{B}$ is defined on $v_1, v_2 \in \mathfrak{v}_b$ by

$$\mathcal{H}_b(v_1,v_2) = \frac{d^2}{dsdt}\ \mathfrak{YM}(\mathfrak{f}(sv_1+tv_2))\Big|_{s=t=0} \quad .$$

When b is a critical point of $\mathfrak{YM}$, then $\mathcal{H}_b$ descends to $T\mathcal{B}_b$ as the true hessian of $\mathfrak{YM}$.

It is necessary to exploit the conformal invariance of the Yang-Mills problem to prove Theorem 2.2. Let $C \simeq SO(4) \times \mathbb{R}^4 \times \mathbb{R}^*$ denote the subgroup of the group of conformal diffeomorphisms of S^4 which fix the south pole, $s \in S^4$. The group C acts on $\mathcal{B}$, and compatibly on $\mathfrak{v}$: For $v = [h,A,\hat{v}] \in \mathfrak{v}$ and $t \in C$, the action sends (t,v) to $tv = [h,t^*A,t^*\hat{v}]$. The functional $\mathfrak{YM}$ is C-equivariant as are $\nabla\mathfrak{YM}$ and $\mathcal{H}$. A smooth, C-invariant metric on $\mathfrak{v}$ comes via the inverse stereographic projection map, $\mathfrak{s}: \mathbb{R}^4 \to S^4\backslash s$. Define the inner product of $\{v_i = [h_i,A_i,\hat{v}_i]\}_{i=1}^2 \in \mathfrak{v}_b$ by

$$\langle v_1, v_2\rangle_b = \langle \nabla_{s^*A}\, s^*\hat{v}_1, \nabla_{s^*A}\, s^*\hat{v}_2\rangle_{2;\mathbb{R}^4} \quad ,$$

where ∇_{s^*A} is the covariant derivative on $s^*\mathrm{Ad}\,P \otimes T^*\mathbb{R}^4$. Let $\|\cdot\|_b$ denote the associated norm on $\mathfrak{v}_b$. From $\|\cdot\|_b$, one obtains C-invariant measures of $\nabla\mathfrak{Y}\mathfrak{M}$ and $\mathcal{H}$. The $\|\cdot\|_b$-dual norm on $\mathfrak{v}^*$ measures $\nabla\mathfrak{Y}\mathfrak{M}$, this is the number

$$\|\nabla\mathfrak{Y}\mathfrak{M}_b\|_* = \sup_{0\neq v\in\mathfrak{v}_b} \|v\|_b^{-1}|\nabla\mathfrak{Y}\mathfrak{M}_b(v)| \quad .$$

For $\mathcal{H}$, a sequence of C-invariant numbers, γ_b^ℓ, are defined for $\ell \in (1,2,\ldots)$ by

$$\gamma_b^\ell = \inf_{E\subset\mathfrak{v}_b} \sup_{0\neq v\in\mathfrak{v}_b} \|v\|_b^{-2}\mathcal{H}_b(v,v) \quad ,$$

where this infimum is over all ℓ-dimensional subspaces $E \subset \mathfrak{v}_b$. Observe that $\mathcal{H}_b(\cdot,\cdot)$ has ℓ or more negative directions on $\mathfrak{v}_b$ if and only if $\gamma_b^\ell < 0$. The assignments of $b \to \|\nabla\mathfrak{Y}\mathfrak{M}_b\|_*$ and $b \to \gamma_b^\ell$ are continuous.

Theorem 2.2 is proved by considering sequences in the space $\bar{\Theta} \equiv \{(\phi,\bar{\phi}) \in \Theta \times \mathcal{B} : \bar{\phi} \in \phi^{-1}(D^\ell)$, and $\mathfrak{Y}\mathfrak{M}(\bar{\phi}) \geq \mathfrak{Y}\mathfrak{M}(\phi(y))$ for all $y \in D^\ell\}$. A sequence $\{\phi_i,\bar{\phi}_i\} \subset \bar{\Theta}$ is called a <u>good</u> sequence if

$$\begin{aligned} &(1) \quad \mathfrak{Y}\mathfrak{M}(\bar{\phi}_i) \geq \mathfrak{Y}\mathfrak{M}(\bar{\phi}_{i+1}) \searrow k + \mathcal{A}(\Theta)\,, \\ &(2) \quad \|\nabla\mathfrak{Y}\mathfrak{M}_{\bar{\phi}_i}\|_* \searrow 0 \ , \text{ and} \\ &(3) \quad \lim_{i\to\infty} \gamma_{\bar{\phi}_i}^{\ell+1} \geq 0 \ . \end{aligned} \tag{2.5}$$

In spite of the failure of the Palais-Smale condition, good sequences always exist in $\bar{\Theta}$.

The convergence of a sequence $\{\bar{\phi}_i\} \subset \mathcal{B}$ which arose from a good sequence $\{\phi_i,\bar{\phi}_i\} \subset \bar{\Theta}$ is analyzed by extending the ideas in [18] and [25], [26]. Here the conformal invariance plays a crucial role. There is a close analogy here with the "bubbling off of S^2's" phenomena in the theory of harmonic maps on S^2, c.f. [17], [14] and Siu and Yau in [20].

THEOREM 2.3. <u>Let</u> $P \to S^4$ <u>be a principal</u> SU(2)-<u>bundle with degree</u> $k \geq 0$. <u>Let</u> $\{\bar{\phi}_i\} \subset \mathcal{B}$ <u>be a sequence which satisfies Eq. (2.5) for some</u> $\ell \geq 0$. <u>There exists a subsequence of</u> $\{\bar{\phi}_i\}$, <u>also denoted</u> $\{\bar{\phi}_i\}$, <u>a</u>

finite set of sequences $\{t_i^\alpha\}_{\alpha=1}^N \subset C$, and a set of pairs $\{(P_\alpha, A_\alpha)\}_{\alpha=1}^N$, where each $P_\alpha \to S^4$ is a principal $SU(2)$ bundle over S^4 with degree k_α, and each A_α is a smooth, connection on P_α and a solution to the Yang-Mills equations on S^4. This data has the properties that are listed below.

(1) For each α, the sequence $\{t_i^{\alpha}\bar{\phi}_i\}$ converges to A_α "strongly in $L^2_{1;loc}$" of $S^4\backslash\{\text{finite set}\}$.

(2) Each (P_α, A_α) is unique up to isomorphism and the action of C. Only for $\alpha = 1$ can A_α be flat.

(3) There exists the following sum rules: $\Sigma_{\alpha=1}^N k_\alpha = k$.

(4) And $\Sigma_{\alpha=1}^N \mathfrak{YM}(A_\alpha) = k + \mathbb{A}(\Theta)$.

(5) The hessian of $\mathfrak{YM}$ at each A_α has less than $\ell + 1$ negative directions.

In Statement (1) above, "strongly in $L^2_{1;loc}$" is the standard $L^2_{1;loc}$ convergence after accounting for the Aut P invariance, c.f. [18], [25] and §5 of [21].

Theorem 2.3 implies Theorem 2.2. If each A_α were minimal for $\mathfrak{YM}$, then for each α, one would have $\mathfrak{YM}(A_\alpha) = |k_\alpha|$ by Theorem 1.3 and [6]. In this case, Statements (3) and (4) would require that

$$\mathbb{A}(\Theta) = \sum_{\alpha=1}^N (|k_\alpha| - k_\alpha) \in 2\mathbb{Z} \quad .$$

The final assertion of Theorem 2.2 follows from Statement (5) above.

III. LONG RANGE FORCES

If the energy of two particles in space is less than twice the energy of one particle alone, a physicist would say that there exists an attractive force between the two particles. Theorem 1.4 is established with a calculation of the "force" between two connections. To compute this force, a digression is required concerning the addition of connections.

Let $P, P' \to S^4$ be principal $SU(2)$ bundles, and let $(b,b') \in (\mathcal{B}(P), \mathcal{B}(P'))$. If the curvature of b is concentrated near the north pole, $n \in S^4$, and if the curvature of b' is concentrated near $s \in S^4$, then it would seem that one could add b to b' by gluing P over the northern hemisphere of S^4 to P' over the southern hemisphere by an identification of P with P' over the equator. This should produce a principal $SU(2)$ bundle, "$P+P'$" over S^4 and a point "$b+b'$" $\in \mathcal{B}(P+P')$.

Let $\theta: S^4 \to S^4$ denote the (orientation reversing) antipodal map. An identification of P and θ^*P' over the equator is naturally defined by b and $\theta b'$. Indeed, let $S^- = S^4 \backslash \{\text{north pole}\}$. Given $(h,A) \in \bar{\mathfrak{C}}$, one obtains a section, $\phi_{(h,A)} \in \Gamma(P|_{S^-})$ by parallel transport of h along the short geodesics through s (great circles). This is the polar gauge [26]. This map from $\bar{\mathfrak{C}}$ to $\Gamma(P|_{S^-})$ is $\mathrm{Aut}\,P$ equivariant, i.e. it defines a section, $\phi(\cdot)$, of the fibre bundle $\bar{\mathfrak{C}} \times_{\mathrm{Aut}\,P} \Gamma(P|_{S^-}) \to \mathcal{B}$. With this section, $b \in \mathcal{B}(P)$ defines a trivialization of P over S^-, while θ^*b' defines likewise a trivialization of θ^*P' over $S^+ = S^4\backslash s$. By identifying these two trivializations in $S^+ \cap S^-$, one obtains a principal $SU(2)$ bundle $P + \theta^*P' \equiv P - P'$ over S^4 [3]. Observe that $\text{degree}(P-P') = \text{degree}\,P - \text{degree}\,P'$.

For $\rho > 0$, let β_ρ be a bump function with support in $\{x \in S^4 : d(x,s) > 1/2 \cdot \rho\}$ which is the identity when $d(x,s) > \rho$.

Suppose that $b = [h,A]$ and $b' = [h',A']$. Define $b + \theta b' = b - b'$ over $S^+ \cap S^-$ by the data $[hh'^{-1}, \beta_\rho \phi^*(h,A)A + \theta^*(\beta_\rho \phi^*(h',A')A')]$. Evidently this definition extends to give $b - b'$ over the whole of S^4 and it defines uniquely the "difference", $b - b' \in \mathcal{B}(P-P')$. For a given $\rho > 0$, the description above defines a smooth map from $\times_2\{(\text{Principal } SU(2) \text{ bundle } P \to S^4, \mathcal{B}(P)\}$ to $\{(P, \mathcal{B}(P))\}$. (Some topological consequences of this map are deduced in [3] and [19].)

When both b and b' have their curvatures concentrated near the north pole, one expects that for small ρ, $\mathcal{YM}(b-b') \sim \mathcal{YM}(b) + \mathcal{YM}(b')$. This turns out to be the case. For Theorem 1.4, it is sufficient to consider the details only for the case when $P' \to S^4$ has degree 1 and when $b' = [h',A'] \in \mathfrak{M}(P')$ is represented by $h' \in P'_s$ and a self-dual connection A' on S^4 with scale size $1 \geq r > 0$, centered at the north pole. (The diffeomorphism $\mathfrak{M}(P') \simeq B^5 \times SO(3)$ sends such $[h',A']$ to $((1-r,0,0,0,0) \in \mathbb{R}^5, h'/\{\pm 1\})$.) Henceforth, (P',b') will be as just described. For such b', and $b = [h,A] \in \mathcal{B}(P)$, one can expand $\mathcal{YM}(b-b')$ in powers of r and ρ to obtain

$$\mathcal{YM}(b-b') - \mathcal{YM}(b) - 1 = z\rho^4 (hh'^{-1} P_+ F^i_{A'}(s) h'h, P_+ F^i_A(s)) \\ + \text{higher order terms in } (r,\rho) \qquad (3.1)$$

Here, $\{P_+F_A^i(s)\}_{i=1}^3 \in \times_3 \mathrm{Ad}\, P_s$ are the components of P_+F_A with respect to an ortho-normal basis of $P_+ \wedge_2 T_s^*$; $(\cdot,\cdot)$ is the invariant inner product on $\mathrm{Ad}\, P_s$; and $0 < z < \infty$ is a constant which does not depend on b, or b'.

The first term on the right hand side of Eq. (3.1) defines a function, f, on $SU(2) \simeq P_s'$. Look at $SU(2) \subset \mathbb{R}^4$ as the unit sphere. From this view, f is a <u>trace-zero</u>, quadratic form on $\mathbb{R}^4$ which is non-vanishing whenever the homomorphism $P_+F_A(s): P_+ \wedge_2 T_s^* \to \mathrm{Ad}\, P_s$ has rank ≥ 2. (One observes explicitly that rank $P_+F_{A'}(s) = 3$.)

Define the set $\mathbb{Q} \subseteq \mathcal{B}$ to be those $[h,A]$ for which $P_+F_A(s): P_+ \wedge_2 T_s^* \to \mathrm{Ad}\, P_s$ has rank ≥ 2. This is an open set, and $\mathcal{B} \setminus \mathbb{Q}$ has codimension at least 4.

As f is traceless, for each $b \in \mathbb{Q}$, there exist $b' \in \mathfrak{M}(P')$ and $\rho > 0$ such that $\mathcal{YM}(b - b') < \mathcal{YM}(b) + 1$. This follows from Eq. (3.1). Now the question arises whether b' can be chosen to be a continuous function of b. The answer is no, as topological obstructions exist. Indeed, b' cannot be chosen continuously even restricting b to certain 2-dimensional submanifolds in $\mathbb{Q}$. However, the quadratic nature of f allows

PROPOSITION 3.1. <u>Let</u> $\gamma: [0,1] \to \mathbb{Q}$ <u>be a continuous curve</u>. <u>There exists</u> $\rho > 0$, <u>and a continuous map</u> $\phi: [0,1] \to \mathfrak{M}(P')$ <u>such that for all</u> $t \in [0,1]$,

$$\mathcal{YM}(\gamma(t) - \phi(t)) < \mathcal{YM}(\gamma(t)) + 1 \quad .$$

Armed with Proposition 3.1 and the theorems in §2, the curve for Theorem 1.4 can be constructed. This curve is described as follows: Let $b_0, b_1 \in \mathfrak{M}(P)$ be given. By Proposition 3.1, there exists $b_0', b_1' \in \mathfrak{M}(P')$ so that $P - P'$ has degree $k - 1$, and for each $i \in (0,1)$, $b_i - b_i' \in \mathcal{B}(P - P')$ satisfies $\mathcal{YM}(b_i - b_i') < k + 1$. With the results in §2 and Theorem 1.2, one can prove with a mini-max argument that for $i \in (0,1)$ there exists $q_i \in C^0([0,1], \mathcal{B}(P - P'))$ with $q_i(0) = b_i - b_i'$, $q_i(1) \in \mathfrak{M}(P - P')$ and such that for all $t \in [0,1]$, $\mathcal{YM}(q_i(t)) < k + 1$. Under the given assumption that $\mathfrak{M}(P - P')$ is path connected, no generality is lost by assuming that $q_0(1) = q_1(1)$. Let $q(t) = q_0 \cdot q_1^{-1}$, where "$\cdot$" is the usual composition for paths, and $q_1^{-1}(t) = q_1(1 - t)$.

Since $\mathcal{B} \setminus \mathbb{Q}$ has codimension at least 4, one can require that $\theta^* q: [0,1] \to \mathbb{Q} \subseteq \mathcal{B}$. Now Proposition 3.1 implies the existence of a curve

$\gamma \in C^0([0,1];\mathfrak{M}(P'))$ and $\rho > 0$ such that $q+\gamma \in C^0([0,1];\mathcal{B}(P))$ satisfies $\mathcal{YM}(q+\gamma) < \mathcal{YM}(q) + 1 < k+2$. Observe that $(q+\gamma)(0) = b_0 - b_0' + \gamma_0$ and $(q+\gamma)(1) = b_1 - b_1' + \gamma_1$.

The point $b_0 - b_0' + \gamma_0 \in \mathcal{B}(P)$ can be thought of as b_0 with an anti-self-dual connection (b_0') of very small scale size grafted on near s; and then a self-dual connection (γ_0) of much smaller scale size grafted on at s again. The configuration $-b_0' + \gamma_0$ is a point in $\mathcal{B}(P_0 = S^4 \times SU(2))$ and it can be connected explicitly to $\mathfrak{M}(P_0)$ = point by a curve on which $\mathcal{YM} < 2$. Equally explicitly, $b_0 - b_0' + \gamma_0$ can be connected to b_0 by a curve $c_0 \in C^0([0,1];\mathcal{B}(P))$ which satisfies $c_0(0) = b_0 - b_0' + \gamma_0$, $c_0(1) = b_0$ and $\mathcal{YM}(c_0(\cdot)) < k+2$. (To a physicist, the curve c_0 models the annihilation of an "instanton" against an "anti-instanton"). Similarly one obtains c_1. The required curve for Theorem 1.4 is $c_1 \cdot (\gamma + q) \cdot c_0^{-1}$.

REFERENCES

1. M.F. Atiyah and R. Bott, "The Yang-Mills equations over Riemann surfaces," Phil. Trans. Roy. Soc. Lond. A308, 523 (1982).
2. M.F. Atiyah, N.J. Hitchin and I.M. Singer, "Self-duality in 4-dimensional Riemannian geometry," Proc. Roy. Soc. A362, 425 (1976).
3. M.F. Atiyah and J.D.S. Jones, "Topological aspects of Yang-Mills theory," Comm. Math. Phys. 61, 97 (1978).
4. M. Berger, Nonlinearity and Functional Analysis, Academic Press, New York (1977).
5. J.P. Bourguignon and H.B. Lawson, Jr., "Yang-Mills theory, its physical origin and differential geometric aspects," in Seminar on Differential Geometry, S.T. Yau, ed., Annals of Math. Studies 102, Princeton, (1982).
6. J.P. Bourguignon, H.B. Lawson, Jr. and J. Simons, "Stability and gap phenomena for Yang-Mills fields," Proc. Natl. Acad. Sci., USA 76, 1550 (1979).
7. H. Brezis and M. Coron, "Multiple solutions of H-systems and Rellich's conjecture," Comm. Pure. Appl. Math. , to appear.
8. H. Brezis and M. Coron, "Large solutions for harmonic maps in two dimensions," preprint.
9. H. Brezis and L. Nirenberg, "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents," Comm. Pure Appl. Math., to appear.
10. S.K. Donaldson, "An application of gauge theory to the topology of 4-manifolds," J. Diff. Geo., to appear. See also, S.K. Donaldson, "The Yang-Mills equations over Kähler manifolds," Thesis, Oxford University (1983).

11. D. Freed, M. Freedman and K.K. Uhlenbeck, "Gauge theories and 4-manifolds," MSRI Berkeley preprint.

12. R. Hartshorne, "Stable vector bundles and instantons," Comm. Math. Phys. 59, 1 (1978).

13. L.L. Ljusternik, Topology of the Calculus of Variations in the Large, (Amer. Math. Soc. Transl. 16), Providence R.I., Amer. Math. Soc. (1966).

14. W. Meeks and S.T. Yau, "The topology of three-dimensional manifolds, and the embedding problems in minimal surface theory," Ann. of Math 112, 441 (1980).

15. R.S. Palais, "Ljusternik-Šnirelman theory on Banach manifolds," Topology 5, 115 (1966).

16. T. Parker, "Gauge theories on four-dimensional Riemannian manifolds," Comm. Math. Phys. 85, 1 (1982).

17. J. Sacks and K.K. Uhlenbeck, "The existence of minimal 2-spheres, Ann. Math. 113, 1 (1981).

18. S. Sedlacek, "A direct method for minimizing the Yang-Mills functional over 4-manifolds",Comm. Math. Phys. 86, 515 (1982).

19. I.M. Singer, "Some remarks on the Gribov amgibuity," Comm. Math. Phys. 60, 7 (1978).

20. Y.T. Siu and S.T. Yau, "Compact Kähler manifolds of positive bisectional curvature," Inven. Math. 59, 189 (1980).

21. C.H. Taubes, "The existence of a non-minimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$",Part 1 in Comm. Math. Phys. 86, 277 (1982) and Part II in Comm. Math. Phys. 86, 299 (1982).

22. C.H. Taubes, "Path connectivity of self-dual moduli spaces," in preparation.

23. C.H. Taubes, "Stability in Yang-Mills theories," Comm. Math. Phys., to appear.

24. C.H. Taubes, "Self-dual connections on non-self-dual 4-manifolds," J. Diff. Geo. 17, 139 (1982).

25. K.K. Uhlenbeck, "Connections with L^p-bounds on curvature," Comm. Math. Phys. 83, 31 (1982).

26. K.K. Uhlenbeck, "Removable singularities in Yang-Mills fields," Comm. Math. Phys. 83, 11 (1982).

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