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M. S. BAOUENDI

LINDA PREISS ROTHSCILD

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ANALYTIC APPROXIMATION FOR HOMOGENEOUS SOLUTIONS
OF INVARIANT DIFFERENTIAL OPERATORS ON LIE GROUPS

M. S. Baouendi*

Linda Preiss Rothschild**

0. Introduction and Statements of Results.

A classical result by Malgrange [3] states that if $P(D)$ is a differential operator with constant coefficients in $\mathbb{R}^n$, then any solution u of the homogeneous equation $P(D)u = 0$ is a limit of exponential-polynomials solutions of the same equation.

Suppose now that $P(x,D)$ is a differential operator with analytic coefficients in an open set of $\mathbb{R}^n$. Assume that the principal symbol is nowhere identically zero. It is natural to ask the following question:

Is it true that any solution of $P(x,D)u = 0$ is locally a limit of real analytic solutions of the same equation?

The answer to this question is not known. However an affirmative answer is given in Baouendi-Treves [2] when P has simple (complex) characteristics. (See also [1] for first order overdetermined systems). We prove in this paper that the answer is also affirmative for left invariant operators defined on a general Lie group.

Theorem 1. Let L be a left invariant differential operator defined on a Lie group G . For every open set $U \subset G$, neighborhood of the identity $e \in G$, there exists another open neighborhood of e , $W \subset G$, such that if u is a distribution on G ($u \in \mathcal{D}'(G)$) satisfying $Lu = 0$ in U , then there exists a sequence u_ν of real analytic functions defined in W and satisfying:

- (i) $Lu_\nu = 0$ in W
- (ii) $\lim u_\nu = u$ in $\mathcal{D}'(W)$.

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Furthermore if u is of class C^k , $k \geq 0$, then the convergence in (ii) is in $C^k(W)$.

Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$, the Lie algebra of G . If α is a multi-index, $\alpha \in \mathbb{Z}_+^n$, as usual set

$$|\alpha| = \sum_{j=1}^n \alpha_j, \quad X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}.$$

Note that a left invariant differential operator on G is of the form

$$(0.1) \quad L = \sum_{|\alpha| \leq m} a_\alpha X^\alpha, \quad a_\alpha \in \mathbb{C}.$$

We can state a somewhat more general result than Theorem 1. Consider a differential operator on $(-T, T) \times G$, ($T > 0$), of the form

$$(0.2) \quad P = \partial_t^m + \sum_{\substack{|\alpha| + j \leq m \\ j < m}} a_{j, \alpha}(t) X^\alpha \partial_t^j,$$

where $a_{j, \alpha}$ are real analytic functions defined on $(-T, T)$.

Theorem 2. Let P be a differential operator on $(-T, T) \times G$ of the form (0.2). For every open set $U \subset G$, neighborhood of e , there exists W , another open neighborhood of e , and $\varepsilon \in (0, T)$, such that if $u \in \mathcal{D}'((-T, T) \times G)$ and satisfies $Pu = 0$ in $(-T, T) \times U$, then there exists a sequence u_ν of real analytic functions in $(-\varepsilon, \varepsilon) \times W$ satisfying

- (i) $Pu_\nu = 0$ in $(-\varepsilon, \varepsilon) \times W$,
- (ii) $\lim u_\nu = u$ in $\mathcal{D}'((-\varepsilon, \varepsilon) \times W)$.

Furthermore if u is of class C^k , then the convergence in (ii) is in $C^k((-\varepsilon, \varepsilon) \times W)$.

I. Proof of Theorem 1.

Before starting the proof we need to introduce some notation. Denote by dg a right Haar measure on G . If $f, h \in L^1(G, dg)$ define the convolution $f * h$ by the integral

$$(I.1) \quad (f \star h)(x) = \int_G f(xg^{-1})h(g)dg.$$

If we set

$$(I.2) \quad \check{f}(x) = f(x^{-1}), \quad \forall x \in G,$$

then making the change of variable $g' = gx^{-1}$, we also get

$$(I.3) \quad (f \star h)(x) = \int_G \check{f}(g)h(gx)dg.$$

Note that if f is a smooth function defined in an open neighborhood V of the identity e , and h is a distribution with compact support in V^{-1} then (I.1) (or (I.3)) is defined for x in an open neighborhood W of e (depending only on V and the support of h , we may take W satisfying $W(\text{supp } h)^{-1} \subset\subset V$).

If L is a left invariant operator on G , using (I.3) we see that

$$(I.4) \quad L(f \star h) = f \star (Lh).$$

Recall that $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$. Let V be a sufficiently small open neighborhood of the identity in G such that the exponential map Exp is an analytic diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto V . For simplicity we assume $V = V^{-1}$. For $x \in V$ we may write

$$x = \text{Exp}(s_1 X_1 + \dots + s_n X_n) = \text{Exp}(s.X)$$

with $s = (s_1, \dots, s_n) \in \mathbb{R}^n$. The map

$$(I.5) \quad S: V \rightarrow \mathbb{R}^n, \quad S(x) = s,$$

is then an analytic diffeomorphism of V onto a neighborhood $\tilde{V}$ of the origin in $\mathbb{R}^n$.

There exists an analytic function σ , $\sigma \neq 0$, defined in $\tilde{V}$ such that if u is, say a continuous function with compact support in V , then

$$(I.6) \quad \int_G u(g)dg = \int_{\mathbb{R}^n} u(S^{-1}(t))\sigma(t)dt.$$

For $v \in \mathbb{Z}_+$ and $x \in V$, set

$$(I.7) \quad f_v(x) = \left(\frac{v}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} e^{-v^2(S(x))^2}.$$

(If $s \in \mathbb{R}^n$, $s^2 = \sum_{j=1}^n s_j^2$). Note that f_v is an analytic function defined in V and satisfies $\check{f}_v = f_v$.

Lemma 1. Let h be a distribution with compact support in V .
There is an open neighborhood of e , $W \subset G$, depending only on the
support of h , such that

$$\lim_{v \rightarrow \infty} (f_v * h)|_W = h|_W \quad \text{in } \mathcal{D}'(W).$$

Moreover if h is in C^k then the convergence is in $C^k(W)$.

Proof: Let W_1 be an open neighborhood of the support of h satisfying

$$\bar{W}_1 \subset V.$$

We may choose an open neighborhood W of e in G satisfying

$$(I.8) \quad W.W_1^{-1} \subset V.$$

(Recall that $V = V^{-1}$).

Assume first that h is a continuous function (with compact support in W_1). Using (I.3), (I.7) and the fact that $\check{f}_v = f_v$ we get for $x \in \bar{W}$.

$$(f_v * h)(x) = \left(\frac{v}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} \int_G e^{-v^2(S(g))^2} h(gx) dg,$$

and making use of (I.5) and (I.6), we obtain for $x \in \bar{W}$

$$(f_v * h)(x) = \left(\frac{v}{\sqrt{\pi}}\right)^n \sigma(0)^{-1} \int_{\mathbb{R}^n} e^{-v^2 s^2} h((\text{Exp } s.X)x) \sigma(s) ds.$$

Changing variables in the latter ($vs = t$) yields

$$(I.9) \quad (f_v * h)(x) = \frac{\sigma(0)^{-1}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-t^2} h((\text{Exp } \frac{t}{v}.X)x) \sigma\left(\frac{t}{v}\right) dt.$$

A limiting argument in (I.9) easily shows that $(f_v * h)|_{\bar{W}}$ converges uniformly to $h|_{\bar{W}}$.

If in addition h is of class C^k , $k > 0$, since we have

$$X^\alpha(f_v * h) = f_v * (X^\alpha h), \quad \forall \alpha \in \mathbb{Z}_+^n,$$

we also get the convergence in $C^k(\bar{W})$.

Assume now that h is a distribution with compact support in W_1 . Let $\phi \in C_0^\infty(W)$. Since $V = V^{-1}$ we get from (I.8)

$$W_1 \cdot W^{-1} \subset V.$$

Therefore it follows from the first part of the proof of this lemma that $f_v * \phi$ converges to ϕ in $C^\infty(W_1)$. On the other hand, using (I.1) and (I.3) we have

$$\int_G (f_v * h)(x) \phi(x) dx = \int_G h(g) (f_v * \phi)(g) dg.$$

This shows that $f_v * h$ converges to h in $\mathcal{D}'(W)$. Q.E.D.

Lemma 2. If the open set V in (I.5) is small enough then for every pair of open neighborhoods of e , V_0 and V_1 , $V_1 \subset V_0 \subset V$, there exists an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^n$ such that if h is a distribution with compact support in V_0 , and $h \equiv 0$ in V_1 , then for every $v \in \mathbb{Z}_+$,

$$(f_v * h) \circ S^{-1}$$

extends holomorphically to $\mathcal{O}$, and converges uniformly to zero in $\mathcal{O}$ as $v \rightarrow \infty$.

Proof: Let us first state the Baker-Campbell-Hausdorff formula in a form which will be needed further (see Varadarajan [4] for example). For $s, t \in \mathbb{R}^n$ sufficiently small we have

$$(I.10) \quad \text{Exp}(s.X) \cdot \text{Exp}(-t.X) = \text{Exp}(u.X)$$

with $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, and for $j = 1, \dots, n$,

$$(I.11) \quad u_j = u_j(s, t) = s_j - t_j + \sum_{\substack{|\alpha| \geq 1 \\ |\beta| \geq 1}} c_{\alpha, \beta, j} t^\alpha s^\beta,$$

where $c_{\alpha, \beta, j} \in \mathbb{R}$ and satisfy

$$|c_{\alpha, \beta, j}| \leq M^{|\alpha| + |\beta| + 1}.$$

Let V be the open set in (I.5) ($V = V^{-1}$). We may assume that V is small enough so that for all $x, g \in V$, if

$$S(x) = s, \quad S(g) = t,$$

then the power series (I.11) is absolutely convergent.

Now let $h \in \mathcal{C}^1(V_0)$, $h \equiv 0$ in V_1 , with $V_1 \subset\subset V_0 \subset\subset V$. Using (I.1) and (I.7) we get, for x near e

$$h_v(x) = (f_v * h)(x) = c_v \int_G e^{-v^2(S(xg^{-1}))^2} h(g) dg$$

with $c_v = \left(\frac{v}{\sqrt{\pi}}\right)^n \sigma(0)^{-1}$. Writing $x = \text{Exp}(s.X)$, $g = \text{Exp}(t.X)$

$\tilde{h} = h \circ S^{-1}$, $\tilde{h}_v = h_v \circ S^{-1}$ and using (I.6) we obtain

$$\tilde{h}_v(s) = c_v \int_{\mathbb{R}^n} e^{-v^2[S(\text{Exp}(s.X)\text{Exp}(-t.X))]^2} \tilde{h}(t) \sigma(t) dt.$$

Making use of (I.10) yields

$$(I.12) \quad \tilde{h}_v(s) = c_v \int_{\mathbb{R}^n} e^{-v^2 u^2} \tilde{h}(t) \sigma(t) dt,$$

where $u = (u_1, \dots, u_n)$ is given by (I.11). Since h vanishes in V_1 , we may assume that

$$\text{supp } \tilde{h} \subset \{t \in \mathbb{R}^n, A < |t| < B\}, \quad A > 0.$$

We must show that $\tilde{h}_v$ defined by (I.12) extends holomorphically to a neighborhood of 0 in $\mathbb{C}^n$ (independent of v), and there converges to 0 as $v \rightarrow \infty$.

Indeed for $s, \tilde{s} \in \mathbb{R}^n$, sufficiently small, we get from (I.12)

$$(I.13) \quad \tilde{h}(s + i\tilde{s}) = c_v \int_{A \leq |t| \leq B} e^{-v^2 v^2} \tilde{h}(t) \sigma(t) dt,$$

with $v = (v_1, \dots, v_n)$, and v_j is the expression obtained by putting $s_j + i\tilde{s}_j$ instead of s_j in (I.11), i.e.

$$(I.14) \quad v_j = s_j + i\tilde{s}_j - t_j + \sum_{\substack{|\alpha| \geq 1 \\ |\beta| \geq 1}} c_{\alpha, \beta, j} t^\alpha (s + i\tilde{s})^\beta.$$

Note that the latter is absolutely convergent for $|t| \leq B$ and s and $\tilde{s}$ sufficiently small. Set

$$Q = \operatorname{Re} v^2 = \operatorname{Re} \left(\sum_{j=1}^n v_j^2 \right).$$

It is easy to check that there is $\delta_0 > 0$ and $C > 0$ such that if $\delta \in (0, \delta_0)$ then for $|s| \leq \delta$, $|\tilde{s}| \leq \delta$ and $A \leq |t| \leq B$ we have

$$Q \geq (A - \delta)^2 - C\delta.$$

Choosing $\delta \in (0, \delta_0)$ small enough we get

$$(I.15) \quad Q \geq \frac{A^2}{2}.$$

Since $\tilde{h}$ is a distribution with compact support in $\{A < |t| < B\}$ it follows from (I.13) that there exists $C > 0$ and $\ell \in \mathbb{Z}_+$ such that for $|s| \leq \delta$, $|\tilde{s}| \leq \delta$

$$(I.16) \quad |\tilde{h}_v(s + i\tilde{s})| \leq C c_v \sup_{\substack{|\alpha| \leq \ell \\ A \leq |t| \leq B \\ |s|, |\tilde{s}| \leq \delta}} |\partial_t^\alpha e^{-v^2 v^2}|.$$

It is clear that the right hand side of (I.16) may be bounded by

$$C' v^N \sup_{\substack{A \leq |t| \leq B \\ |s|, |\tilde{s}| \leq \delta}} (e^{-v^2 Q})$$

where $C' > 0$ and $N \in \mathbb{Z}_+$ are independent of v . Therefore (I.15) and (I.16) imply that for $|s| \leq \delta$, $|s'| \leq \delta$

$$(I.17) \quad |\tilde{h}_v(s + i\tilde{s})| \leq C' v^N e^{-v^2 A^2/2}.$$

(I.17) yields the desired result by taking

$$\emptyset = \{s + i\tilde{s} \in \mathbb{C}^n, |s| < \delta, |\tilde{s}| < \delta\}. \quad \text{Q.E.D.}$$

We are now ready to prove Theorem 1. Let u be as in Theorem 1 i.e.

$$u \in \mathcal{D}'(G), Lu = 0 \text{ in } U, \quad e \in U \subset G.$$

Let V be a sufficiently small open neighborhood of e , $V \subset G$, in which Lemmas 1 and 2 are valid. Take $\zeta \in C_0^\infty(V)$, $\zeta \equiv 1$ near e . Set

$$(I.18) \quad h = \zeta u, \quad r = Lh.$$

Both h and r are distributions with compact supports in V . Furthermore $r \equiv 0$ in some neighborhood V_1 of e , $V_1 \subset\subset V$. Since L commutes with the convolution with f_v we get from (I.18).

$$(I.19) \quad L(f_v * h) = f_v * r.$$

By Lemma 1, we know that $f_v * h$ converges to h in a neighborhood W of e . Lemma 2 implies that $f_v * r$ extends holomorphically to a complex neighborhood of e (independent of h and v) and there converges to zero. By the Cauchy-Kovalevski theorem and by shrinking W if needed, we may find a sequence k_v of analytic functions in W converging to 0 (in the space of analytic functions in W) and satisfying

$$(I.20) \quad Lk_v = f_v * r.$$

[In fact we can require that the Cauchy data of k_v be zero on a non-characteristic analytic hypersurface passing through e].

Put

$$u_v = f_v * h - k_v.$$

It follows from (I.19) and (I.20) that

$$Lu_{\nu} = 0 \quad \text{in } W.$$

On the other hand

$$(I.21) \quad \lim u_{\nu} = h \quad \text{in } \mathcal{D}'(W);$$

since $h = u$ near e (where $\zeta \equiv 1$, see (I.8)), the proof of Theorem 1, when u is a distribution, is complete.

If u is of class C^k , it follows from Lemma 1 that the convergence in (I.21) is in $C^k(W)$. Q.E.D.

II. Proof of Theorem 2.

The proof of Theorem 2 is similar to the proof of Theorem 1.

Let $u \in \mathcal{D}'((-T, T) \times G)$ satisfying

$$Pu = 0 \quad \text{in } (-T, T) \times U, \quad e \in U \subset G.$$

Without loss of generality, by shrinking U and the interval $(-T, T)$ if needed, we may assume

$$(II.1) \quad u \in C^m((-T, T); H^{-N}(U))$$

($N \in \mathbb{Z}_+$, $H^{-N}(U)$ is the usual negative Sobolev space in U).

Let V be an open neighborhood of e in which Lemmas 1 and 2 are valid. Take $\zeta \in C_0^\infty(V)$, $\zeta \equiv 1$ near e , and set

$$(II.2) \quad \zeta u = h, \quad Ph = r.$$

It follows from (II.1) and (II.2) that we have

$$h \in C^m((-T, T), H_{\text{comp}}^{-N}(V)), \quad r \in C^0((-T, T), H_{\text{comp}}^{-N-m}(V)),$$

furthermore

$$r(t, \cdot) \equiv 0 \quad \text{near } e.$$

Let f_ν be defined by (I.7), since P (defined by (0.2)) commutes with the convolution with f_ν (convolution on G , t being a parameter) we get from (II.2)

$$(II.3) \quad P(f_\nu * h) = f_\nu * r.$$

Inspection of the proofs of Lemmas 1 and 2 shows that

$$(II.4) \quad \lim f_\nu * h = h \quad \text{in } \mathcal{D}'((-T, T) \times W),$$

and that $f_\nu * r$ extends as an element of

$$C^0((-T, T), \mathcal{H}(\mathcal{O}))$$

and converges to 0 in this space ($\mathcal{H}(\mathcal{O})$ is the space of bounded holomorphic functions in $\mathcal{O} \subset \mathbb{C}^n$).

Using a refinement of the Cauchy-Kovalevsky theorem, and contracting W if needed, we may find $\varepsilon > 0$ (independent of h and ν) and a sequence

$$k_\nu \in C^m((-\varepsilon, \varepsilon), \mathcal{A}(W))$$

($\mathcal{A}(W)$ is the space of real analytic functions in W) converging to zero in that space and satisfying

$$(II.5) \quad \left\{ \begin{array}{l} Pk_\nu = f_\nu * r \quad \text{in } (-\varepsilon, \varepsilon) \times W \\ \partial_t^j k_\nu|_{t=0} = 0, \quad j = 0, \dots, m-1. \end{array} \right.$$

If we set

$$u_\nu = f_\nu * h - k_\nu,$$

it follows from (II.3) and (II.5) that we have

$$(II.6) \quad Pu_\nu = 0 \quad \text{in } (-\varepsilon, \varepsilon) \times W.$$

On the other hand we have

$$u_\nu \in C^m((-\varepsilon, \varepsilon), \mathcal{A}(W)).$$

Since $\partial_t^j u_\nu|_{t=0} = f_\nu * (\partial_t^j h)|_{t=0} \in \mathcal{A}(W)$, uniqueness for the Cauchy problem, in conjunction with (II.6), implies that u_ν is analytic in $(-\varepsilon, \varepsilon) \times W$. Q.E.D.

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M. S. Baouendi
Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA

Linda Preiss Rothschild
Department of Mathematics
University of California
San Diego, CA 92023
USA