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Vanishing theorems on singular spaces

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VANISHING THEOREMS ON SINGULAR SPACES

J.H.M. Steenbrink.

1. INTRODUCTION

The classical vanishing theorem of Kodaira-Akizuki-Nakano [1] of which we will describe a generalization to singular spaces is the following.

THEOREM. *Let* X *be an* n*-dimensional compact complex manifold,* L *an ample line bundle on* X. *Then*

$$H^q(X,\Omega_X^p \otimes L) = 0 \qquad \textit{for } p+q > n .$$

A partial generalization is furnished by Grauert and Riemenschneider [5]:

THEOREM. *Let* X *be a compact complex space of dimension* n, L *an ample line bundle on* X *and* $\pi : \tilde{X} \to X$ *a proper birational morphism such that* $\tilde{X}$ *is smooth. Then*

a) $H^q(X,\pi_*\Omega_{\tilde{X}}^n \otimes L) = 0 \qquad \textit{for } q > 0$,

b) $R^q\pi_*\Omega_{\tilde{X}}^n = 0 \qquad \textit{for } q > 0$.

This gives no information about the Ω^p for $p \neq n$. In fact, Ramanujam [11] observes that vanishing of $H^q(X,\pi_*\Omega_{\tilde{X}}^p \otimes L)$ in the terminology of the theorem of Grauert Riemenschneider need not hold.
Two observations are fundamental for the understanding of the problem. The first one, due to Grauert and Riemenschneider, is that to define Ω_X^n for an n-dimensional singular space X one must specify which properties of Ω^n one wants to remain valid: either its rôle as a dualizing sheaf or its function in the vanishing theorems. They show that their vanishing theorem is no longer true if one replaces $\pi_*\Omega_{\tilde{X}}^n$ by Grothendieck's dualizing sheaf ω_X. Because the two vanishing results above are closely related to and have been proved by Hodge theory, it is desirable to focus attention to the Hodge theoretic properties of the sheaves Ω^p. On the other hand, it appears that for a singular space Ω^p need not generalize to one single sheaf but to a complex of sheaves with $\mathcal{O}$-linear differential and coherent cohomology sheaves. This idea is present in the work of Du Bois [4]. Following P. Deligne's work on mixed Hodge structures

he shows , that to each complex algebraic variety X one may associate a filtered complex $(K_X^{\cdot},F)$ which resolves the constant sheaf $\mathbb{C}_X$ and such that the complexes $Gr_F^p K_X^{\cdot}$ have O_X - linear differentials and coherent analytic cohomology sheaves. Moreover $(K_X^{\cdot},F)$ is functorial and in a sense unique up to isomorphism in the filtered derived category $D^+F(X,\mathbb{C})$. In the smooth case $K_X^{\cdot}$ is realized by the holomorphic De Rham complex and F is its "filtration bête" $\sigma_{\geq}$:

$$\sigma_{\geq p}\Omega_X^{\cdot} = \{0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots\} .$$

Guillén, Navarro Aznar and Puerta proved the following theorem ([7,Th. 6.6.1, Th. 6.7.1]) :

MAIN THEOREM. *Let* X *be a compact complex variety of dimension* n, L *an ample line bundle on* X *and* $(K_X^{\cdot},F)$ *the filtered De Rham complex of* X. *Then*

a) $\mathbb{H}^m(X,Gr_F^p K_X^{\cdot} \otimes L) = 0$ *for* $m > n$,

b) $H^m(Gr_F^p K_X^{\cdot}) = 0$ *for* $m < p$ *or* $m > n$.

Here $\mathbb{H}$ denotes hypercohomology and H stands for cohomology sheaf. We will deduce the main theorem from

THEOREM 2. *Let* X *be an* n-*dimensional complex projective variety,* $\Sigma \subset X$ *such that* $X \setminus \Sigma$ *is nonsingular,* L *an ample line bundle on* X *and* $\pi : \tilde{X} \to X$ *a proper birational mapping such that* $\tilde{X}$ *is nonsingular,* $E = \pi^{-1}(\Sigma)$ *is a divisor with normal crossings on* $\tilde{X}$ *and* π *maps* $\tilde{X} \setminus E$ *isomorphically to* $X \setminus \Sigma$. *Then*

a) $H^q(\tilde{X},J_E\Omega_{\tilde{X}}^p(\log E) \otimes \pi^*L) = 0$ *for* $p+q > n$,

b) $R^q\pi_* J_E\Omega_{\tilde{X}}^p(\log E) = 0$ *for* $p+q > n$.

Here $\Omega_{\tilde{X}}^{\cdot}(\log E)$ is the logarithmic De Rham complex and J_E is the ideal sheaf of the divisor E. Observe that statement a) is equivalent to

a') $H^q(\tilde{X},\Omega_{\tilde{X}}^p(\log E) \otimes \pi^*L^{-1}) = 0$ *for* $p+q < n$

by Serre duality. If in a') one takes q=0, p < n, one obtains a special case of Bogomolov's vanishing theorem (see [2,13]).
The purpose of this note is, to make an advertisement for the important results obtained in [7] and to present a relatively simple proof of the main theorem in

the spirit of Ramanujam's proof of the Kodaira-Akizuki-Nakano vanishing theorem. For simplicity we restrict ourselves to the case of a very ample line bundle L. The general case can be obtained by passing to a finite covering, ramified along a section of a high multiple of L. For more vanishing theorems the reader is referred to [7] and [13].

2. THE FILTERED DE RHAM COMPLEX.

We give a short description of the construction of the filtered complex $(K^{\cdot}_X, F)$.

(2.1) Let X be a complex algebraic variety of dimension n. A *simplicial space of level* k over X is a sequence $X_0,\dots,X_k$ of complex algebraic varieties together with maps $\varepsilon_{ij} : X_i \to X_{i-1}$ for $0 \le j \le i \le k$, where we put $X = X_{-1}$, satisfying $\varepsilon_{ij}\varepsilon_{i+1,j+1} = \varepsilon_{ij}\varepsilon_{i+1,j}$ for all $j < i$:

$$\cdots X_2 \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} X_1 \rightrightarrows X_0 \longrightarrow X$$

By composing any sequence of $\varepsilon_{q,j}$'s, q=0,...,i one obtains well-defined maps

$$\varepsilon_i : X_i \longrightarrow X \ .$$

Let $\Delta^p = \{(t_0,\dots,t_p) \in \mathbb{R}^{p+1} \mid \Sigma_{i=0}^p t_i = 1,\ t_i \ge 0 \text{ all } i\}$ be the standard p-simplex. For all $j \le i$ one has the map $\varepsilon^{i,j}: \Delta^{i-1} \to \Delta^i$ given by

$$\varepsilon^{i,j}(t_0,\dots,t_{i-1}) = (t_0,\dots,t_{j-1},0,t_j,\dots,t_{i-1}) \ .$$

(2.2) The *geometric realization* $|X_{\cdot}|$ of the simplicial space $X_{\cdot}$ over X is the space, obtained from the disjoint union of all $X_i \times \Delta^i$, i=0,...,k by identification of the points

$$(x, \varepsilon^{ij}(t)) \text{ and } (\varepsilon_{ij}(x), t)$$

for $x \in X_i, t \in \Delta^{i-1}, i \ge 1$ and $0 \le j \le i$. One has a natural map

$$|\varepsilon| : |X_{\cdot}| \longrightarrow X$$

which is continuous.

We call $X_{\cdot} \to X$ a *simplicial resolution of* X if the X_i are smooth, the morphisms

ε_{ij} are proper and $|\varepsilon|$ has contractible fibers.

In [7] the theory of mixed Hodge structures for complex algebraic varieties has been built up using simplicial resolutions as above, which are obtained from "cubical schemes", in contrast to Deligne's approach which uses a stronger kind of simplicial schemes with maps both ways, which are necessarily infinite. Of course the resulting mixed Hodge structures are the same.

(2.3) PROPOSITION. *Let* X *be an* n*-dimensional complex algebraic variety. Then there exists a simplicial resolution* $X_{.}$ *of* X *such that dim* $X_i \leq n-i$.
PROOF. See [7,§2].

REMARK. Suppose that $X_{.} \to X$ is a simplicial resolution and that $Z \subset X$ is a closed subvariety which is transverse to all morphisms $\varepsilon_i : X_i \to X$ (e.g. a sufficiently general hyperplane section if X is quasiprojective). Put $Z_i = Z \times_X X_i$. Then the Z_i form in a natural way a simplicial resolution of Z.

(2.4) If Y is a smooth complex variety, we let $E_Y^{\cdot}$ denote the complex valued C^∞ De Rham complex of Y. It has a decomposition into Hodge types

$$E_Y^m = \bigoplus_{p+q=m} E_Y^{p,q} \quad .$$

If X is a singular variety, we construct an analogous complex of sheaves as follows. We first choose a simplicial resolution $X. \to X$. Then we let

$$K_X^{\cdot} = \bigoplus_{i,m} \varepsilon_{i*} E_{X_i}^m \quad .$$

The differential in $K_X^{\cdot}$ has the form $d=d'+d''$ where d' is ordinary differentiation of C^∞ forms and

$$d'' = \sum (-1)^{i+j} \varepsilon_{i+1,j}^*: \varepsilon_{i*}E_{X_i}^m \to \varepsilon_{i+1*}E_{Xi+1}^m$$

We filter $K_X^{\cdot}$ by

$$F^pK_X^m = \bigoplus_{r\geq p} \bigoplus_{i} \varepsilon_{i*}E_{X_i}^{r,m-i-r}$$

Then $(K_X^{\cdot},F)$ is an incarnation of the filtered De Rham complex of X.

REMARK. This definition differs from the one used by Du Bois. He considers essentially $\mathbb{R}\varepsilon_*\Omega^\cdot_{X_\cdot}$ where $\Omega^\cdot$ denotes algebraic differential forms. One has a filtered quasi-isomorphism

$$\mathbb{R}\varepsilon_*\Omega^\cdot_{X_\cdot^{an}} \longrightarrow K^\cdot_X \quad .$$

(2.5) PROPERTIES (see [4,§4]).

1. The differentials $d_i : K^i_X \to K^{i+1}_X$ are differential operators of order one.
2. The differentials on the graded complexes

$$\mathrm{Gr}^p_F K^\cdot_X = F^p K^\cdot_X / F^{p+1} K^\cdot_X$$

   are $\mathcal{O}_X$-linear and their cohomology sheaves are coherent analytic sheaves on X.
3. $K^\cdot_X$ is a resolution of the constant sheaf $\mathbb{C}$ on X.
4. If X is a *compact* algebraic variety over $\mathbb{C}$, the spectral sequence of hypercohomology

$$E_1^{pq} = \mathbb{H}^{p+q}(X,\mathrm{Gr}^p_F K^\cdot_X) \Rightarrow \mathbb{H}^{p+q}(X,K^\cdot_X) = H^{p+q}(X,\mathbb{C})$$

   degenerates at E_1 and abuts to Deligne's Hodge filtration: $E_1^{pq} \simeq \mathrm{Gr}^p_F H^{p+q}(X,\mathbb{C})$.
5. Up to isomorphism in the filtered derived category $D_+F(X,\mathbb{C})$ of X, the complex $(K^\cdot_X,F)$ does not depend on the choice of the simplicial resolution $X_\cdot$ of X . In particular the coherent sheaves

$$\mathcal{H}^q \mathrm{Gr}^p_F K^\cdot_X$$

   are invariants of X as an analytic space.

(2.6) SMALL INCARNATIONS.

For some special types of spaces, $K^\cdot_X$ can be described up to quasi-isomorphism without passing to a simplicial resolution.

- If E is a variety with normal crossings, then

$$(K^\cdot_E,F) \underset{\text{qiso}}{\approx} (\Omega^\cdot_E/\text{torsion}, \sigma)$$

  where σ is the "filtration bête". If E lies on a complex manifold Y with dim Y = dim E+1 one has

$$\Omega^\cdot_E/\text{torsion} \simeq \Omega^\cdot_Y / J_E \Omega^\cdot_Y(\log E)$$

(Friedman).

- If X has only toroidal singularities,

$$(K^{\cdot}_X, F) \underset{\text{qiso}}{\approx} (j_*\Omega^{\cdot}_U, \sigma)$$

where $j : U \to X$ denotes the inclusion of the regular locus of X.

- The condition that $Gr^0_F K^{\cdot}_X$ be a resolution of $\mathcal{O}_X$ defines "Du Bois singularities". See [12,§3] where it is shown that, if a normal Gorenstein surface X satisfies this condition, then the only singularities of X can be rational double points, simply elliptic points or cusps.

- Suppose X has dimension n and $X_{.} \to X$ is a simplicial resolution such that $\dim X_i < n$ for $i > 0$. Then

$$Gr^n_F K^{\cdot}_X = \varepsilon_{0*}\mathcal{E}^{n\cdot}_{X_0}\ [-n]$$

where $[-n]$ means a shift of n places to the right. Remark that $X_0 \to X$ may be any resolution of X and that $\mathcal{E}^{n\cdot}_{X_0}$ is a fine resolution of $\Omega^n_{X_0}$.
Hence

$$H^i(Gr^n_F K^{\cdot}_X) \simeq R^{i-n}\varepsilon_{0*}\Omega^n_{X_0} \ .$$

By statement b) of the theorem of Grauert and Riemenschneider this vanishes for $i \neq n$. Hence for any resolution $\pi : \tilde{X} \to X$ one has

$$Gr^n_F K^{\cdot}_X \underset{\text{qiso}}{\approx} \pi_*\Omega^n_{\tilde{X}}\ [-n] \ .$$

3. THE FILTERED DE RHAM COMPLEX OF A PAIR.

(3.1) Let $f : Y \to X$ be a morphism of complex algebraic varieties. Then there exist simplicial resolutions $Y_{.} \to Y$ and $X_{.} \to X$ and a commutative diagram

$$\begin{array}{ccc} Y_{.} & \xrightarrow{f_{.}} & X_{.} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

where $f_.$ is a morphism of simplicial spaces. By pulling back differential forms one obtains a mapping

$$f^*\colon K^\cdot_X \longrightarrow f_*K^\cdot_Y$$

which is compatible with the filtration F. We let $K^\cdot_{X,Y}$ denote the cone of f^*, i.e.

$$K^m_{X,Y} = K^m_X \oplus f_*K^{m-1}_Y$$

$$d(\xi,\eta) = (-d\xi, d\eta + f^*(\xi)).$$

It carries a filtration F which is given by

$$F^pK^m_{X,Y} = F^pK^m_X \oplus f_*F^pK^{m-1}_Y .$$

We do not claim uniqueness of $(K^\cdot_{X,Y},F)$ in $D^+F(X,\mathbb{C})$ though this is probably true.

(3.2) EXAMPLE. Let Y be smooth and $E \subset Y$ a divisor with normal crossings. Then $K^\cdot_Y \simeq \Omega^\cdot_Y$ and $K^\cdot_E \simeq \Omega^\cdot_E$/torsion. As the map

$$f^* : \Omega^\cdot_Y \to \Omega^\cdot_E/\text{torsion}$$

is surjective, its cone is filtered quasi-isomorphic to $\mathrm{Ker}(f^*)$. Hence

$$K^\cdot_{Y,E} \simeq J_E\, \Omega^\cdot_Y(\log E)$$

and F corresponds again to the "filtration bête".

(3.3) PROPOSITION. Let X be a complex algebraic variety, $\Sigma \subset X$ a closed subvariety such that $X \smallsetminus \Sigma$ is smooth, $\pi : Y \to X$ a proper birational map such that Y is smooth, $E = \pi^{-1}(\Sigma)$ is a divisor with normal crossings on Y and π maps $Y \smallsetminus E$ isomorphically to $X \smallsetminus \Sigma$. Then for all p

$$\mathrm{Gr}^p_F K^\cdot_{X,\Sigma} \simeq \mathbb{R}\pi_* J_E \Omega^p_Y(\log E) \qquad \text{in } D^+(X,\mathbb{C}).$$

PROOF. The morphism $\pi : (Y,E) \to (X,\Sigma)$ of pairs induces a morphism

$$\pi^* : K^\cdot_{X,\Sigma} \to \pi_*K^\cdot_{Y,E}$$

which is a filtered quasi-isomorphism by [4, Prop. 3.9].

(3.4) COROLLARY. *With notations as above, for any line bundle* L *on* X *one has*

$$\mathbb{H}^m(X, Gr_F^p K_X^{\cdot} \otimes L) \simeq H^{m-p}(Y, J_E \Omega_Y^p(\log E) \otimes \pi^* L)$$

and

$$\mathcal{H}^m(Gr_F^p K_{X,\Sigma}^{\cdot}) \simeq R^{m-p}\pi_* J_E \Omega_Y^p(\log E) \ .$$

§4. VANISHING THEOREMS.

(4.1) We first reformulate our Main Theorem for pairs (X,Σ):

THEOREM 1. *Let* X *be a complex projective variety of dimension* n, $\Sigma \subset X$ *a closed subvariety such that* $X \setminus \Sigma$ *is smooth,* L *an ample line bundle on* X. *Then*

a) $\mathbb{H}^m(X, Gr_F^p K_{X,\Sigma}^{\cdot} \otimes_{\mathcal{O}_X} L) = 0$ *for* $m > n$,

b) $\mathcal{H}^m(Gr_F^p K_{X,\Sigma}^{\cdot}) = 0$ *for* $m < p$ *or* $m > n$.

By Corollary (3.4) this follows from the more down-to-earth Theorem 2 as formulated in the introduction.

(4.2) We first deduce the Main Theorem from Theorem 1. Observe that for each p one has in $D^+(X,\mathbb{C})$ the exact triangle

$$(*) \qquad Gr_F^p K_{X,\Sigma}^{\cdot} \to Gr_F^p K_X^{\cdot} \to i_* Gr_F^p K_\Sigma^{\cdot} \to Gr_F^p K_{X,\Sigma}^{\cdot}[1]$$

where $i : \Sigma \to X$ is the inclusion. Hence the sequence

$$\mathcal{H}^m(Gr_F^p K_{X,\Sigma}^{\cdot}) \to \mathcal{H}^m(Gr_F^p K_X^{\cdot}) \to \mathcal{H}^m(i_* Gr_F^p K_\Sigma^{\cdot})$$

is exact for all m. If $m > n$, the first term is zero by statement b) from Theorem 1 and the last term is zero by induction on the dimension. Hence statement b) of the Main Theorem follows. Statement a) is proved by taking the tensor product of (*) with L and considering part of the long exact hypercohomology sequence in a similar way.

(4.3) Because Theorem 1 follows from Theorem 2, we only have to prove Theorem 2. We first consider statement a'):

$$H^q(\tilde{X},\Omega^p_{\tilde{X}}(\log E) \otimes \pi^*L^{-1}) = 0 \qquad \text{for } p+q<n\ .$$

PROOF. Put $\omega = \pi^*L$. We restrict ourselves to the case that L is very ample. Then there exists $Y \in |L|$ which is transverse to all mappings $E_{i_1} \cap \ldots \cap E_{i_p} \to X$ if $E = E_1 \cup \ldots \cup E_r$. This implies that $\tilde{Y} = \pi^{-1}(Y)$ is smooth and $\tilde{Y} \cup E$ is a divisor with normal crossings on $\tilde{X}$. Hence $D = \tilde{Y} \cap E$ is a divisor with normal crossings on $\tilde{Y}$, mapping to $S = \Sigma \cap Y$. Remark that $Y\setminus S$ is smooth, isomorphic to $\tilde{Y}\setminus D$ via π. We have exact sequences (cf. [11, p.43])

$$(1)\ 0 \to \Omega^p_{\tilde{X}}(\log E) \otimes \omega^{-1} \to \Omega^p_{\tilde{X}}(\log E) \to \Omega^p_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{Y}} \to 0$$

$$(2)\ 0 \to \Omega^{p-1}_{\tilde{Y}}(\log D) \otimes \omega^{-1} \to \Omega^p_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{Y}} \to \Omega^p_{\tilde{Y}}(\log D) \to 0$$

By [8, Thm. 2] the pair $(X-\Sigma, Y-S)$ is (n-1)-connected.

Hence the restriction mappings

$$H^k(X\setminus\Sigma,\mathbb{C}) \to H^k(Y\setminus S,\mathbb{C})$$

are isomorphisms for $k < n-1$ and injective for $k=n-1$. By [3] these mappings are morphisms of mixed Hodge structures. Taking Gr^p_F at both sides gives:
the mappings:

$$a_{pq}\ :\ H^q(\tilde{X},\Omega^p_{\tilde{X}}(\log E)) \to H^q(\tilde{Y},\Omega^p_{\tilde{Y}}(\log D))$$

are isomorphisms for $p+q < n-1$ and injective for $p+q = n-1$. Moreover $a_{pq} = b_{pq} \circ c_{pq}$ with

$$b_{pq}\ :\ H^q(\tilde{Y},\Omega^p_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{Y}}) \to H^q(\tilde{Y},\Omega^p_{\tilde{Y}}(\log D))$$

$$c_{pq}\ :\ H^q(\tilde{X},\Omega^p_{\tilde{X}}(\log E)) \to H^q(\tilde{Y},\Omega^p_{\tilde{X}}(\log E) \otimes \mathcal{O}_{\tilde{Y}})$$

obtained from the sequences (2) and (1) respectively. By induction we may assume that

$$H^q(\tilde{Y},\Omega_{\tilde{Y}}^{p-1}(\log D) \otimes \omega^{-1}) = 0 \qquad \text{for } p+q < n$$

so b_{pq} is an isomorphism for $p+q < n-1$ and injective for $p+q = n-1$. Hence the same holds for c_{pq}, which implies statement a').

(4.4) We now prove statement b):

$$R^q\pi_*J_E\ \Omega_{\tilde{X}}^p(\log E) = 0 \qquad \text{for } p+q > n\ .$$

Take Y as above and let $U = X \setminus Y$. It clearly suffices to show that for all such U

$$\Gamma(U,R^q\pi_*J_E\Omega_{\tilde{X}}^p(\log E)) = 0 \qquad \text{for } p+q > n\ .$$

Let $\tilde{U} = \pi^{-1}(U) = \tilde{X}\setminus\tilde{Y}$. Because U is affine, the Leray spectral sequence for $\pi : \tilde{U} \to U$ degenerates and we obtain the isomorphism

$$\Gamma(U,R^q\pi_*J_E\ \Omega_{\tilde{X}}^p(\log E)) \simeq H^q(\tilde{U},J_E\Omega_{\tilde{X}}^p(\log E)_{|\tilde{U}})\ .$$

We let $A^p = J_E\Omega_{\tilde{X}}^p(\log E+\tilde{Y})$, $B^p = J_E\Omega_{\tilde{X}}^p(\log E)$ and $C^p = J_D\Omega_{\tilde{Y}}^p(\log D)$.
We must show that $H^q(\tilde{U},B^p{}_{|\tilde{U}}) = 0$ for $p+q > n$. As $A^p{}_{|\tilde{U}} = B^p{}_{|\tilde{U}}$ this is equivalent to $H^q(\tilde{U},A^p{}_{|\tilde{U}}) = 0$ for $p+q > n$.
Because π induces an isomorphism from $\tilde{X}\setminus(\tilde{Y}\cup E)$ onto $X\setminus(Y\cup\Sigma)$, by [9] we obtain that

$$H^m(\tilde{X}\setminus\tilde{Y},E\setminus\tilde{Y}) \simeq H^m(X\setminus Y,\Sigma\setminus Y) \quad \text{for all } m\ .$$

Because $X\setminus Y$ and $\Sigma\setminus Y$ are affine this implies that

$$H^m(\tilde{X}\setminus\tilde{Y},\ E\setminus\tilde{Y}) = 0 \qquad \text{for } m > n\ .$$

Again considering Gr_F^p we obtain that

$$H^q(\tilde{X},A^p) = 0 \quad \text{for } p+q > n\ .$$

We have the exact sequence

$$H^q(\tilde{X},A^p) \to H^q(\tilde{U},A^p) \to H^{q+1}_{\tilde{Y}}(\tilde{X},A^p) \to H^{q+1}(\tilde{X},A^p)$$

so our claim will follow from the following lemma.

(4.5) LEMMA. $H^{q+1}_{\tilde{Y}}(\tilde{X},A^p) = 0$ *for* $p+q > n$.

PROOF. Taking residues along $\tilde{Y}$ gives an exact sequence

$$0 \to B^p \to A^p \to C^{p-1} \to 0 .$$

The connecting homomorphism gives a map

$$d_{pq} : H^q(\tilde{Y},C^{p-1}) \simeq H^q_{\tilde{Y}}(\tilde{X},C^{p-1}) \to H^{q+1}_{\tilde{Y}}(\tilde{X},B^p) .$$

Claim: $d_{p,q}$ is an isomorphism for $p+q > n+1$ and surjective for $p+q = n+1$. Clearly the lemma follows from this.
Proof of the claim: one has by [6,Thm. 2.8]

$$H^{q+1}_{\tilde{Y}}(\tilde{X},B^p) \simeq \varinjlim_k \mathrm{Ext}^{q+1}_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{k\tilde{Y}},B^p) .$$

As $\mathcal{O}_{k\tilde{Y}} \simeq \mathcal{O}_{\tilde{X}}/\omega^{-k}$ one obtains

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{k\tilde{Y}},B^p) = 0 \qquad \text{for } i \neq 1$$

$$\underline{\mathrm{Ext}}^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{k\tilde{Y}},B^p) \simeq B^p \otimes \omega^k/B^p .$$

Hence

$$H^{q+1}_{\tilde{Y}}(\tilde{X},B^p) \simeq \varinjlim_k H^q(\tilde{X},B^p \otimes \omega^k/B^p) .$$

As in (4.3) one has sequences

$$(3)_k \quad 0 \to C^{p-1} \otimes \omega^{k-1} \to B^p \otimes \omega^k \otimes \mathcal{O}_{\tilde{Y}} \to C^p \otimes \omega^k \to 0 .$$

By induction hypothesis $H^q(C^r \otimes \omega^s) = 0$ if $s \geq 1$, $q+r > n-1$. This implies that

$$H^q(B^p \otimes \omega^k \otimes \mathcal{O}_{\tilde{Y}}) = 0 \quad \text{if } p+q > n,\ k \geq 2 .$$

Hence $H^q(B^p \otimes \omega \otimes \mathcal{O}_{\tilde{Y}}) \xrightarrow{\sim} H^{q+1}_{\tilde{Y}}(\tilde{X},B^p)$. Moreover sequence $(3)_1$ gives a natural map

$$H^q(C^{p-1}) \to H^q(B^p \otimes \omega \otimes \mathcal{O}_{\tilde{Y}})$$

which is surjective if p+q = n+1 and an isomorphism for p+q > n+1. One easily sees that this map corresponds to $d_{p,q}$. Hence the claim follows and therewith our Main Theorem.

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