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INDEX THEOREM FOR
CONSTRUCTIBLE SHEAVES

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§0 - INTRODUCTION

0.1. Let X be a complex manifold of dimension n and let $\underline{\mathcal{M}}$ be a holonomic module over the ring $\mathcal{D}_X$ of differential operators on X . Then the Rham complex $\mathrm{DR}(\underline{\mathcal{M}})$ of $\underline{\mathcal{M}}$ has constructible sheaves as its cohomology groups, and its local index $\sum (-1)^i \dim H^i(\mathrm{DR}(\underline{\mathcal{M}}))_x$ at a point x can be expressed in terms of the characteristic cycle $\mathrm{Ch}(\underline{\mathcal{M}})$ of $\underline{\mathcal{M}}$ (Kashiwara [3], Brylinski-Dubson-Kashiwara [1]). Recently Dubson [2] found a beautiful formula to describe this.

THEOREM - If X is a compact complex manifold, we have

$$\sum (-1)^i \dim H^i(X; \mathrm{DR}(\underline{\mathcal{M}})) = (-1)^n \mathrm{Ch}(\underline{\mathcal{M}}) \cdot T_X^* X.$$

Here the last term means the intersection number of two n -cycles in T^*X .

0.2. The purpose of this lecture is to generalize his result to the real case.

Let X be a real analytic manifold of dimension n and F a constructible sheaf on X . First we shall define the characteristic cycle $\widetilde{\mathrm{SS}}(F)$ of F as a $\pi^{-1}\omega_X$ -valued n -cycle in T^*X . Here ω_X denotes the orientation sheaf of X and $\pi : T^*X \rightarrow X$ is the cotangent bundle to X . In order to define this, we use the micro-local theory of sheaves developed in Kashiwara-Schapira [4].

Secondly we prove the index theorem.

THEOREM - Let F be a constructible sheaf, and $\varphi : X \rightarrow \mathbb{R}$ a C^2 -function. Set $Y_\varphi = \{d\varphi(x) ; x \in X\} \subset T^*X$. We assume that $\{x \in \mathrm{supp} F ; \varphi(x) \leq t\}$ is compact for any t and that $\mathrm{SS}F \cap Y_\varphi$ is compact. Then $\dim H^j(X; F) < \infty$ for any j and we have

$$\sum (-1)^j \dim H^j(X; F) = (-1)^{n(n+1)/2} \widetilde{\mathrm{SS}}(F) \cdot Y_\varphi.$$

The proof uses the micro-local version of Morse's theory. Similarly to the Morse function, we deform φ a little in a generic position so that Y_φ intersects $\mathrm{SS}F$ transversally. Then we consider $H^j(\{x ; \varphi(x) < t\} ; F)$ and vary t . Then the cohomology groups change at points $t \in \varphi(\pi(Y_\varphi \cap \mathrm{SS}F))$, and the obstruction can be calculated locally and coincides with the intersection number of Y_φ

and $\widetilde{SS}(F)$ at $p \in SSF \cap Y_\varphi$ with $t = \varphi \pi(p)$.

§1 - SUBANALYTIC CHAINS

1.1. For a topological manifold X , let us denote by ω_X the orientation sheaf of X . If X is oriented then $\omega_X \cong \mathbb{Z}_X$ and this isomorphism changes the signature when we take the opposite orientation of X .

1.2. If X is a differentiable manifold of dimension n and if θ is a nowhere vanishing n -form on X , then we shall denote by $\text{sgn } \theta$ the section of ω_X given by the orientation that θ determines. Hence we have

$$(1.2.1) \quad \text{sgn } \varphi \theta = \text{sgn } \varphi \text{sgn } \theta$$

where $\text{sgn } \varphi = \pm 1$ if $\pm \varphi > 0$.

1.3. From now on, we assume that X is a real analytic manifold. For an integer r , let us denote by $E_r(X)$ the set of pairs (Y, s) of a subanalytic locally closed r -dimensional real analytic submanifold Y of X and a section s of ω_Y . We define the equivalence relation $\sim$ on $E_r(X)$ as follows : $(Y_1, s_1) \sim (Y_2, s_2)$ if and only if there exists a subanalytic locally closed r -dimensional real analytic submanifold $\bar{Y}$ such that $Y \subset Y_1 \cap Y_2$, $s_1|_Y = s_2|_Y$ and $\overline{\text{supp } s_1} = \overline{\text{supp } s_2} = \overline{\text{supp } s_1} \cap \bar{Y}$.

We denote by $C_r(X)$ the set of equivalence classes in $E_r(X)$ and an equivalence class is called *subanalytic r -chain*. Remark that its support is not assumed to be compact.

We can define the boundary operator

$$\partial : C_r(X) \longrightarrow C_{r-1}(X),$$

so that $\partial \partial = 0$.

1.4. One can see easily that $C_r : U \mapsto C_r(U)$ is a fine sheaf on X and we have the exact sequence

$$(1.4.1) \quad 0 \longrightarrow \omega_X \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0$$

This follows for example from the fact that any subanalytic set admits a subanalytic triangulation.

1.5. For a sheaf F on X , we set $C_r(F) = C_r \otimes F$. By (1.4.1), $\omega_X \otimes F$ is quasi-isomorphic to the complex of soft sheaves

$$(1.5.1) \quad C_n(F) \rightarrow C_{n-1}(F) \rightarrow \dots \rightarrow C_0(F).$$

We set

$$(1.5.2) \quad C_r(X; F) = \Gamma(X; C_r(F))$$

and call its elements F -valued subanalytic r -chains. We have isomorphisms

$$(1.5.3) \quad H_r^{\text{inf}}(X; F) \stackrel{\text{def}}{=} H_r(C_\bullet(X; F)) = H^{n-r}(X; F \otimes \omega_X).$$

$$(1.5.4) \quad H_r(X; F) \stackrel{\text{def}}{=} H_r(\Gamma_c(X; C_\bullet(F))) = H_c^{n-r}(X; F \otimes \omega_X).$$

1.6. Assume further that F is locally constant. For a subanalytic r -dimensional real analytic submanifold Y of X and for a section s of $F \otimes \omega_Y$ over Y , the pair (Y, s) determines an F -valued subanalytic r -chain.

1.7. The following criterion for a chain to be a cycle is evident.

LEMMA 1.1 - Let α be a subanalytic r -chain, $\varphi : X \rightarrow \mathbb{R}^r$ be a real analytic map. We assume that

- (i) $\text{Supp } \alpha \rightarrow \mathbb{R}^r$ is a finite map,
- (ii) $\text{Supp } \partial \alpha \rightarrow \mathbb{R}^r$ is an immersion,
- (iii) the intersection number of α and $\varphi^{-1}(t)$ is constant in $t \in \mathbb{R}^r \setminus \varphi(\text{Supp } \partial \alpha)$.

Then α is a cycle, i.e. $\partial \alpha = 0$.

§2 - SYMPLECTIC GEOMETRY

2.1. Let X be an n -dimensional real analytic manifold of dimension n and $\pi : T^*X \rightarrow X$ the cotangent bundle to X . Let θ_X denote the canonical 1-form on T^*X . Then $(d\theta_X)^n$ is nowhere vanishing and this gives the orientation of T^*X .

2.2. Now, let Y be a real analytic submanifold of X . Let T_Y^*X be the conormal bundle to Y . Then we have the canonical isomorphism

$$(2.2.1) \quad \omega_{T_Y^*X} \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_{T_Y^*X}.$$

Since the choice of signature is important in the future arguments, we shall write this explicitly. Let $(x_1, \dots, x_n)$ be a local coordinate system of X such that Y is given by $x_1 = \dots = x_r = 0$, and let $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ be the coordinates of T^*X such that $\theta_X = \sum \xi_j dx_j$. Then the section $(-1)^r \operatorname{sgn}(d\xi_1 \dots d\xi_r dx_{r+1} \dots dx_n) \otimes \operatorname{sgn}(dx_1 \dots dx_n)$ of $\omega_{T_Y^*X} \otimes \pi^{-1}\omega_X$ does not depend on the choice of coordinates and it determines the isomorphism (2.2.1).

2.3. Let Λ be a subanalytic conic locally closed Lagrangian subvariety of T^*X such that the projection $\Lambda \rightarrow X$ has a constant rank. Then we have $\omega_\Lambda \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_\Lambda$. In fact, locally, Λ is an open subset of T_Y^*X for a real analytic submanifold Y of X and we can apply 2.2. Therefore Λ defines the $\pi^{-1}\omega_X$ -valued n -chain in T^*X (see 1.6), which we shall denote by $[\Lambda]$.

§3 - CHARACTERISTIC CYCLE

3.1. Let us fix a commutative field k once for all, and vector spaces mean vector spaces over k . Let X be a real analytic manifold of dimension n . Let $D(X)$ be the derived category of the abelian category of sheaves of vector spaces on X .

An object F of $D(X)$ is called *constructible* if the following conditions are satisfied.

(3.1.1) $H^j(F) = 0$ except for finitely many j 's.

(3.1.2) There exists a subanalytic locally finite decomposition $X = \bigcup X_\alpha$ of X such that $H^j(F)|_{X_\alpha}$ is a locally constant sheaf of finite rank for any j and any α .

We denote by $D_C^b(X)$ the full subcategory of $D(X)$ consisting of constructible complexes.

3.2. For the notion of micro-support and its properties, we refer to [4]. We just mention the following properties.

For $F \in \text{Ob}(D^+(X))$, we can define the micro-support $\text{SS}(F)$ of F as a closed conic subset of T^*X .

PROPOSITION 3.1 - Let $F \in \text{Ob}(D^+(X))$, φ a $\mathbb{C}^1$ -function on X and let $t_1 \leq t_2$ be two real numbers. Assume that $\varphi|_{\text{Supp } F} : \text{Supp } F \rightarrow \mathbb{R}$ is proper and that $d\varphi(x) \notin \text{SS}(F)$ for any $x \in X$ with $t_1 \leq \varphi(x) < t_2$. Then the restriction homomorphism

$$H^j(\{x; \varphi(x) < t_2\}; F) \rightarrow H^j(\{x; \varphi(x) < t_1\}; F)$$

is an isomorphism for any j .

PROPOSITION 3.2. - If $F \in \text{Ob}(D_{\mathbb{C}}^b(X))$, then $\text{SS}(F)$ is a closed subanalytic Lagrangian subset of T^*X .

3.3. A morphism $u : F \rightarrow F'$ in $D^+(X)$ is called an isomorphism at $p \in T^*X$, if, for a distinguished triangle $F \xrightarrow{u} F' \rightarrow F'' \rightarrow F[1]$, we have $p \notin \text{SS}(F'')$. We denote by $D^+(X; p)$ the category obtained by localizing $D^+(X)$ by the isomorphisms at p (see [4]).

In particular, if φ is a $\mathbb{C}^1$ -function such that $d\varphi(\pi(p)) = p$, $\varphi(\pi(p)) = 0$, then $F \mapsto \mathbb{R}^{\Gamma_{\varphi^{-1}(\mathbb{R}^+)}}(F)_{\pi(p)}$ is a functor from $D^+(X; p)$. Here $\mathbb{R}^+$ signifies the set of non-negative numbers.

PROPOSITION 3.3 - Let $F \in \text{Ob}(D_{\mathbb{C}}^b(X))$ and Y a real analytic submanifold. If $\text{SS}(F) \subset T_Y^*X$ on a neighborhood of $p \in T_Y^*X$, then we have

$$F \cong \underline{V}_Y \quad \text{in } D^+(X; p)$$

where V is a bounded complex of finite-dimensional vector spaces and V_Y is the constant sheaf on Y with V as fiber.

3.4 Let F be an object of $D_{\mathbb{C}}^b(X)$. Then $\Lambda = \text{SS}(F)$ is a subanalytic Lagrangian subvariety. Hence there exists a locally finite family $\{\Lambda_{\alpha}\}$ of real analytic subsets of T^*X satisfying the following conditions.

(3.3.1) Λ_{α} is subanalytic and connected.

(3.3.2) There exists a real analytic submanifold Y_{α} of X such that Λ_{α} is an open subset of $T_{Y_{\alpha}}^*X$.

(3.3.3) $\Lambda \subset \bigcup_{\alpha} \Lambda_{\alpha}$.

(3.3.4) $\Lambda_{\alpha} \cap \Lambda_{\beta} = \emptyset$ if $\alpha \neq \beta$.

Then by proposition 3.3, for $p \in \Lambda_{\alpha}$ there exists a bounded

complex V_α of finite-dimensional vector spaces such that $F \cong \bigoplus_\alpha V_\alpha$ in $D^+(X; p)$. Then $\chi(V_\alpha) = \sum (-1)^j \dim H^j(V_\alpha)$ is locally constant in p and hence determined by α . We set $m_\alpha = \chi(V_\alpha)$.

DEFINITION 3.4 - We define the $\pi^{-1}\omega_X$ -valued n -chain $\widetilde{SS}(F)$ by

$$(3.3.5) \quad \widetilde{SS}(F) = \sum_\alpha m_\alpha [\Lambda_\alpha]$$

It is almost obvious that this chain does not depend on the choice of $\{\Lambda_\alpha\}$. We shall call this the *characteristic cycle* of F . Later we shall show that $\widetilde{SS}(F)$ is in fact an n -cycle.

§4. INDEX THEOREM

4.1. Let X be a real analytic manifold of dimension n . For a real valued C^2 -function φ on X we set

$$(4.1.1) \quad Y_\varphi = \{ d\varphi(x) ; x \in X \} \subset T^*X \quad \text{and}$$

$$(4.1.2) \quad Y_\varphi^a = \{ -d\varphi(x) ; x \in X \} \subset T^*X.$$

Then Y_φ and Y_φ^a are isomorphic to X and hence we can regard them as $\pi^{-1}\omega_X$ -valued n -cycles in T^*X .

4.2. Now, we state the following three main theorems, whose proof is given in the next three sections.

THEOREM 4.1 - For $F \in \text{Ob}(D_C^b(X))$, $\widetilde{SS}(F)$ is an n -cycle, i.e., $\partial \widetilde{SS}(F) = 0$.

THEOREM 4.2 - Let φ be a C^2 -function and $F \in \text{Ob}(D_C^b(X))$. We assume

$$(4.2.1) \quad \text{For any } t \in \mathbb{R}, \{x \in \text{Supp } F ; \varphi(x) \leq t\} \quad \text{is compact.}$$

$$(4.2.2) \quad Y_\varphi \cap \widetilde{SS}(F) \text{ is compact.}$$

Then, $\dim H^j(X; F) < \infty$ for any j and we have

$$\chi(X; F)_{\text{def}} \sum (-1)^j \dim H^j(X; F) = (-1)^{n(n+1)/2} \widetilde{SS}(F) \cdot Y_\varphi.$$

THEOREM 4.3 - Let φ and F be as in the preceding. We assume (4.2.1) and the following condition.

$$(4.2.3) \quad Y_\varphi^a \cap \widetilde{SS}(F) \text{ is compact.}$$

Then $\dim H_C^j(X; F) < \infty$ for any j and we have

$$\chi_C(X; F) \stackrel{\text{def}}{=} \sum (-1)^j \dim H_C^j(X; F) = (-1)^{n(n+1)/2} \widetilde{SS}(F) \cdot Y_\varphi^a.$$

Remark that Theorem 4.1, $\pi^{-1}(\omega_X) \otimes \pi^{-1}(\omega_X) \cong \mathbb{Z}_T^* X$ and the condition (4.2.2) or (4.2.3) permit us to define the intersection number $\widetilde{SS}(F) \cdot Y_\varphi$ or $\widetilde{SS}(F) \cdot Y_\varphi^a$.

§5 - PROOF OF MAIN THEOREMS (I)

5.1 We shall prove first the local version of Theorem 4.2 in a generic case. Let F be an object of $D_C^b(X)$, and we choose $\{\Lambda_\alpha\}$ and $\{Y_\alpha\}$ as in 3.4. Let x_0 be a point of X and φ a C^2 -function on X such that

$$(5.1.1) \quad \varphi(x_0) = 0,$$

$$(5.1.2) \quad d\varphi(x_0) \in \Lambda_\alpha \text{ and } Y_\varphi \text{ intersects transversally } \Lambda_\alpha \text{ at } p = d\varphi(x_0).$$

PROPOSITION 5.1 - Under these conditions we have

$$\chi(R\Gamma_{\varphi^{-1}(\mathbb{R}^+)}(F)_{x_0}) = (-1)^{n(n+1)/2} (\widetilde{SS}(F) \cdot Y_\varphi)_p.$$

Here the last term means the intersection number of $\widetilde{SS}(F)$ and Y_φ at $p = d\varphi(x_0)$.

PROOF - We shall take a local coordinate system $(x_1, \dots, x_n)$ of X such that Y_α is given by $x_1 = \dots = x_r = 0$ and $x_0 = 0$. Then we have

$$T_p(T_{Y_\alpha}^* X) = \{ (x, \xi) ; x_1 = \dots = x_r = \xi_{r+1} = \dots = \xi_n \}$$

and

$$T_p(Y_\varphi) = \{ (x, \xi) ; \xi_j = \sum_k \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(0) x_k \}.$$

The transversality condition (5.1.2) implies that the Hessian matrix $(\frac{\partial^2 \varphi}{\partial x_j \partial x_k}(0))_{r < j, k \leq n}$ is non-degenerate. Hence by Morse's lemma, after a change of local coordinates, we may assume that

$$\varphi|_{Y_\alpha} = \sum_{j>r} a_j x_j^2 \quad \text{for } a_j \in \mathbb{R} \setminus \{0\}.$$

Let V be a bounded complex of vector spaces such that $F \cong \underline{V}_{Y_\alpha}$ in $D^+(X;p)$. Then as stated in 3.3, we have

$$(5.1.3) \quad \mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_0} \cong \mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(\underline{V}_{Y_\alpha})_{x_0}.$$

Let us note the following lemma.

LEMMA 5.2 - Let $Q(x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$, q the number of negative eigenvalues of Q . Then for any vector spaces V , we have

$$\begin{aligned} H_{Q^{-1}(\mathbb{R}^+)}^j(\mathbb{R}^n; V_{\mathbb{R}^n}) \\ = H_{Q^{-1}(\mathbb{R}^+)}^j(V_{\mathbb{R}^n})_0 = \begin{cases} V & \text{for } j=q \\ 0 & \text{for } j \neq q \end{cases}. \end{aligned}$$

Hence we have, by denoting $q = \# \{j; a_j < 0\}$,

$$H^k(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_0}) \cong H_{\varphi^{-1}\mathbb{R}^+}^k(\underline{V}_{Y_\alpha})_{x_0} = H^{k-q}(V).$$

Therefore we obtain

$$(5.1.4) \quad \chi(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_0}) = (-1)^q \chi(V) = (-1)^q m_\alpha.$$

On the other hand, we have

$$(\widetilde{SS}(F) \cdot Y_\varphi)_p = m_\alpha ([T_{Y_\alpha}^* X] \cdot Y_\varphi)_p,$$

and we can easily verify

$$([T_{Y_\alpha}^* X] \cdot Y_\varphi)_p = (-1)^{n(n+1)/2 + q}$$

This completes the proof of Proposition 5.1.

Q.E.D.

5.2. Now we assume the condition (4.2.1) and the following conditions :

$$(5.2.1) \quad SSF \cap Y_\varphi \subset \bigcup_{\alpha} \Lambda_\alpha$$

$$(5.2.2) \quad SSF \text{ and } Y_\varphi \text{ intersect transversally.}$$

$$(5.2.3) \quad \#(SSF \cap Y_\varphi) < \infty$$

PROPOSITION 5.3 - Under these conditions we have $\dim H^k(X; F) < \infty$

and

$$\chi(X; F) = (-1)^{n(n+1)/2} \widetilde{SSF} \cdot Y_{\varphi} \quad .$$

PROOF - Set $\Omega_t = \{x; \varphi(x) < t\}$ and $Z_t = \{x; \varphi(x) \leq t\}$, and $\varphi \pi(Y_{\varphi} \cap SSF) = \{t_1, \dots, t_N\}$ with $t_1 < \dots < t_N$. We also set $t_0 = -\infty$, $t_{N+1} = \infty$, $\Omega_j = \Omega_{t_j}$ and $Z_j = Z_{t_j}$. Then by Proposition 3.1, we have

$$H^k(\Omega_{j+1}; F) \cong H^k(\Omega_t; F) \text{ for } t_{j+1} \geq t > t_j \text{ and } 0 \leq j \leq N.$$

Taking the inductive limit with respect to t we obtain

$$(5.2.4) \quad H^k(\Omega_{j+1}; F) \xrightarrow{\sim} H^k(Z_j; F)$$

Then by the following well-known lemma, we have

$$\dim H^k(\Omega_{j+1}; F) = \dim H^k(Z_j; F) < \infty$$

LEMMA - If K is a compact set and if U is an open neighborhood of K , then the image of $H^k(U; F) \rightarrow H^k(K; F)$ is finite-dimensional.

Since $\Omega_{N+1} = X$ and $Z_0 = \emptyset$, (5.2.4) implies

$$(5.2.5) \quad \chi(X; F) = \sum_{j=1}^N (\chi(Z_j; F) - \chi(\Omega_j; F)) \quad .$$

Now we have a distinguished triangle

$$\mathbb{R}\Gamma(Z_j \setminus \Omega_j; \mathbb{R}T_{X \setminus \Omega_j}(F)) \rightarrow \mathbb{R}\Gamma(Z_j; F) \rightarrow \mathbb{R}\Gamma(\Omega_j; F)$$

Hence we obtain

$$(5.2.6) \quad \chi(Z_j; F) - \chi(\Omega_j; F) = \chi(\mathbb{R}\Gamma(Z_j \setminus \Omega_j; \mathbb{R}T_{X \setminus \Omega_j}(F))) \quad .$$

By the definition of the micro-support, we have

$$\text{supp } \mathbb{R}\Gamma_{X \setminus \Omega_j}(F)|_{\varphi^{-1}(t_j)} \subset \pi(Y_{\varphi} \cap SSF) \quad .$$

Hence we obtain

$$(5.2.7) \quad \begin{aligned} \mathbb{R}\Gamma(Z_j \setminus \Omega_j; \mathbb{R}T_{X \setminus \Omega_j}(F)) &= \\ \bigoplus \mathbb{R}\Gamma_{X \setminus \Omega_j}(F)_x & \\ x \in \pi(Y_{\varphi} \cap SSF) \cap \varphi^{-1}(t_j) & \end{aligned}$$

The identities (5.2.5), (5.2.6) and (5.2.7) imply

$$\chi(X; F) = \sum_{\substack{x \in \pi(Y_\varphi \cap \text{SSF}) \\ \varphi(x) = t_j}} \chi(\mathbb{R}T_{X \setminus \Omega_j}(F)_x) .$$

Thus Proposition 5.3 follows from Proposition 5.1. Q.E.D.

§6 - PROOF OF MAIN THEOREMS (II)

6.1. We shall prove Theorem 4.1. We give only an outline of the proof.

Since $\widetilde{\text{SS}}(F \otimes k_{\{0\}}) = \widetilde{\text{SS}}(F) \times T_{\{0\}}^* \mathbb{R}$, it is sufficient to show that $\widetilde{\text{SS}}(F)$ is a cycle outside the zero section.

The support of $\beta = \partial \widetilde{\text{SS}}(F)$ is an $(n-1)$ -dimensional subanalytic subset contained in $\bigcup_{\alpha} \partial \Lambda_{\alpha}$. Taking a smooth point p of $\text{supp } \beta \setminus T_X^* X$, we shall derive the contradiction by the use of Lemma 1.1 and Proposition 5.3.

6.2. Let us take a local coordinate system $(x_1, \dots, x_n)$ of X such that $p = (0, \xi_0)$ and that the map $(x, \xi) \mapsto \xi$ from T^*X to $\mathbb{R}^n$ gives a local embedding from $\text{supp } \beta$ into $\mathbb{R}^n$ and a finite map from SSF into $\mathbb{R}^n$.

Set $\varphi(x, y) = \frac{1}{2} x^2 + xy$ and $\varphi_y(x) = \varphi(x, y)$.

Then we have

$$\text{SSF} \cap Y_{\varphi_y} \cap \{x; |x| = \varepsilon\} = \emptyset \text{ for } |y| \leq \varepsilon \text{ and } 0 < \varepsilon \ll 1 .$$

Therefore, if $|y| \ll \varepsilon$ and if Y_{φ_y} satisfies the conditions

(5.2.1) - (5.2.3), then we have, by Proposition 5.3

$$\chi(\{x; |x| < \varepsilon\}; F) = (-1)^{n(n+1)/2} \widetilde{\text{SS}}(F) \cdot Y_{\varphi_y} .$$

In particular, $\widetilde{\text{SS}}(F) \cdot Y_{\varphi_y}$ does not depend on y .

The relation $\xi = \text{grad}_x \varphi_y = x + y$ gives the projection $g : T^*X \rightarrow \mathbb{R}^n$ by $g(x, \xi) = \xi - x$. Since $g^{-1}(y) = Y_{\varphi_y}$, $g^{-1}(y) \cdot \widetilde{\text{SS}}(F)$ is constant in y .

Therefore we can apply Lemma 1.1 to see $\partial \widetilde{\text{SS}}(F) = 0$.

§7 - PROOF OF MAIN THEOREMS (III)

7.1. In order to prove Theorem 4.2, we shall note the following

LEMMA 7.1. (i) Let Λ be an n -dimensional subanalytic conic real analytic submanifold of T^*X . Then $\{\varphi; Y_\varphi \text{ and } \Lambda \text{ intersect transversally}\}$ is dense in the space $C^\infty(X)$ of C^∞ -functions on X with respect to the C^2 -topology.

(ii) Let Z be an $(n-1)$ -dimensional subanalytic conic subset of T^*X . Then $\{\varphi; Y_\varphi \cap Z = \emptyset\}$ is a dense subset of $C^\infty(X)$.

They can be shown by using Baire's category theorem similarly to the proof of the existence theorem of Morse's function.

Let φ and F satisfy the conditions in Theorem 4.2. Then there exists a function φ' close to φ which satisfies the conditions (5.2.1) - (5.2.3). Hence Proposition 5.3 can be applied to see $\chi(X; F) = (-1)^{n(n+1)/2} \widetilde{SS}(F) \cdot Y_{\varphi'}$.

Since Y_φ and $Y_{\varphi'}$ are homotopic, we have

$$\widetilde{SS}(F) \cdot Y_\varphi = \widetilde{SS}(F) \cdot Y_{\varphi'}$$

This shows Theorem 4.2.

7.2. Theorem 4.3 can be proven in a similar argument or by reducing to Theorem 4.2 by the use of the Poincaré duality and the following proposition, which can be shown easily.

PROPOSITION 7.2 - For $F \in \text{Ob}(D_C^b(X))$, we have

$$\widetilde{SS}(\mathcal{R}\underline{\text{Hom}}_X(F, k_X)) = a^*(\widetilde{SS}(F))$$

where a is the antipodal map of T^*X .

§8 - APPLICATIONS

8.1. The following theorem follows immediately from Theorem 4.2.

THEOREM 8.1 - Let X be a compact complex manifold, and $F \in \text{Ob}(D_C^b(X))$. Then

$$\chi(X; F) = (-1)^{n(n+1)/2} \widetilde{SS}(F) \cdot T_X^* X.$$

8.2. When X is a complex manifold and $\underline{m}$ is a holonomic module over the ring $\mathcal{D}_X$ of differential operators. Then $SS(DR(\underline{m}))$ coincides with the characteristic variety $\text{Ch}(\underline{m})$ of $\underline{m}$ and $\widetilde{SS}(DR(\underline{m}))$ coincides with the characteristic cycle $\underline{\text{Ch}}(\underline{m})$ of $\underline{m}$. Hence the results in this paper can be easily applied to holonomic modules.

8.3. Let φ be a real-valued real analytic function defined on X and $x_0 \in X$.

$$(8.3.1) \quad \varphi(x) > 0 \quad \text{for } x \in X \setminus \{x_0\}.$$

LEMMA 8.2. For any subanalytic closed conic Lagrangian set Λ , $d\varphi(x_0)$ is an isolated point of $\Lambda \cap Y_\varphi$.

PROOF - Otherwise there exists a real analytic path $x = x(t)$ such that $x(0) = x_0$; $x(t) \neq x_0$ for $t \neq 0$ and $d\varphi(x(t)) \in \Lambda$. Since Λ is Lagrangian, $\theta = d\varphi(x(t)) = 0$. Hence $\varphi(x(t))$ is a constant function, which is a contradiction. Q.E.D.

Along with this lemma, the following theorem follows immediately from Theorems 8.2 and 8.3.

THEOREM 8.3 - Let $F \in \text{Ob}(D_C^b(X))$ and let φ satisfy (8.3.1). Then we have

$$(8.3.1) \quad \chi(F_{x_0}) = (-1)^{n(n+1)/2} (\widetilde{SS}(F) \cdot Y_\varphi)_{x_0},$$

$$(8.3.2) \quad \chi(\mathbb{R}\Gamma_{\{x_0\}}(X; F)) = (-1)^{n(n+1)/2} (\widetilde{SS}(F) \cdot Y_\varphi^a)_{x_0}.$$

Here $(.)$ means the intersection number of two cycles at $x_0 \in T_X^* X \cong X \subset T^* X$.

8.4. A $\mathbb{Z}$ -valued function φ on X is called *constructible* if there exists a subanalytic stratification $X = \bigcup X_\alpha$ of X such that $\varphi|_{X_\alpha}$ is constant. We define the $\pi^{-1}\omega_X$ -valued n -cycle

$$(8.4.1) \quad c(\varphi) = \sum_\alpha \varphi(X_\alpha) \widetilde{SS}(Q_{X_\alpha}).$$

Then it is immediate that this does not depend on the choice of stratification.

Let us denote by $C(X)$ the space of $\mathbb{Z}$ -valued constructible functions on X . Let $K(D_C^b(X))$ be the additive group generated by $Ob(D_C^b(X))$ with the relation

$$[F] = [F'] + [F'']$$

for distinguished triangles $F' \rightarrow F \rightarrow F'' \rightarrow F'[1]$.

For $F \in Ob(D_C^b(X))$ we define the constructible function $\chi(F)$ by $X \ni x \mapsto \chi(F_x)$. Then this passes through the quotient and we obtain the commutative diagram

$$(8.4.2) \quad \begin{array}{ccc} K(D_C^b(X)) & \xrightarrow{\chi} & C(X) \\ \tilde{SS} \searrow & & \swarrow c \\ & Z_n(T^*X; \pi^{-1}\omega_X) & \end{array}$$

Here $Z_n(T^*X; \pi^{-1}\omega_X)$ denotes the space of $\pi^{-1}\omega_X$ -valued subanalytic n -cycles.

EXAMPLE 8.5.

(i) Let Y be a closed r -codimensional submanifold of X and χ_Y the characteristic function of Y . Then

$$c(\chi_Y) = [T_Y^*X]$$

(ii) Set $X = \mathbb{R}$, $Z_{\pm} = \{x; \pm x > 0\}$, $Z_0 = \{0\}$.

We define the 1-cycles $\alpha_{\pm}$ and $\beta_{\pm}$ by

$$\alpha_{\pm} = \{(x, \xi); \xi = 0, \pm x > 0\} \text{ with } \text{sgn } dx \otimes \text{sgn } dx,$$

$$\beta_{\pm} = \{(x, \xi); x = 0, \pm \xi > 0\} \text{ with } \text{sgn } d\xi \otimes \text{sgn } dx.$$

Then we have

$$c(\chi_{Z_+}) = \alpha_+ + \beta_-,$$

$$c(\chi_{Z_-}) = \alpha_- + \beta_+ \text{ and}$$

$$c(\chi_{Z_0}) = -\beta_+ - \beta_-.$$

(iii) Set $X = \mathbb{R}^n$, $q(x) = x_1^2 - x_2^2 - \dots - x_n^2$ ($n \geq 2$),

$$dx' = dx_2 \wedge \dots \wedge dx_n, \quad dx = dx_1 \wedge dx',$$

$$Z_{\pm} = \{x \in X; q(x) \geq 0, \pm x_1 \geq 0\},$$

$$Z_0 = \{x \in X; q(x) \leq 0\},$$

$$\text{and } U_{\epsilon} = \text{Int } Z_{\epsilon} \quad (\epsilon = \pm, 0).$$

We define the n -cycles in T^*X by

$$\alpha_{\epsilon} = \{(x, \xi); x \in U_{\epsilon}, \xi = 0\} \quad \text{with } \text{sgn } dx \otimes \text{sgn } dx,$$

$$\beta_{\epsilon} = \{(x, \xi); x = 0, \xi \in U_{\epsilon}\} \quad \text{with } \text{sgn } d\xi \otimes \text{sgn } dx,$$

$$\text{for } \epsilon = \pm, 0, \text{ and}$$

$$\gamma_{\epsilon_1, \epsilon_2} = \{(x, \xi); \epsilon_1 x_1 > 0, \epsilon_2 \xi_1 > 0, \xi_j/x_j = -\xi_1/x_1 \\ \text{for } j \geq 2, q(x) = 0\}$$

$$\text{with } \text{sgn}(d\xi_1 \wedge dx') \otimes \text{sgn } dx, \text{ for } \epsilon_1, \epsilon_2 = \pm 1.$$

Then we have

$$c(\chi_{Z_{\pm}}) = \alpha_{\pm} - \gamma_{\pm, \pm} + (-)^n \beta_{\pm},$$

$$c(\chi_{U_{\pm}}) = \alpha_{\pm} + \gamma_{\pm, \mp} + \beta_{\mp},$$

$$c(\chi_{Z_0}) = \alpha_0 - \gamma_{+, -} - \gamma_{-, +} - \beta_+ - \beta_- \quad \text{and}$$

$$c(\chi_{U_0}) = \alpha_0 + \gamma_{+, +} + \gamma_{-, -} - (-)^n \beta_+ - (-)^n \beta_-.$$

§9 - VARIATIONS OF MAIN THEOREMS

9.1. Let f be a real analytic function on X . We define, for $F \in \text{Ob}(D(X))$,

$$(9.1.1) \quad \mu_f(F) = \mathbb{R}\Gamma_{f^{-1}(\mathbb{R}^+)}(F)|_{f^{-1}(0)}.$$

Let $F \in \text{Ob}(D_c^b(X))$ and Ω an open subset of $f^{-1}(0)$.

We assume

$$(9.1.2) \quad \Omega \cap \text{supp } F \text{ is relatively compact.}$$

$$(9.1.3) \quad \text{SSF} \cap Y_f \cap \pi^{-1}(\partial\Omega) = \emptyset.$$

Then we have the following

THEOREM 9.1 - Under these conditions we have $\dim H^k(\Omega; \mu_f(F)) < \infty$

and

$$\chi(\Omega ; \mu_f(F)) = (-1)^{n(n+1)/2} (\widetilde{SSF} \cap \Omega) \cdot (Y_f \cap \Omega) \quad .$$

This theorem can be shown by deforming f to a generic position with respect to SSF .

9.2. Let F and F' be two objects of $D_C^b(X)$ and φ a C^1 -function on T^*X . We assume the following

(9.2.1) $\Omega = \{p \in T^*X ; \varphi(p) < 0\}$ is relatively compact in T^*X .

(9.2.2) $C_p(SS(F'), SS(F)) \not\supset H_{\varphi}(p)$ for any $p \in \varphi^{-1}(0)$.

Here C_p means the normal cone (see [4]), and H_{φ} means the Hamiltonian vector field of φ . We set

$$SS(F)^{\varepsilon} = e^{-\varepsilon H_{\varphi}(SSF)}$$

$$\text{and } \widetilde{SS}(F)^{\varepsilon} = e^{-\varepsilon H_{\varphi}(\widetilde{SSF})}$$

Then (8.6.2) implies for $0 < \varepsilon \ll 1$

$$(SS(F)^{\varepsilon} \cap \Omega) \cap (SS(F') \cap \Omega) = \emptyset \quad .$$

THEOREM 9.2 - Under these conditions we have

$$\dim H^k(\Omega ; \mu_{\text{hom}}(F, F')) < \infty$$

and

$$\chi(\Omega ; \mu_{\text{hom}}(F, F')) = (-1)^{n(n+1)/2} (\widetilde{SS}(F') \cap \Omega) \cdot (\widetilde{SS}(F)^{\varepsilon} \cap \Omega) \quad .$$

For the definition of μ_{hom} , we refer to [4] . This theorem can be shown by reducing to Theorem 9.1 with the aid of contact transformations.

If we assume instead of (9.2.2)

(9.2.3) $C_p(SS(F'), SSF) \not\supset H_{\varphi}(p)$ for any $p \in \varphi^{-1}(0)$.

Then we have

THEOREM 9.3 - Under (9.2.1) and (9.2.3) we have

$$\dim H_C^k(\Omega ; \mu_{\text{hom}}(F, F')) < \infty$$

and

$$\chi_C(\Omega ; \mu_{\text{hom}}(F, F')) = (-1)^{n(n+1)/2} (\widetilde{SS}(F') \cap \Omega) \cdot (\widetilde{SS}(F)^{-\varepsilon} \cap \Omega) \quad .$$

Remark that if we take as F the constant sheaf k_X , then we can recover Theorems 4.2 and 4.3.

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Corrections to "Microlocal study of Sheaves", M.Kashiwara, P.Schapira.
Astérisque 128, 1985.

- 1) p.48, 1.-6 ; p.85, 1.-8, -9 ; p.86, 1.-2 ; p.191, 1.-8, -5 :
read "... $\mathbb{L}_{\mathcal{O}}^! \underline{Z}_M^* X$ "
- 2) p.40, 1.-3 : p.47, 1.-9 :
read "... convex proper cone of..."
- 3) p.40, 1.-2 : read "... $\mathbb{R}\Gamma(\text{Int}(A^{\text{oa}}), \underline{F})$..."
- 4) p.47, 1.-6 : read "... $\cap \text{Int } Z^{\text{oa}}$..."
- 5) p.189, 1.4 : read "... is punctually endowed..."
- 6) p.119, 1. 4, 1.6 : read " $\alpha \geq 3$ ", " a C^2 -function"