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by

Alan ADOLPHSON*

1. INTRODUCTION.

Let p be a prime, $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{Z}_p$ the ring of p-adic integers. Every $x \in \mathbb{Z}_p$ may be written uniquely in the form

$$x = \sum_{i=0}^{\infty} x_i p^i \quad , \tag{1}$$

where the x_i are rational integers, $0 \leqslant x_i \leqslant p-1$. Define a function $\varphi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ by

$$\varphi(x) = \sum_{i=1}^{\infty} x_i p^{i-1} \quad .$$

For any positive integer n , let $\varphi^{(n)}$ denote the composition of φ with itself n times. In [1] we proved :

THEOREM 1. Let $F : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ be a continuous, non-vanishing function satisfying for all positive integers n

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(2) <u>If</u> $a \in \mathbb{Z}_p$, $\varphi^{(n)}(a) = a$, <u>then</u> $\prod_{i=0}^{n-1} F(\varphi^{(i)}(a)) = 1$.

<u>Then there exists a continuous, non-vanishing function</u> $G : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ <u>such that for all</u> $x \in \mathbb{Z}_p$,

(3) $$F(x) = G(x)/G(\varphi(x)) .$$

The purpose of this note is to show that if F is locally analytic, the G may be taken to be locally analytic also. Our method of proof may be used to give a simpler proof of theorem 1. More precisely, our construction of G from F produces a locally analytic function if F is locally analytic and produces a continuous function if F is continuous. We discuss some motivation behind this result in Section 3. For a fuller discussion of motivation, see [1] and [2].

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2. MAIN THEOREM.

Let Ω be a complete, algebraically closed field containing $\mathbb{Q}_p$. For $a \in \Omega$, ρ a positive real number, let

$$D(a,\rho^-) = \{x \in \Omega \mid |x-a| < \rho\} .$$

We shall use $W_\rho(\mathbb{Z})$ to denote the union of all disks $D(z,\rho^-)$, $z \in \mathbb{Z}$. Clearly, $W_\rho(\mathbb{Z})$ may be expressed as the disjoint union of finitely many of the indicated disks. We shall say that a function F on $W_\rho(\mathbb{Z})$ is locally analytic if F can be expressed as a convergent power series on each of these disks, i.e., for each $z \in \mathbb{Z}$,

$$F(x) = \sum_{n=0}^{\infty} a_n(z)(x-z)^n \qquad (a_n(z) \in \Omega)$$

for all $x \in D(z,\rho^-)$.

We extend φ to $W_1(\mathbb{Z})$ as follows. Given $z \in \mathbb{Z}$, there exists a unique $z_0 \in \mathbb{Z}$, $0 \leqslant z_0 \leqslant p-1$, such that $z \equiv z_0 \pmod{p}$. If $x \in W_1(\mathbb{Z})$, then $x \in D(z,1^-)$ for some $z \in \mathbb{Z}$ and we define

$$\varphi(x) = \frac{x-z_0}{p} . \tag{4}$$

Thus $\varphi : W_1(\mathbb{Z}) \longrightarrow W_p(\mathbb{Z})$. For $\rho \leqslant 1$, $W_\rho(\mathbb{Z}) \subseteq W_1(\mathbb{Z})$ so we may restrict φ to $W_\rho(\mathbb{Z})$. If $x \in W_\rho(\mathbb{Z})$, then $x \in D(z,\rho^-)$ for some $z \in \mathbb{Z}$, hence by (4),

$$\left| \varphi(x) - \frac{z-z_0}{p} \right| = \left| \frac{x-z}{p} \right|$$

i.e., $\varphi : D(z,\rho^-) \longrightarrow D((z-z_0)/\ D(z-z_0)/p,(p\rho)^-$. Thus for $\rho \leqq 1$,

$$\varphi : W_\rho(\mathbb{Z}) \longrightarrow W_{p\rho}(\mathbb{Z}) . \tag{5}$$

THEOREM 2. *Fix* $\rho \leqslant 1$ *and let* $F : W_\rho(\mathbb{Z}) \longrightarrow \Omega$ *be a non-vanishing, locally analytic function satisfying for all positive integers* n

$$\text{(6)}\quad \textit{If}\quad a \in \mathbb{Z}_p,\ \varphi^{(n)}(a) = a,\ \textit{then}\ \prod_{i=0}^{n-1} F(\varphi^{(i)}(a)) = 1 .$$

Then there exists a non-vanishing, locally analytic function $G : W_{p\rho}(\mathbb{Z}) \longrightarrow \Omega$ *such that for all* $x \in W_\rho(\mathbb{Z})$,

$$F(x) = G(x)\,/\,G(\varphi(x)) . \tag{7}$$

Remark. If $F : W_\rho(\mathbb{Z}) \longrightarrow \Omega$ is a locally analytic function that does not vanish on $\mathbb{Z}_p$, then there exists $\rho' > 0$ (but possibly $< \rho$) such that F is non-vanishing on $W_{\rho'}(\mathbb{Z})$. Thus the desired function G exists, but may only be defined and locally analytic on $W_{p\rho'}(\mathbb{Z})$.

Proof. For each rational integer α, $0 \leqslant \alpha \leqslant p-1$, we shall construct a locally analytic function $G_\alpha : W_{p\rho}(\mathbb{Z}) \longrightarrow \Omega$ which satisfies the conclusion of the theorem. Consider the map $\sigma_\alpha : W_{p\rho}(\mathbb{Z}) \longrightarrow W_\rho(\mathbb{Z})$ defined by

$$\sigma_\alpha(x) = \alpha + px .$$

For each α , σ_α is an analytic right inverse to φ , i.e., $\varphi \circ \sigma_\alpha$ is the identity on $W_{p\rho}(\mathbb{Z})$. Consider the infinite product

(8) $$G_\alpha(x) = \prod_{i=1}^{\infty} F(\sigma_\alpha^{(i)}(x))^{-1} .$$

We shall show that this infinite product converges uniformly on $W_{p\rho}(\mathbb{Z})$, hence $G_\alpha(x)$ is a non-vanishing, locally analytic function on $W_{p\rho}(\mathbb{Z})$. For this it suffices to show that $\{F(\sigma_\alpha^{(i)}(x))\}_{i=1}^{\infty}$ converges uniformly to the constant function 1 on $W_{p\rho}(\mathbb{Z})$.

One computes directly that

$$\sigma_\alpha^{(i)}(x) = \alpha + p\alpha + p^2\alpha + \ldots + p^{i-1}\alpha + p^i x ,$$

hence $\sigma_\alpha^{(i)}(x) - \alpha/(1-p) = p^i(x-\alpha/(1-p))$. In particular, we have for all $x \in W_{p\rho}(\mathbb{Z})$

(9) $$|\sigma_\alpha^{(i)}(x) - \frac{\alpha}{1-p}| \leq \frac{1}{p^i} \max\{1, p\rho\} .$$

But $\varphi(\alpha/(1-p)) = \alpha/(1-p)$, so we have by (6) that $F(\alpha/(1-p)) = 1$. It now follows from (9) and the continuity of F at $\alpha/(1-p))$ that $\{F(\sigma_\alpha^{(i)}(x))\}_{i=1}^{\infty}$ converges uniformly to the constant function 1 on $W_{p\rho}(\mathbb{Z})$.

It remains to show that (7) holds for $G = G_\alpha$. Let $x \in \alpha + pW_{p\rho}(\mathbb{Z})$. For such x , $\sigma_\alpha(\varphi(x)) = x$, i.e., σ_α is also a left inverse to φ on this set. Hence for $x \in \alpha + pW_{p\rho}(\mathbb{Z})$

$$G_\alpha(\varphi(x)) = \prod_{i=1}^{\infty} F(\sigma_\alpha^{(i-1)}(x))^{-1}$$

$$= F(x)^{-1} G_\alpha(x) ,$$

i.e., $F(x) = G_\alpha(x)/G_\alpha(\varphi(x))$. Note that $G_\alpha(\varphi(x))$ and hence $G_\alpha(x)/G_\alpha(\varphi(x))$ are locally analytic on $W_\rho(\mathbb{Z})$. Since $\alpha + p\mathbb{Z}_p \subseteq \alpha + pW_{p\rho}(\mathbb{Z})$, the equality $F(x) = G_\alpha(x)/G_\alpha(\varphi(x))$ for all $x \in W_\rho(\mathbb{Z})$ follows from :

LEMMA 1. <u>Let</u> $F_1, F_2 : W_\rho(\mathbb{Z}) \longrightarrow \Omega$ <u>be locally analytic functions, non-vanishing on</u> $\mathbb{Z}_p$, <u>that coincide on</u> $\alpha + p\mathbb{Z}_p$ <u>and satisfy</u> (6). <u>Then</u> $F_1(x) = F_2(x)$ <u>for all</u> $x \in W_\rho(\mathbb{Z})$.

Proof. For $n = 0,1,2,\ldots,$ put

$A_n = \{x \in \mathbb{Z}_p \mid$ The expansion (1) satisfies $x_i = \alpha$ for $i \geq n\}$.

We have clearly $\{\alpha/(1-p)\} = A_0 \subset A_1 \subset A_2 \subset \ldots$. Let $A = \bigcup_{n=0}^{\infty} A_n$. It suffices to show $F_1(x) = F_2(x)$ for all $x \in A$, because the set A has a limit point in each of the disks that make up $W_\rho(\mathbb{Z})$ and F_1 and F_2 are analytic on each of these disks. To show equality on A, it suffices to show that F_1 and F_2 coincide on each A_n. The proof is by induction on n.

When $n=0$, we have $F_1(\alpha/(1-p)) = 1 = F_2(\alpha/(1-p))$ by (6) since $\varphi(\alpha/(1-p)) = \alpha/(1-p)$. Supposing that they coincide on A_n, we show they agree on A_{n+1} also. Let $x \in A_{n+1}$. Then

$$x = x_0 + x_1 p + \ldots + x_n p^n + \alpha p^{n+1} + \alpha p^{n+2} + \ldots .$$

Define

$$x^{(i)} = (x_0 + \ldots + x_n p^n + \alpha p^{n+1} + \ldots + \alpha p^{n+i}) / (1-p^{n+i+1}) .$$

Then $\varphi^{(n+i+1)}(x^{(i)}) = x^{(i)}$, so by (6)

$$\prod_{j=0}^{n+i} F_1(\varphi^{(j)}(x^{(i)})) = 1 = \prod_{j=0}^{n+i} F_2(\varphi^{(j)}(x^{(i)})) .$$

But for $j = n+1, n+2, \ldots, n+i$, $\varphi^{(j)}(x^{(i)}) \in \alpha + p\mathbb{Z}_p$, so by hypothesis $F_1(\varphi^{(j)}(x^{(i)})) = F_2(\varphi^{(j)}(x^{(i)}))$. By the non-vanishing of F_1 and F_2 on $\mathbb{Z}_p$,

$$\prod_{j=0}^{n} F_1(\varphi^{(j)}(x^{(i)})) = \prod_{j=0}^{n} F_2(\varphi^{(j)}(x^{(i)})) . \tag{10}$$

For $j = 1,2,\ldots,n$, put

$$y^{(j)} = x_j + x_{j+1}p + \ldots + x_n p^{n-j} + \alpha p^{n-j+1} + \alpha p^{n-j+2} + \ldots .$$

Then $\lim_{i\to\infty} \varphi^{(j)}(x^{(i)}) = y^{(j)}$, $\lim_{i\to\infty} x^{(i)} = x$, so by (10) and the continuity of F_1, F_2

$$F_1(x) \prod_{j=1}^{n} F_1(y^{(j)}) = F_2(x) \prod_{j=1}^{n} F_2(y^{(j)}) .$$

But $y^{(j)} \in A_{n-j+1} \subseteq A_n$ for $j = 1,2,\ldots n$, so by the induction hypothesis and the non-vanishing of F_1 and F_2 on $\mathbb{Z}_p$ we conclude $F_1(x) = F_2(x)$. Q.E.D.

Since A is dense in $\mathbb{Z}_p$ one has the following version of lemma 1 for continuous functions :

LEMMA 1'. Let $F_1, F_2 : \mathbb{Z}_p \longrightarrow \Omega$ be continuous, non-vanishing functions that coincide on $\alpha + p\mathbb{Z}_p$ and satisfy (6). Then $F_1(x) = F_2(x)$ for all $x \in \mathbb{Z}_p$.

If F is any continuous, non-vanishing function on $\mathbb{Z}_p$ satisfying $F(\alpha/(1-p)) = 1$, the argument used in the proof of theorem 2 shows that the function $G_\alpha(x)$ defined by (8) is continuous and non-vanishing on $\mathbb{Z}_p$ and satisfies $F(x) = G_\alpha(x)/G_\alpha(\varphi(x))$ for $x \in \alpha + p\mathbb{Z}_p$. Theorem 1 then follows immediately from lemma 1'.

3. EXAMPLE.

For $s = 1,2,\ldots$, put $\rho_s = p^{e_s - 1}$, where $e_s = 1-p^{-s}(s+1+(p-1)^{-1})$, and put $\rho_\infty = 1$. Not that for $s = 1,2,\ldots,\infty$,

$$W_{\rho_s}(\mathbb{Z}) = \bigcup_{\alpha=0}^{p-1} D(\alpha, \rho_s^-) .$$

For each s, $s = 1,2,\ldots,\infty$, Baldassarri [2] contructs a non-vanishing locally analytic function $\Gamma_{D,s}$ on $W_{\rho_s}(\mathbb{Z})$. In particular $\Gamma_{D,1} = \Gamma_p$, Morita's p-adic gamma function. Baldassarri shows there is a constant $\gamma_s \in \Omega$, ord $\gamma_s = (p-1)^{-1}$, such that if we define for $x \in D(\alpha, \rho_s^-)$

$$\tilde{\Gamma}_{D,s}(x) = \gamma_s^{p-1-\alpha}\Gamma_{D,s}(x) ,$$

then

$$g_f(a,\pi) = \prod_{i=0}^{f-1} \tilde{\Gamma}_{D,s}(-\varphi^{(i)}(-a)) , \tag{11}$$

where $-a = j/(1-p^f)\,(j \in \mathbb{Z}\,,\, 0 \leqslant j \leqslant p^f-1)$ and $g_f(a,\pi)$ is the Gauss sum defined by [2, equation (0.1)] . Note that the set of rational numbers $-a$ just described is exactly the set of fixed points of $\varphi^{(f)}$.

If $s < s'$, then $W_{\rho_s}(\mathbb{Z}) \subseteq W_{\rho_{s'}}(\mathbb{Z})$. By (11), the ratio $F_{s,s'}(x) = \tilde{\Gamma}_{D,s}(x)/\tilde{\Gamma}_{D,s'}(x)$ is a non-vanishing, locally analytic function on $W_{\rho_s}(\mathbb{Z})$ satisfying for each non-negative integer f

(12) If $a \in \mathbb{Z}_p$, $\varphi^{(f)}(-a) = -a$, then $\prod_{i=0}^{f-1} F_{s,s'}(-{}^{(i)}(-a)) = 1$.

A simple change of variable in theorem 2 shows that there exists a non-vanishing, locally analytic function $G_{s,s'}$ on $W_{p\rho_s}(\mathbb{Z})$ such that for all $x \in W_{\rho_s}(\mathbb{Z})$,

$$\tilde{\Gamma}_{D,s}(x) = \tilde{\Gamma}_{D,s'}(x)G_{s,s'}(x) \,/\, G_{s,s'}(-\varphi(-x)) .$$

It should be possible to compute $G_{s,s'}$ explicitly.

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