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EXPANSION-COEFFICIENTS AS APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS

By

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INTRODUCTION.

One of the fundamental themes of Dwork's work is the study of the variation with parameters of the zeta function in a parameterized family of varieties, and of the p-adic cohomology which gives rise to them. A basis example of such a family is the Legendre family of elliptic curves

$$E : y^2 = x(x-1)(x-\lambda)$$

over the ring

$$R = \mathbb{Z}[1/2,\lambda][1/\lambda(1-\lambda)].$$

It was (implicitly) known to Gauss that the differential of the first kind $\omega = dx/y$, viewed as lying in $H^1_{DR}(E/R)$, is annihilated by the action of the hypergeometric differential operator with parameters $(1/2,1/2,1)$,

$$\mathcal{D} = \lambda(1-\lambda)(\frac{d}{d\lambda})^2 + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4},$$

acting on $H^1_{DR}(E/R)$ via the Gauss-Manin connection.

One of the early indications of the possible existence of a p-adic theory was Igusa's discovery ([3]) that in any characteristic $p \neq 2$, the Hasse-invariant $A_p(\lambda) \in \mathbb{F}_p[\lambda]$ of the <u>same</u> Legendre curve, considered in characteristic p, provided an $\mathbb{F}_p$-polynomial solution of the same differential equation :

$$\mathcal{D}(A_p(\lambda)) = 0 \quad \text{in} \quad \mathbb{F}_p[\lambda] \ .$$

One knows that the $A_p(\lambda)$ for variable p may all be obtained simultaneously from the expansion coefficients of ω at the origin. Explicitly, the quantity

$$t = 1/\sqrt{x}$$

provides a formal parameter at the origin for E/R, in terms of which the expansion of ω is given by

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

$$= \frac{-2\,dt}{\sqrt{(1-t^2)(1-\lambda t^2)}}$$

$$= -2 \sum_{n\geq 0} P_{2n+1}(\lambda) t^{2n+1} \frac{dt}{t} ,$$

with expansion coefficients

$$P_{2n+1}(\lambda) = (-1)^n \sum_{a+b=n} \binom{-1/2}{a}\binom{-1/2}{b} \lambda^b \ .$$

Thus we have

$$P_{2n+1}(\lambda) \in \mathbb{Z}[1/2][\lambda] \ , \ \deg P_{2n+1} = n \ .$$

The relation of the Hasse invariant $A_p(\lambda)$ to the P_n's is simply

$$A_p(\lambda) \equiv P_p(\lambda) \quad \text{mod } p \ .$$

From this point of view, Igusa's observation becomes the statement that

$$\nabla(\mathcal{D})(\omega) = 0 \quad \text{in} \quad H^1_{DR}(E/R)$$

$$\mathcal{D}(P_p) \equiv 0 \text{ mod } pR, \text{ for all } p \neq 2 \ .$$

By explicit computation of the P_N in this case, the reader can convince himself that one has the more general congruence-differential equation

$$\mathcal{D}(P_N) \equiv 0 \quad \text{mod } N.R$$

for every integer N .

The purpose of this note is to point out that this is a completely general phenomenon.

THE MAIN RESULT.

Let A be a noetherian ring, R a smooth A-algebra, and X a smooth R-scheme, purely of relative dimension $N \geq 1$. We suppose given

- a marked R-valued point $0 \in X(R)$;
- a set $(T_1,\ldots,T_N)$ of local coordinates on X at 0 ;
- a (possibly empty) finite set of R-smooth divisors $D_j \subset X$ which have normal crossings relative to R, and which satisfy the following condition : for each j, either D_j is disjoint from $0 \in X(R)$, or, in a small enough Zariski neighborhood of 0, D_j is defined by $T_i = 0$ (for some $i = i(j)$).

On X we dispose of the following complexes, in increasing order of generality :

$\Omega^{\bullet}_{X/R}$, the de Rham complex of X/R

$\Omega^{\bullet}_{X/R}(\log \cup D_j)$, the logarithmic de Rham complex of X/R with respect to the D_j's

and for any collection of integers n_j, one for each D_j, the complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j) \otimes (\bigotimes_j I^{-1}(D_j)^{\otimes n_j}) ,$$

where $I^{-1}(D_j)$ denotes the inverse ideal sheaf of D_j in X. The third of these includes the previous two as the specal case "no D_j's at all" and "all n_j's = 0".

We denote by

$$H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

the hypercohomology groups of X with coefficients in the complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j) \otimes (\otimes I^{-1}(D_j)^{\otimes n_j}) .$$

The Katz-Oda construction [4] of the Gauss-Manin connection carries over mutatis mutandis to this case (simply filter the X/A-

analogue,

$$\Omega^{\bullet}_{X/A}(\log \cup D_j) \otimes (\otimes\, I^{-1}(D_j)^{\otimes n_j}) \ ,$$

by the ideals generated by the pull-backs of the $\Omega^{i}_{R/A}$, and proceed as in [4]). Thus the cohomology groups

$$H^{p}_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

have the structure of R-modules (not necessarily of finite type unless X is assumed proper over R) endowed with the integrable Gauss-Manin connection ∇ relative to A. For any "P-D differential operator" (in the sense of [O]) $\mathcal{D}$ on R relative to A, we can thus speak of the A-linear endomorphism

$$\nabla(\mathcal{D}) \ : \ H^{p}_{DR}(X(\log \cup D_j + \sum n_j D_j)/ R) \circlearrowleft \ .$$

Let us denote by $\hat{X}$ the formal completion of X along O. In terms of the local coordinates $T_1,\dots,T_N$, we have an isomorphism of pointed formal R-schemes

$$\mathrm{Spf}(R[[T_1,\dots,T_N]]) \xleftarrow{\ \sim\ } \hat{X} \ .$$

Each divisor D_j is either disjoint from $\hat{X}$, or is defined in $\hat{X}$ by an equation $T_i = 0$. Therefore if we invert $T_1,\dots,T_N$, we obtain a natural R-linear morphism of complexes

$$\Gamma(X,\Omega^{\bullet}_{X/R}(\log \cup D_j + \sum n_j D_j)) \longrightarrow (\Omega^{\bullet}_{R[[T_1,\dots,T_N]]/R})[1/T_1,\dots,1/T_N] \ ,$$

which we call "formal expansion at O".

THEOREM. <u>Let</u> $\mathcal{D}$ <u>be a</u> P-D <u>differential operators on</u> R <u>relative to</u> A, $p \geq 1$ <u>an integer</u>, <u>and</u>

$$\omega \in H^{0}(X \ , \ \Omega^{p}_{X/R}(\log \cup D_j + \sum n_j D_j))$$

<u>a closed</u> p-<u>form on</u> X <u>with the allowed poles along the</u> D_j. <u>Let us denote by</u> $a(\underline{k},\underline{w}) \in R$ <u>the coefficients of the formal expansion at</u> O <u>of</u> ω :

$$\omega \sim \sum_{1 \leq k_1 < \dots < k_p \leq N} \ \sum_{w \in \mathbb{Z}^N} a(k,w) \prod_{i=1}^{N} T_i^{w_i} \prod_{v=1}^{p} \frac{dT_{k_v}}{T_{k_v}} \ .$$

<u>If the cohomology class</u> $\tilde{\omega}$ <u>of</u> ω <u>is killed by</u> $\mathcal{D}$, <u>i.e.</u>, <u>if</u>

$$\nabla(\mathcal{D})(\tilde{\omega}) = 0 \quad \text{in} \quad H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R) ,$$

<u>then each expansion coefficient</u> $a(\underline{k},\underline{w}) \in R$ <u>satisfies the congruence differential equation</u>

$$\mathcal{D}(a(\underline{k},\underline{w})) \equiv 0 \mod \sum_{v=1}^{p} w_{k_v} R .$$

<u>Proof</u> : By the functoriality of the Gauss-Manin connection we may replace X by an affine open neighborhood of 0 which is etale over $\mathbb{A}^N_R$ by $(T_1,\dots,T_N)$, and in which the D_j's, if there are any, are defined by $T_i = 0$ for various i . Increasing the number of D's , we may suppose we have $D_1,\dots,D_N$, with D_i defined by $T_i = 0$. Because X is etale over $\mathbb{A}^N_R$, any differential operator $\mathcal{D}$ on R has a <u>unique</u> extension $\tilde{\mathcal{D}}$ to X which on the subring $R[T_1,\dots,T_N]$ is given by

$$\tilde{\mathcal{D}}(\sum a(\underline{w})\underline{T}^{\underline{w}}) = \sum \mathcal{D}(a(\underline{w})) . \underline{T}^{\underline{w}} .$$

The $\mathcal{O}_X$-modules

$$\Omega^p_{X/R}(\log \cup D_j + \sum n_j D_j)$$

are $\mathcal{O}_X$-free, with basis

$$\frac{1}{\prod_{j=1}^{N} T_j^{n_j}} \frac{dT_{k_1}}{T_{k_1}} \wedge \dots \wedge \frac{dT_{k_p}}{T_{k_p}} , \qquad 1 \leq k_1 < \dots < k_p \leq N .$$

We extend $\tilde{\mathcal{D}}$ to the entire complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j + \sum n_j D_j)$$

by defining, for $f \in \mathcal{O}_X$,

$$\tilde{\mathcal{D}}(f . \frac{1}{\prod_j T_j^{n_j}} . \prod_v \frac{dT_{k_v}}{T_{k_v}}) = \tilde{\mathcal{D}}(f) . \frac{1}{\prod_j T_j^{n_j}} \prod_v \frac{dT_{k_v}}{T_{k_v}} .$$

It is transparent from the definitions that this action of $\tilde{\mathcal{D}}$ induces $\nabla(\mathcal{D})$ on the cohomology. Therefore, if ω has formal expansion

$$\omega \sim \sum_{\underline{k},\underline{w}} a(\underline{k},\underline{w}) \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}} ,$$

it is obvious by T-adic continuity that $\tilde{\mathcal{D}}(\omega)$ has formal expansion

$$\tilde{\mathcal{D}}(\omega) \sim \sum_{\underline{k},\underline{w}} \mathcal{D}(a(\underline{k},\underline{w})) \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}} .$$

This being the case, the hypothesis

$$\nabla(\mathcal{D})(\tilde{\omega}) = 0 \quad \text{in} \quad H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

guarantees that the formal differential form over R

$$\sum_{\underline{k},\underline{w}} \mathcal{D}(a(\underline{k},\underline{w})) \cdot \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}}$$

is formally exact. Writing it as the exterior derivative of a formal (p-1)-form over R and equating coefficients yield the asserted congruences on the $\mathcal{D}(a(\underline{k},\underline{w}))$. Q.E.D.

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