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P-ADIC THETA SERIES WITH INTEGRAL COEFFICIENTS

Valentino CRISTANTE

O. INTRODUCTION.

Let  $R$  be the ring of the integers of a local field  $K$ , let  $k$  be its residue field, and assume  $k$  be perfect of characteristic  $p \neq 0$ . If  $A$  is an abelian variety over  $K$  with good reduction mod  $p$ , we will denote by  $A_0$  its reduced variety, by  $e$  and  $e_0$  the identity of  $A$  and  $A_0$  respectively, by  $\theta_0$  the local ring of  $A$  at  $e_0$  and by  $S$  its completion. So, if  $A$  has dimension  $n$ ,  $S = R[[t_1, \dots, t_n]]$ , where  $(t_1, t_2, \dots, t_n)$  is a set of uniformizing parameters of  $A$  at  $e_0$ .

Now, if  $X$  is a divisor of  $A$ , rational over  $K$ , and if we denote by  $\theta_X$  a theta of in  $S_K = K[[t_1, \dots, t_n]]$  (we are assuming that the polar part of  $X$  doesn't go through  $e$ ), a natural question arises : is it possible to choose  $\theta_X$  in  $S$  ? The answer, in general, is no. In fact, if  $\theta_X \in S$ , the image of  $\theta_X$  in  $S_0 = \hat{S} \hat{\otimes} k$  would be a theta of the image  $X_0$  of  $X$  in  $A_0$ . But, as shown in [7], if  $A_0$  is not ordinary, or if  $X_0 \not\equiv 0$ , the thetas of  $X_0$  live in a ring quite bigger than  $S_0$ . So, if we are looking for a positive answer to our former question, we must assume  $A_0$  be ordinary. In fact, with this assumption, denoted by  $D = (D_1, \dots, D_n)$  a basis of the  $R$ -module of the invariant derivation of  $A$ , and by  $(\eta_{1,X}, \dots, \eta_{n,X})$  the  $n$ -uple of integrals of the second kind corresponding to the couple  $(X, D)$  (see section 3. for a precise definition), we will show that the system of differential equations

$$O.1. \quad D_i \theta - \theta \eta_{i,X} = 0, \quad i = 1, 2, \dots, n,$$

has solutions in  $S$ . However, we will not use a direct approach to O.1. In fact, if we denote by  $p_i$ ,  $i = 1, 2, 3$ , the projections from  $A^3$

to  $A$ , and if  $p_i^*$  are the corresponding applications from  $S$  to  $S \hat{\otimes} S \hat{\otimes} S$ , the system 0.1 is equivalent to the functional equation

$$0.2 \quad \frac{((p_1+p_2+p_3)^*\theta)(p_1^*\theta)(p_2^*\theta)(p_3^*\theta)}{((p_1+p_2)^*\theta)((p_1+p_3)^*\theta)((p_2+p_3)^*\theta)} = F,$$

where  $F$  is an equation of the divisor

$$Y = (p_1+p_2+p_3)^{-1}X + p_1^{-1}X + p_2^{-1}X + p_3^{-1}X - (p_1+p_2)^{-1}X - (p_1+p_3)^{-1}X - (p_2+p_3)^{-1}X$$

of  $A^3$ . Now, in view of the cohomological properties of  $F$ , the equation 0.2 is not only much more easier to solve than 0.1, but also allows to understand that 0.1 has solutions even in a more general situation.

After the construction of the solutions of 0.2, we will show how these are related to the canonical decomposition of  $H_{DR}^1(A)$  (see [9] and [4]), and finally we'll give some explicit computation for the elliptic curves.

An analogous construction has been done by P. Norman using different techniques ; here I'd like to thank him for the useful conversations we had on these topics.

## 1. SPLITTING OF BI-MULTIPLICATIVE CO-CYCLES.

Let  $R$  be a commutative ring with identity,  $t = (t_1, \dots, t_n)$  a set of indeterminates over  $R$ , and let  $S = R\langle t \rangle$  be a  $R$ -bi-algebra. For short, the image of  $t$  in  $S \hat{\otimes} S$  given by the coproduct will be denoted by  $t_1 \dot{+} t_2$ .

1.1. DEFINITION. An element  $H = H(t_1, t_2, t_3) \in S \hat{\otimes} S \hat{\otimes} S$  is called a symmetric, bi-multiplicative (resp. bi-additive) co-cycle of  $S$  if

- i)  $H(0, t_2, t_3) = 1$  (resp.  $H(0, t_2, t_3) = 0$ ) ;
- ii)  $H(t_1, t_2, t_3) = H(t_{\sigma_1}, t_{\sigma_2}, t_{\sigma_3})$ , for each permutation  $\sigma \in \mathcal{S}_3$  ;
- iii)  $H(t_1 \dot{+} t_2, t_3, t_4)H(t_1, t_2, t_4) = H(t_1, t_2 \dot{+} t_3, t_4)H(t_2, t_3, t_4)$  (resp.  $H(t_1 \dot{+} t_2, t_3, t_4) + H(t_1, t_2, t_4) = H(t_1, t_2 \dot{+} t_3, t_4) + H(t_2, t_3, t_4)$ ).

Moreover, if there exists an element  $h \in S$  such that

$$1.2 \quad \frac{h(t_1 \dot{+} t_2 \dot{+} t_3) h(t_1) h(t_2) h(t_3)}{h(t_1 \dot{+} t_2) h(t_1 \dot{+} t_3) h(t_2 \dot{+} t_3)} = H$$

(resp.  $h(t_1 \dot{+} t_2 \dot{+} t_3) + h(t_1) + \dots - h(t_2 \dot{+} t_3) = H$ ) the co-cycle  $H$  is called a co-boundary of  $S$ . Later on the left hand side of 1.2 will be denoted by  $\mathfrak{D}_\mu^2 h$  (resp.  $\mathfrak{D}_\alpha^2 h$ ).

For instance, if  $R$  and  $t$  have the same meaning as in the introduction, and if  $X$  is a divisor of  $A$  such that its reduced mod  $p$  doesn't go through  $e_0$ , one can choose for  $F$  (cfr. 02) a symmetric, bi-mult. co-cycle of  $S$  (see [7] and [5]). So our main goal in this section will be the proof of the following.

1.3. THEOREM. Let  $R$  be the ring of the Witt vectors with components in the algebraically closed field  $k$  of characteristic  $p \neq 0$ , and  $S = R[t]$  be a multiplicative bi-algebra. Then each symmetric, bi-multiplicative co-cycle of  $S$  is a co-boundary of  $S$ .

The assumption about the algebraic closure of  $k$  seems necessary if we like results which can be applied to each divisor. Later on we will show that symmetric divisor possess theta series with integral coefficients even if  $k$  is only a perfect field.

In order to prove 1.3 we need some results which are given in theorem A.4 and section 2 of [5]. With our actual language they can be formulated in the following way :

1.4. THEOREM. If  $R$  is a  $\mathbb{Q}$ -algebra, each symmetric, bi-multiplicative (resp. bi-additive) co-cycle  $H$  of  $S$  is a co-boundary of  $S$ .

1.5. THEOREM. If  $R$  is an algebraically closed field of characteristic  $p \neq 0$ , and if  $S$  is a multiplicative bi-algebra, then each bi-multiplicative co-cycle  $H$  of  $S$  is a co-boundary of  $S$ .

If  $R$  is algebraically closed field of characteristic 0, and if  $A$  is an abelian variety over  $R$ , result 1.4, under the assumption that  $H$  be a rational function on  $A^3$ , was first proved in [2].

Since the symmetric, bi-additive co-cycles are more easy to use, we start with the following result :

1.6. PROPOSITION. Let S be as in 1.3 ; then each symmetric bi-additive co-cycle of  $S_0 = S \hat{\otimes} k$  is a co-boundary of  $S_0$ .

In fact, as the following arguments will show, from 1.6 we deduce the following

1.7. PROPOSITION. Let S be as in 1.3 ; then each symmetric, bi-additive co-cycle of S is a co-boundary of S.

An finally, from 1.7 we can get 1.3.

Proof of (1.6  $\implies$  1.7). Let  $H \in S \hat{\otimes} S \hat{\otimes} S$  be a symmetric, bi-additive co-cycle of S. Denote by  $H_0$  the image of H in  $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$  ; now as  $H_0$  is a co-boundary of  $S_0$ , there exists an element  $h_0 \in S_0$ , such that  $\mathcal{D}_\alpha^2 h_0 = H_0$ . If h is an element of S whose image in  $S_0$  is  $h_0$ , and if  $H_1 = \mathcal{D}_\alpha^2 h$ , we have  $H \equiv H_1 \pmod{p}$  ; and then, since  $\frac{1}{p} (H - H_1)$  is a symmetric, bi-additive co-cycle of S, our procedure may be repeated. As a consequence,

$$H = H_1 + pH_2 + p^2H_3 + \dots ,$$

is a co-boundary of S, Q.E.D. .

Proof of (1.7  $\implies$  1.3). Let  $F \in S \hat{\otimes} S \hat{\otimes} S$  be a symmetric, bi-multiplicative co-cycle of S, and denote by  $F_0$  its canonical image in  $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$ . By 1.4 we know that  $F_0$  is a co-boundary of  $S_0$ , so there exists  $\theta_0$  in  $S_0$ , s.t.  $\mathcal{D}_\mu^2 \theta_0 = F_0$ . Now, denote by  $S^+$  the kernel of the coidentity of S, and let  $\theta'$  be an element of S,  $\theta' \equiv 1 \pmod{S^+}$ , whose image in  $S_0$  is  $\theta_0$ . If we denote by  $F_1$  the co-boundary  $\mathcal{D}_\mu^2 \theta'$  of S, we have

$$F/F_1 \equiv \pmod{p} ,$$

and therefore

$$\log F = \log F_1 + pH ,$$

where  $H$  is a symmetric, bi-additive co-cycle of  $S$ . Now, by 1.7 there exists an element  $h \in S$ , s.t.  $\mathfrak{D}_\alpha^2 h = H$ ; and it is clear that  $\theta = \theta' \exp ph$  is an element of  $S$  which satisfies the equation  $\mathfrak{D}_\mu^2 \theta = F$ ,  
 Q.E.D. .

Now will give a lemma which will be used the proof of 1.6.

1.8. LEMMA. Let  $B$  be an integral domain of characteristic  $p \neq 0$ ,  $B[t_1, \dots, t_n]$  a multiplicative bi-algebra; then each symmetric additive co-cycle of  $B[t]$  is a co-boundary.

Proof. This result is probably well known; nevertheless we'll give here a direct proof. Let  $g$  be such a co-cycle. Using the co-cycle property  $g(t_1 \dot{+} t_2, t_3) + g(t_1, t_2) = g(t_1, t_2 \dot{+} t_3) + g(t_2, t_3)$ , it is immediate to see that  $(p \smile \hat{\circ} p \smile)g$  is a co-boundary ( $p \smile =$  multiplication by  $p$ ), i.e. there exists an element  $\tau \in B[t]$  such that

$$\tau(t_1 \dot{+} t_2) - \tau(t_1) - \tau(t_2) = g(p \smile t_1, p \smile t_2).$$

From the last formula we deduce that

$$1.9. \quad D\tau - \varepsilon(D\tau) = 0$$

for each invariant derivation  $D$  of  $B[t]$ , where  $\varepsilon$  is the co-identity. But, as  $B[t]$  is multiplicative, 1.9 implies that  $D\tau = 0$ , and so  $\tau = p \smile \sigma$ , for  $\sigma \in B[t]$ . In conclusion  $\sigma(t_1 \dot{+} t_2) - \sigma(t_1) - \sigma(t_2) = g(t_1, t_2)$ ,

Q.E.D. .

Proof of 1.6. Let  $H \in S_0 \hat{\circ} S_0 \hat{\circ} S_0$  be a symmetric bi-additive co-cycle of  $S_0$ ; then by 1.8 there exists a (unique) element  $\varphi$  in  $S_0 \hat{\circ} S_0$ , such that

$$1.10. \quad \varphi(t_1, t_2 + t_3) - \varphi(t_1, t_2) - \varphi(t_1, t_3) = H.$$

Now, if  $\mu \in S_0 \hat{\circ} S_0 \hat{\circ} S_0$  is the element defined by

$$\mu(t_1, t_2, t_3) = \varphi(t_1, t_2 + t_3) + \varphi(t_2, t_3) - \varphi(t_1 + t_2, t_3) - \varphi(t_1, t_2),$$

as a consequence of the co-cycle properties of  $H$  (see def. 1.1) we have

$$\mu(t_1, t_2, t_3 + t_4) = \mu(t_1, t_2, t_3) + \mu(t_1, t_2, t_4) .$$

But, since  $S_0 \hat{\otimes} k[t_1, t_2]$  is a multiplicative  $k[t_1, t_2]$ -bi-algebra, from the last formula we deduce that  $\mu = 0$ . As a consequence, recalling also point ii) of 1.1, we conclude that  $\varphi$  is a symmetric, additive co-cycle of  $S_0$ ; so using 1.8 again we have

$$\varphi(t_1, t_2) = \tau(t_1 + t_2) - \tau(t_1) - \tau(t_2) ,$$

and finally  $\mathcal{D}_\alpha^2 \tau = H$ , Q.E.D..

1.11. Remark. Let  $S$  as in 1.3, and  $u = (u_1, \dots, u_n)$  be a basis of the integrals of the first kind of  $S$ , i.e. a basis of the  $R$ -module of the additive elements  $u$  of  $S_K = S \hat{\otimes} K$  such that  $Du$  is in  $S$  for each invariant derivation  $D$  of  $S$ . If  $F$  is a symmetric, bi-multiplicative co-cycle of  $S$ , and if  $\theta \in S$  is a solution of the equation

$$1.12. \quad \mathcal{D}_\mu^2 \theta = F ,$$

each solution of 1.12 in  $S_K$  is of the form  $\theta \exp(L(u) + Q(u))$ , where  $L(u)$  and  $Q(u)$  are linear and respectively quadratic forms of the  $u_i$ 's. Now, since in  $S$  there is no element of the form  $\exp Q(u)$  (see [MA]), we conclude that all solutions of 1.12 in  $S$  are of the form  $\theta \exp L(u)$ .

Now we'll show that, if 1.12 admits a even solution, i.e. invariant with respect to the inversion of  $S_K$ , it is sufficient to assume  $k$  perfect; more precisely we have the following.

1.13. THEOREM. Let  $k$  be a perfect field of characteristic  $p \neq 0, 2$ , let  $R$  be the ring of the Witt vectors with components in  $k$ , and  $S = R[t_1, \dots, t_n]$  be a bi-algebra of multiplicative type. Then if  $F$  is a symmetric, bi-multiplicative co-cycle of  $S$ , such that  $F(t_1, t_2, t_3) = F({}^{\sim}t_1, {}^{\sim}t_2, {}^{\sim}t_3)$  ( ${}^{\sim}t$  is the image of  $t$  given by the inversion of  $S$ ), the equation 1.12 has a unique solution  $\tilde{\theta} \in S$  which satisfies the relation

$$1.14. \quad \tilde{\theta}(t) = \tilde{\theta}(-t) .$$

Moreover if  $\theta \in S_K$  ( $K = \text{Frac} R$ ) is a solution of 1.12 which satisfies 1.14 we have

$$1.15. \quad \hat{\theta} = \lim_{n \rightarrow \infty} \theta / (p\iota)^{-n} p^{2n};$$

finally, the direct relation between  $\hat{\theta}$  and  $F$  is the following :

$$1.16. \quad \hat{\theta} = \lim_{n \rightarrow \infty} (p\iota)^{-n} \left( 1 / \prod_{j=1}^{p^n-1} F(t, \iota^j t) \right) p^{n-j}$$

where the limits 1.15 and 1.16 are considered in the topology of  $\varinjlim (S \xrightarrow{p\iota} S \xrightarrow{p\iota} \dots)$  given by the system  $I_{m,n} = t^m S + p^n S$  of ideals of  $S$ .

Proof. Let  $\bar{R}$  be the ring  $W(\bar{k})$  of the Witt vectors with components in the algebraic closure  $\bar{k}$  of  $k$ . By 1.3 we know that there exists a solution  $\theta(t)$  of 1.12 in  $\bar{R}[[t]]$ ; but in view of the properties of  $F$ , also  $\theta(-t)$  is a solution of 1.12, and so  $\theta(t)^{1/2} \theta(-t)^{1/2}$  is a solution of 1.12 which satisfies 1.14. Now, each element of  $\bar{R}[[t]]$  which satisfies 1.14 is a even power series of  $u$  (see 1.11); therefore it can't be multiplied by an exponential of a linear form  $L(u)$  without loosing the property 1.14. As a consequence 1.12 has a unique solution  $\hat{\theta}$  in  $\bar{R}[[t]]$  such that  $\hat{\theta}(t) = \hat{\theta}(-t)$ . Now we'll show that  $\hat{\theta}$  is in  $R[[t]]$ . In fact by 1.1, if  $\theta$  is a solution of 1.12 in  $S_K$  which satisfies 1.14, we have

$$1.17. \quad \theta(t)^{p^{2n}} / (p\iota)^n \theta(t) = \prod_{j=1}^{p^n-1} F(t, \iota^j t) p^{n-j},$$

for each  $n \geq 1$ . So the remaining part of the theorem will be proved if we verify that

$$1.18. \quad \hat{\theta}(t) = \lim_{n \rightarrow \infty} \theta(t) / (p\iota)^{-n} \theta(t) p^{2n}.$$

Now the relation between  $\theta$  and  $\hat{\theta}$  must be  $\theta = \hat{\theta} \exp Q(u)$ , where  $Q(u)$  is a quadratic form.

But  $\lim_{n \rightarrow \infty} (p\iota)^{-n} \hat{\theta} p^{2n} = 1$ , and  $\lim_{n \rightarrow \infty} (p\iota)^{-n} (\exp Q(u)) p^{2n} = \exp W(u)$ , Q.E.D..

1.19. Remark. With the notation of 1.17 also the limit

$$1.20. \quad \lim_{n \rightarrow \infty} \prod_{j=1}^{p^n-1} F(t, \iota^j t) (p^{n-1}) / p^{2n}$$



exists in  $S_K$  : it gives the (unique) solution  $\theta_0$  of 1.12 in  $S_K$  which satisfies 1.14 and the initial condition

$$\epsilon(D'D \log \theta) = 0 ,$$

for each couple  $(D, D')$  of invariant derivations of  $S_K$ . In fact, in order to show that 1.20 exists, we remark that by 1.14,  $\theta = 1 + Q(u) + \dots$ , where  $Q$  is a quadratic form ; as a consequence

$$\lim_{n \rightarrow \infty} \frac{1}{p^{2n}} \log(p\iota)^{n\theta} = Q(u) ,$$

and finally

$$\theta_0 = \theta / \exp Q(u) .$$

This is the procedure used in [10] ; but in general  $\theta_0$  isn't in  $S$ .

1.15. Remark. Let  $\mathcal{Y}$  be the completion of the perfect closure of  $S_0 = S \hat{\otimes} k$  and  $\text{Biv}(\mathcal{Y})$  the completion of the ring of Witt bivectors with components in  $\mathcal{Y}$ . Using the methods described in [12] (see in particular th. 8.1) one can define a canonical embedding  $j$  of a subring of  $S_K$ , containing all solutions of 1.12, in  $\text{Biv}(\mathcal{Y})$ . In such situation  $\mathcal{Y}$  is characterized by the property  $j_{\theta}^{\sim} \in W(\mathcal{Y})$ . Since 1.12 has solutions with this peculiarity also when  $S_0$  is an affine algebra of a general B-T group, it would be interesting to describe the functions (series) which correspond to them.

## 2. THETA SERIES.

In this section we'll translate the previous results in a geometric language.

2.1. THEOREM. If  $k, K$  and  $R$  have the same meaning as in 1.13, if  $A$  is an abelian variety over  $K$  with good reduction mod  $p$ , and if the reduced variety  $A_0$  is ordinary ; then each divisor  $X$  of  $A$ , rational over  $K$ , has a theta series in  $\bar{R}((t))$ , where  $\bar{R}$  is the ring of the Witt vectors of the algebraic closure  $\bar{k}$  of  $k$ , and  $t = (t_1, \dots, t_n)$  is a set of uniformizing parameters of  $A$  at the identity point  $e_0$  of  $A_0$ . Moreover if  $X$  is totally symmetric, i.e. if there exists  $X'$ , s.t.

$X = X' + (-\iota)^{-1}X'$ ,  $X$  possesses a theta series in  $R((t))$ ,  $\tilde{\theta}_X$  which satisfies the relation  $\tilde{\theta}_X(t) = \tilde{\theta}_X(-t)$ . The series  $\tilde{\theta}$  is determined up to a constant.

Proof. We'll assume that the support of the reduced divisor  $X_0$  doesn't intersect  $e_0$ ; in fact the result in general is an immediate consequence of this particular situation (see remark 2.2). If  $Y$  has the same meaning as in the introduction, as remarked in section 1, we can choose as an equation of  $Y$  a symmetric, bi-multiplicative co-cycle  $F$  of  $S$ . At this point it is clear that the first part of the theorem is a consequence of 1.3. Now if  $X$  is totally symmetric,  $F(\dot{t}_1, \dot{t}_2, \dot{t}_3)$  is, as  $F$ , an equation of  $Y$ , which satisfies i) of 0.1, and so  $F(\dot{t}_1, \dot{t}_2, \dot{t}_3) = F(t_1, t_2, t_3)$ . As a consequence, the second part our theorem follows immediately by the first part of 1.13, Q.E.D..

2.2. Remark. The assumption  $\text{supp } X_0 \cap E_0 = \emptyset$  used in the proof of 2.1 is not necessary. In fact each divisor  $X'$  of  $A$  rational over  $K$  can be written as  $X' = X'' + (f)$ , where  $X''$  satisfies the assumption and  $f$  is in  $R((t))$ . In this case we define  $\theta_{X'} = \theta_{X''} f$ . In particular, if  $X = X' + (-\iota)^{-1}X'$ ,  $\tilde{\theta}_X(t) = \tilde{\theta}_{X'' + (-\iota)^{-1}X''}(t) f(t)f(\dot{t})$  is determined up to a multiplicative constant. Finally, if the polar part  $X'''$  of  $X$  satisfies the previous assumption, i.e.  $\text{supp } X''' \cap e_0 = \emptyset$ ,  $\bar{R}((t))$  and  $R((t))$  can be replaced by  $\bar{R}[\dot{t}]$  and  $R[\dot{t}]$  respectively.

### 3. THE CANONICAL SPLITTING OF $H_{DR}^1(A)$ ASSOCIATED TO $\tilde{\theta}$ .

With the notations and assumptions of 2., we recall that to  $A$  and  $S$  are associated the free  $R$ -modules  $H_{DR}^1(A)$  and  $H_{DR}^1(S)$  of rank  $2n$  and  $n$  respectively. For our purposes, the more convenient description of them is the following (see [3] and [4]) :

we start with two sub- $R$ -modules of  $S_K = K[\dot{t}]$  : the first is

$$I_2(A) = \{f \in S_K \mid df \text{ is a diff. of } S, \text{ and } f(t_1 \dot{t}_2) - f(t_1) - f(t_2) \in K(A^2)\};$$

the second is

$$I_2(S) = \{f \in S_K \mid df \text{ is a diff. of } S, \text{ and } f(t_1 + t_2) - f(t_1) - f(t_2) \in S \hat{\otimes} S\}.$$

Clearly  $I_2(A)$  contains the local ring  $\mathcal{O}_0$  of  $A$  at  $e_0$ , and  $I_2(S)$  contains  $S$ . With these notations, we have :

$$H_{DR}^1(A) = I_2(A)/\mathcal{O}_0 \quad \text{and} \quad H_{DR}^1(S) = I_2(S)/S.$$

Now let  $I_1$  be the sub-R-module of  $I_2(A)$  (and of  $I_2(S)$ ) given by the additive elements :

$$I_1 = \{f \in I_2(A) \mid f(t_1 + t_2) - f(t_1) - f(t_2) = 0\}.$$

It is well known that  $I_1$  is a free R-module of rank  $n$ , and that  $I_1 \cap S = \{0\}$ . Therefore, by a comparison of the ranks, we conclude that the canonical map of  $I_2(A)$  in  $H_{DR}^1(S)$  is surjective, that  $I_2(A) = I_1 \oplus (I_2(A) \cap S)$ , and finally that

$$3.1. \quad H_{DR}^1(A) = I_1 \oplus (I_2(A) \cap S) / \mathcal{O}_0 :$$

this is the canonical splitting of  $H_{DR}^1(A)$ . Now we'll show how the sub-R-module  $N = (I_2(A) \cap S) / \mathcal{O}_0$  of  $H_{DR}^1(A)$  is related to the theta series.

3.2. THEOREM. Let  $A$  be an abelian variety as in 2.1 ; let  $X > 0$  be a totally symmetric, ample divisor of  $A$  rational over  $K$ , and and  $\tilde{\theta}$  (one of) its theta series in  $S$ . If  $\text{Lie}(S)$  denotes the R-module (dual of  $I_1$ ) of the invariant derivations of  $S$ , then the image of the map  $\lambda : D \rightarrow D \log \tilde{\theta}$  of  $\text{Lie}(S)$  in  $S$  is contained in  $I_2(A)$ . Moreover, if  $N_{\tilde{\theta}}$  denotes the image of  $\lambda(\text{Lie}(S))$  in  $H_{DR}^1(A)$ , we have

$$N = \{f \mid f \in H_{DR}^1(A), p^n f \in N_{\tilde{\theta}}, \text{ for some } n \in \mathbb{N}\}.$$

Proof. As in the proof of 2.1 we'll assume, for simplicity, that  $\text{Supp } X_0 \cap e_0 = \emptyset$ . So we can assume  $\tilde{\theta} \equiv 1 \pmod{S^+}$ , and therefore

$$3.3. \quad F = \mathcal{D}_{\mu}^{2 \tilde{\theta}}$$

is a symmetric, bi-multiplicative co-cycle of  $S$  which is in  $K(A^3)$ . Now, if we transform both terms of 3.3 by the operator  $(L \hat{\otimes} L \hat{\otimes} \epsilon D) \log$ ,

and successively by  $(\iota \hat{\otimes} \varepsilon D')$ , where  $D, D' \in \text{Lie}(S)$ , we have

$$3.4. (\iota \hat{\otimes} \iota \hat{\otimes} \varepsilon D) \log F(t_1, t_2, t_3) = (D \log \hat{\theta}) (t_1 + t_2) - (D \log \hat{\theta}) (t_1) - (D \log \hat{\theta}) (t_2) + \varepsilon (D \log \hat{\theta}), \text{ and}$$

$$3.5. (\iota \hat{\otimes} \varepsilon D' \hat{\otimes} \varepsilon D) \log F(t_1, t_2, t_3) = (DD' \log \hat{\theta}) (t_1) - \varepsilon (DD' \log \hat{\theta}),$$

which say precisely that  $D \log \hat{\theta}$  is in  $I_2(A)$ . Since  $\hat{\theta} \in S$ ,  $\lambda(\text{Lie}(S))$  is contained in  $S$ , and so  $N_{\hat{\theta}}$  is contained in  $N$ . Finally, since  $X$  is ample  $\lambda(\text{Lie}(S))$  is a free  $R$ -module which doesn't intersect  $\theta_0$  (see [1] and [6]), so by comparing the ranks we conclude that  $N_{\hat{\theta}}$  is isogenous to  $N$ , Q.E.D..

3.6. Remark. If  $\Delta$  is the determinant of the map  $\text{Lie}(S) \rightarrow N$ ,  $\|1/\Delta\|_p$  is the separable degree of the polarization associated to  $X_0$  (cfr. [MA]). So, in particular, if  $A$  is principally polarized, one can choose  $X$  in such a way that  $N = N_{\hat{\theta}}$ .

3.7. Remark. Let  $G_1$  and  $G_{\hat{\varepsilon}t}$  be the local and the étale component of the Barsotti-Tate group  $G$  of the reduced abelian variety  $A_0$ . By results on the crystalline cohomology (see [3] and [9<sub>a</sub>]),  $H_{DR}^1(A)$  is canonically isomorphic to the Dieudonné module  $D(G)$  and  $H_{DR}^1(S)$  is canonically isomorphic to the Dieudonné module  $D(G_1)$ . Moreover the canonical map from  $H_{DR}^1(A)$  onto  $H_{DR}^1(S)$  corresponds to the projection  $D(G) = D(G_1) \oplus D(G_{\hat{\varepsilon}t}) \rightarrow D(G_1)$ ; therefore  $N_{\hat{\theta}}$  as Dieudonné module, is isogenous to  $D(G_{\hat{\varepsilon}t})$ . As a consequence, if  $V$  denotes the Verschiebung of  $H_{DR}^1(A)$  and  $\overline{D \log \hat{\theta}}$  the image of  $D \log \hat{\theta}$  in  $H_{DR}^1(A)$ , we have  $\lim_{i \rightarrow \infty} V^i(\overline{D \log \hat{\theta}}) = 0$ , for each  $D \in \text{Lie}(S)$ .

#### 4. AN EXAMPLE.

Let  $\mathbb{F}_p$  be the Galois field with  $p$  elements,  $p \neq 2$ , and let  $\lambda_0$  be an indeterminate over  $\mathbb{F}_p$ . We shall denote by  $k$  the perfect field  $\mathbb{F}_p(\lambda_0, \lambda_0^{1/p}, \lambda_0^{1/p^2}, \dots)$ , by  $R = W(k)$  the ring of the Witt vectors

with components in  $k$ , and by  $\lambda$  an element of  $R$  whose image in  $k$  is  $\lambda_0$ . Now we consider the cubic  $E_\lambda$  over  $K = \text{Frac}R$ , whose affine equation is

$$i) \quad y^2 = (1-x^2)(1-\lambda x).$$

If we choose as identity the point  $e$  of coordinates  $x=0, y=1$ ,  $(E_\lambda, e)$  is an abelian variety which satisfies the request of 3.2. Moreover  $2e$  is a totally symmetric divisor which gives a principal polarization, and so the image of  $\text{Dlog}_{2e}^\vee$  in  $H_{\text{DR}}^1(E_\lambda)$  spans  $N$  (see th. 3.2). This, in view of 3.7, is equivalent to saying that the image  $\overline{\text{Dlog}_{2e}^\vee}$  is an eigenvector of the Frobenius of  $H_{\text{DR}}^1(E_\lambda)$  corresponding to a unit eigenvalue. So, as remarked also by Norman (see [11]),  $\overline{\text{Dlog}_{2e}^\vee}$  spans the Dwork's sub-crystal of  $H_{\text{DR}}^1(E_\lambda)$  (see [8] and [9]). The aim of this example is to give an explicit computation for  $\overline{\text{Dlog}_{2e}^\vee}$ .

Since  $x$  is a uniformizing parameter of  $E_\lambda$  at  $e_0$ , a basis of  $H_{\text{DR}}^1(E_\lambda)$  is given by the canonical images of two series  $u$  and  $v$  of the following type :

$$u = \sum_{i=1}^{\infty} (c_i/i)x^i \quad \text{and} \quad v = \sum_{i=1}^{\infty} (b_i/i)x^i, \quad \text{where } c_i \text{ and } b_i \text{ are in } R.$$

In particular we can choose  $u$  and  $v$  in such a way that  $du = dy/y$  and  $dv = xdx/y$ ; with this choice  $b_i = c_{i-1}$ , if  $i > 1$ , and  $b_1 = 0$ .

Now let  $\hat{\theta}(x) \in R[[x]]$  be a theta series of  $2e$  (see th. 3.2) and let  $D$  be the derivation of  $S$  defined by  $Du = 1$ . By 3.2

$$ii) \quad D\hat{\theta} / \hat{\theta} \equiv v + au, \quad \text{mod } R((x)),$$

where  $a$  is in  $R$ . Since  $D\hat{\theta} / \hat{\theta} - 2Dx/x \in R[[x]]$ , we deduce that

$$iii) \quad v + au \equiv 0, \quad \text{mod } R[[x]].$$

The relation iii), as shown in [4], allows to compute  $a$  :

$$a = - \lim_{i \rightarrow \infty} c_{i-1} / c_i \cdot p^{i-1} / p^i.$$

*P-ADIC THETA SERIES*

To finish, let us show how the image of  $v+au$  may be recovered from each theta,  $\theta$ , of  $2e$  which satisfies the property  $\theta(x) = \theta(-x)$ . As we have shown in 1.13, there exists a constant  $b \in K$ , such that

$$iv) \quad D\theta/\theta + bu \equiv \overset{\sim}{D}\overset{\sim}{\theta}/\overset{\sim}{\theta}, \quad \text{mod } R((x)),$$

and so

$$v) \quad D\theta/\theta + bu \equiv 0, \quad \text{mod } R((x)).$$

The relation v) determines  $b$ . In fact if  $z = \exp u - 1$ , and if

$$\log(\theta/x^2) = \sum_{i=1}^{\infty} a_i z^i, \quad v) \quad \text{is equivalent to}$$

$$vi) \quad (1+z) \sum_{i=1}^{\infty} i a_i z^{i-1} + b \sum_{i=1}^{\infty} (-1)^{i-1} z^i / i \equiv 0, \quad \text{mod } R[[x]];$$

and therefore

$$b = - \lim_{i \rightarrow \infty} p^i ((p^i + 1) a_{p^i+1} + p^i a_{p^i}).$$

BIBLIOGRAPHY

- [MA] I. BARSOTTI, *Metodi analitici per varietà abeliane in caratteristica positiva*. Cap. 1,2 ; Cap 3,4 ; Capitolo 5 ; Capitolo 6 ; Capitolo 7 ; Ann. Scuola Norm. Sup. Pisa, 18 (1964), pp. 1-25 ; 19 (1965) pp. 277-330 ; 19 (1965) pp. 481-512 ; 20 (1966) pp. 101-137 ; 20 (1966) pp. 331-365.
- [1] I. BARSOTTI, *Repartitions on abelian varieties*, Illinois Journ. of Math. 2 (1958) pp. 43-69.
- [2] I. BARSOTTI, *Considerazioni sulle funzioni theta*, Symp. Math., 3 (1970) pp. 247-277.
- [3] I. BARSOTTI, *Bivettori*, Symp. Math., 24 (1981) pp. 23-63.
- [4] P.R. CIBOTTO, *Congruences for abelian integrals*, Bollettino U.M.I. (15) 18-A (1981) pp. 431-433.

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- [5] M. CANDILERA e V. CRISTANTE, Biextensions associated to divisors on abelian varieties and theta functions, to appear.
- [6] V. CRISTANTE, Classi differenziali e forma di Riemann, Ann. Scuola Norm. Sup. Pisa, serie IV, Vol. IV (1977) pp. 1-12.
- [7] V. CRISTANTE, Theta functions and Barsotti-Tate groups, Ann. Scuola Norm. Sup. Pisa, serie IV, Vol. VII (1980) pp. 181-215.
- [8] B. DWORK, p-adic cycles, Pub. Math. I.H.E.S., 37 (1968) pp. 327-415.
- [9] N. KATZ, Travaux de Dwork, Exposé 409, Seminaire Bourbaki 1971/72, LNM 317, Springer (1973) pp. 167-200.
- [9a] W. MESSING, The crystals associated to Barsotti-Tate groups : with applications to abelian schemes, LNM 264, Springer (1970).
- [10] A. NERON, Fonctions thêta p-adiques, Symp. Math., 24 (1981) pp. 315-345.
- [11] P. NORMAN, p-adic theta functions, to appear.
- [12] I. BARSOTTI, Variet  abeliane su corpi p-adici, parte I<sup>o</sup>, Symp. Math. 1 (1968) pp. 109-173.

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