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FREE BOUNDARY PROBLEM IN FLUID DYNAMICS

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PART I: ONE FLUID PROBLEMS.

Let N be a $C^{1+\alpha}$ curve $x = g(y)$ ($b \leq y < \infty$) with $b > 0$, $g(b) = 0$ and $\lim_{y \rightarrow \infty} g(y) = -\infty$. Consider the 2-dimensional symmetric jet problem: Find a function u and a curve Γ in $\{0 < y \leq b\}$ satisfying

(1.1) $N \cup \Gamma$ is a continuous curve, such that with G denoting the domain bounded by $N \cup \Gamma$ and the x -axis I_0 , the following properties hold:

(1.2) $0 < u < 1$ in G ,

(1.3) $\Delta u = 0$ in G ,

(1.4) $u = 0$ on I_0 ,

(1.5) $u = 1$ on $N \cup \Gamma$,

(1.6) $\frac{\partial u}{\partial \nu} = \lambda$ on $N \cup \Gamma$;

finally, Γ is required to be smooth, say in $C^{1+\delta}$, and

(1.7) $N \cup \Gamma$ is a C^1 curve in $B_\delta(A)$, u is in $C^1(\overline{G} \cap B_\delta(A))$

for some $\delta > 0$, where $A = (0, b)$ and $B_r(X)$ denotes the disc of radius r and center X . The parameter λ is to be determined together with u, Γ .

This problem was solved by Leray [21], Serrin [22] and others; see [11] [20] and the references given there. The methods used by most authors are based on

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reducing the problem by means of the hodograph variables to a nonlinear integral equation, and then solving the equation with the aid of the Leray-Schauder fixed point theorem. There are however some other methods, based on variational principles; see Garabedian and Spencer [18] and Garabedian, Lewy and Schiffer [17].

A new variational approach was introduced by Alt, Caffarelli and Friedman (ACF) [2]; it exploits the paper of Alt and Caffarelli [1] who studied the following variational problem. Let

$$(1.7) \quad J(v) = \int_{\Omega} \left(|\nabla v|^2 + Q(x) I_{\{v>0\}} \right) dx$$

and let

$$K = \left\{ v \in H^{1,2}(\Omega), v = u^0 \text{ on a part } S \text{ of } \partial\Omega \right\},$$

where Q is a given positive C^α function in $\bar{\Omega}$, $\Omega \subset \mathbb{R}^n$ and $u^0 \geq 0$. Consider the problem of finding $u \in K$ such that

$$J(u) = \min_{v \in K} J(v) .$$

It is proved in [1] that any minimizer u is Lipschitz continuous, and the free boundary $\Gamma = \partial\{u>0\}$ is smooth if $n = 2$ (analytic if Q is analytic).

Further

$$\frac{1}{r} \int_{B_r(x^0)} u \geq c \quad (c > 0) \quad \text{if } x_0 \in \Gamma \quad (\text{non-degeneracy})$$

Theorem 1.1 [2]. *There exists a unique solution of problem (1.1)-(1.7).*

Uniqueness is well known [19] and is based on a comparison argument. To prove existence, set

$$I_1 = \{(x,b); x > 0\}$$

and denote by Ω the domain bounded by $N \cup I_1$ and I_0 . For any $\mu > 0$, set

$$\Omega_\mu = \Omega \cap \{x > -\mu\} ,$$

$$K_\mu = \left\{ v \in H^{1,2}(\Omega_\mu); v = 0 \text{ on } I_0, v = 1 \text{ on } N \cup I_1, v = h_\mu(y) \text{ on } x = -\mu \right\},$$

where $h_\mu(y)$ is increasing, $h_\mu(0) = 0$, $h_\mu = Q$ at $N \cap \{x = -\mu\}$.

Consider the truncated functional

$$(1.8) \quad J_{\lambda,\mu}(v) = \int_{\Omega_\mu} |\nabla v - \lambda I_{\{v < 1\}} \cap E^e|^2 dx dy$$

where $e = (0,1)$, $E = \{(x,y); -\infty < x < \infty, 0 < y < b\}$. If $\lambda > 1/b$ then we can construct a v_0 in K_μ with

$$v_0 = \min\{\lambda y, 1\} \text{ if } x > 1.$$

For such v_0 , $J_{\lambda,\mu}(v_0) < \infty$. Consequently, there exists a minimizer $u_{\lambda,\mu}$ of

$$v \rightarrow J_{\lambda,\mu}(v), \quad v \in K_\mu.$$

Lemma 1.2. *The minimizer is unique.*

Indeed, suppose u_1, u_2 are two minimizers and introduce

$$u_1^\varepsilon(x,y) = u_1(x-\varepsilon,y) \text{ and}$$

$$v_1 = u_1^\varepsilon \wedge u_2, \quad v_2 = u_1^\varepsilon \vee u_2.$$

Denote by J^ε the functional $J = J_{\lambda,\mu}$ corresponding to the translation $x \rightarrow x + \varepsilon$ of Ω_μ, K_μ . One verifies that

$$J^\varepsilon(u_1^\varepsilon) + J(u_2) = J^\varepsilon(v_1) + J(v_2),$$

which implies that $J(u_2) = J(v_2)$, i.e., $u_1^\varepsilon \vee u_2$ is a minimizer. Consequently

$u_1^\varepsilon \vee u_2$ is smooth, which implies that either $u_1^\varepsilon \geq u_2$ or $u_1^\varepsilon \leq u_2$ everywhere.

Since $u_1^\varepsilon < u_2$ near the boundary, we deduce that $u_1^\varepsilon < u_2$ throughout the domain.

Taking $\varepsilon \rightarrow 0$ we get $u_1 \leq u_2$, and similarly $u_2 \leq u_1$.

Taking $u_1 = u_2$ in the above argument we get:

$$(1.9) \quad \frac{\partial}{\partial x} u_{\lambda,\mu} \geq 0.$$

Thus the analytic free boundary $\Gamma = \Gamma_{\lambda, \mu}$ has the form

$$(1.10) \quad \Gamma_{\lambda, \mu} : x = f_{\lambda, \mu}(y) .$$

Next we have:

Lemma 1.3. $f_{\lambda, \mu}(y)$ is continuous and finite if and only if $h < y < b$, where $h = 1/\lambda$.

Lemma 1.4. $\lambda \rightarrow f_{\lambda, \mu}(b)$ is continuous.

Using the Lipschitz continuity estimate on $u_{\lambda, \mu}$ and non-degeneracy, we establish:

Lemma 1.5. If λ is sufficiently large then $f_{\lambda, \mu}(b) < 0$; if $\lambda > 1/b$ and $\lambda - 1/b$ is small enough, then $f_{\lambda, \mu}(b) > 0$.

From Lemmas 1.4, 1.5 we deduce that there is a value $\lambda = \lambda(\mu)$ such that $f_{\lambda, \mu}(b) = 0$, i.e., there is a "continuous fit" at A. For this value of λ , $(u_{\lambda, \mu}, \Gamma_{\lambda, \mu})$ "almost" solve the jet problem. In order to complete the construction of a solution we let $\mu \rightarrow \infty$, $\lambda(\mu) \rightarrow \lambda$ and denote the limiting $u_{\lambda, \mu}, \Gamma_{\lambda, \mu}$ by u, Γ . Then (1.1)-(1.6) are satisfied. Finally:

Lemma 1.6. Continuous fit implies "smooth fit", i.e., it implies (1.7).

The above outline of the proof of Theorem 1.1 is a special case of ACF [2] who actually considered the corresponding 3-dimensional axially symmetric jet problem, whereby one replaces

$$\Delta u \text{ by } \mathcal{L}u \equiv \Delta u - \frac{1}{y} u_y ,$$

and

$$\frac{\partial u}{\partial v} = \lambda \text{ by } \frac{1}{y} \frac{\partial u}{\partial v} = \lambda .$$

For this problem the classical methods do not apply.

The variational approach of ACF [2] has been extended to more difficult jet

problems. We shall mention a few. In ACF [3] the plane asymmetric jet problem is considered, whereby the nozzle N is made up of two curves N_1 and N_2 , given by $y = g_1(x)$ ($-\infty < x \leq 0$) and $g_1(x) > g_2(x)$. Setting $A_i = (0, g_i(0))$, one seeks two curves Γ_1, Γ_2 of the form $\Gamma_i: y = f_i(x)$ ($0 \leq x < \infty$) with $f_1(x) > f_2(x)$, such that,

$$f_i(0) = g_i(0) \quad ,$$

and, with G denoting the domain bounded by $N_1 \cup \Gamma_1$, $N_2 \cup \Gamma_2$,

$$-1 < u < 1 \quad , \quad \Delta u = 0 \quad \text{in } G \quad ,$$

$$u = 1 \quad \text{on } \Gamma_1 \quad , \quad u = -1 \quad \text{on } \Gamma_2 \quad ,$$

$$\left| \frac{\partial u}{\partial \nu} \right| = \lambda \quad \text{on } \Gamma_1 \cup \Gamma_2 \quad ,$$

$$\lim_{x \rightarrow \infty} f_1'(x) = \lim_{x \rightarrow \infty} f_2'(x) \quad ,$$

and both limits exist, and, finally, there is smooth fit at A_1 and A_2 .

In this problem both λ and the direction of the jet at $x = \infty$ are not given, and are to be found together with u, Γ_1, Γ_2 .

One works with the truncated functional

$$\int_{\Omega_\mu} |\nabla v - \lambda I_{\{|v| < 1\}} \cap E e^\perp|^2 dx dy$$

where $E = \{x > 0\}$, e is unspecified unit vector (e_1, e_2) with $e_1 > 0$, and e^\perp is obtained by rotating e counterclockwise by $\pi/2$.

Notice that the present situation is a 2-parameter problem, with parameters λ, e ; these must be chosen so as to achieve continuous (and smooth) fit at A_1 and A_2 . For details see ACF [3].

In ACF [4] a jet in a gravity field is considered. Here, roughly speaking,

$$\left| \frac{\partial u}{\partial \nu} \right| = \lambda \sqrt{|y|}$$

on the free boundary, which means that we replace λ by $\lambda\sqrt{|y|}$ in the variational functional. An interesting "threshold" phenomenon occurs: If $\alpha \leq 120^\circ$ then there exists a $Q_0 > 0$ such that a solution exists if and only if the flux Q is $> Q_0$ (for $\alpha > 120^\circ$ there exists a solution for any $Q > 0$); see Figure 1.

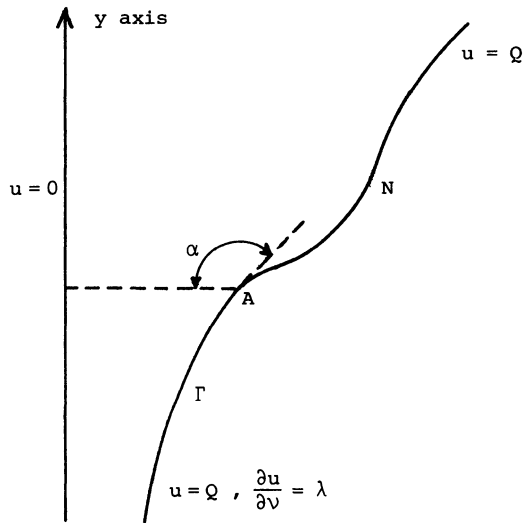


Figure 1

We turn to the axially symmetric cavity problem past a nose $N : x = g(y)$ ($0 \leq y \leq 1$), with $g(y) < 0$, $g(0) = -a$. Set $I_- = \{(x, 0), x \leq -a\}$. We seek a curve Γ initiating at $(0, b)$ and lying in $\{x > 0, y \geq b\}$ such that, if G is the domain bounded by $N \cup \Gamma$ and I_- , then:

$$\begin{aligned} (1.11) \quad & Lu = 0, \quad u > 0 \quad \text{in } G, \\ & u = 0 \quad \text{on } \partial G, \\ & \frac{1}{y} \cdot \frac{\partial u}{\partial \nu} = 1 \quad \text{on } \Gamma, \end{aligned}$$

FREE BOUNDARY PROBLEMS IN FLUID DYNAMICS

$$\frac{\nabla u}{y} \rightarrow (0,1) \text{ if } x^2 + y^2 \rightarrow \infty, (x,y) \in G$$

and there is smooth fit at $(0,b)$.

This problem was solved by Caffarelli and Friedman [12] (see also [14] and [9]). One first considers a truncated problem, such as the one obtained by looking at the same problem in a channel of width $2H$, setting $u = Qy^2/2$ on the boundary of the channel. Then, with a suitable $Q = Q(H)$ one can achieve a solution with smooth fit. Finally one takes $H \rightarrow \infty$ and shows that $Q(H) \rightarrow 1$. This fact is used in order to prove the last condition in (1.11) for the limiting (u,Γ) . In this manner we obtain a solution of (1.11) with smooth fit at $(0,b)$; further, for large x , Γ has the form

$$\Gamma : y = f(x) \text{ with } f'(x) \rightarrow 0 \text{ if } x \rightarrow \infty .$$

The above results have recently been extended in ACF [9] to compressible flows; the corresponding quasi-linear functional was studied in ACF [8].

We finally mention that axially symmetric cavities for rotational flow in a channel were studied by Friedman [15], and irrotational flow in a channel with oscillatory wall was studied by Friedman and Vogel [16].

PART II: TWO FLUID PROBLEMS.

Let $N_1 : y = H$, I_0 the x -axis, and $N_2 : y = g(x)$ ($-\infty < x \leq 0$) with $0 < g(x) < H$, and set $A = (0, g(0))$. Consider the following problem: Find a curve Γ ,

$$(2.1) \quad \Gamma : y = f(x) \quad (0 \leq x < \infty), \quad f(0) = g(0), \quad f \text{ continuous}, \quad 0 < f(x) < H,$$

and a function u , $u = u_1$ in the domain Ω_1 bounded by N_1 and $N_2 \cup \Gamma$, $u = u_2$ in the domain Ω_2 bounded by I_0 and $N_2 \cup \Gamma$, such that the following is true:

$$\left\{ \begin{array}{l} u_1 = Q \text{ on } N_1 \\ u_1 = u_2 = 0 \text{ on } N_2 \cup \Gamma, \\ u_2 = -1 \text{ on } I_0, \\ 0 < u_1 < Q \text{ in } \Omega_1, \\ -1 < u_2 < 0 \text{ in } \Omega_2, \\ \Delta u_i = 0 \text{ in } \Omega_i; \end{array} \right.$$

further,

$$(2.3) \quad \nabla u \text{ is bounded in } B_\delta(A), \quad N_2 \cup \Gamma \text{ has tangent at } A$$

for some $\delta > 0$, and

$$(2.4) \quad |\nabla u_1|^2 - |\nabla u_2|^2 = \lambda \text{ on } \Gamma.$$

Here λ is a constant which must be found together with u and Γ .

The jump relation (2.4) is the new feature of this problem. In ACF [5] the functional

$$J(v) = \iint \left[|\nabla v|^2 + Q^2(x) \lambda^2(v) \right] dx$$

was introduced with

$$\lambda^2(v) = \begin{cases} \lambda_1^2 & \text{if } v < 0 \\ \lambda_2^2 & \text{if } v > 0 \end{cases}$$

and $\lambda_1^2 - \lambda_2^2 \neq 0$, with the idea of applying it to study 2-fluid problems, such as (2.1)-(2.4). It was proved that any minimizer is Lipschitz continuous, a non-degeneracy property holds, and the free boundary is in C^1 in case $n = 2$. These results are used in order to establish:

Theorem 2.1 (ACF [6]). *There exists a unique solution of the problem (2.1)-(2.4), and Γ is in C^1 .*

To prove the theorem we introduce a truncated functional

$$J_{\lambda, \mu}(v) = \int_{\Omega_{\mu}} |\nabla v - (\lambda_2 I_{\{v < 0\}} + \lambda_1 I_{\{v > 0\}} + \lambda_0 I_{\{v = 0\}}) I_{\{x > 0\}} e|^2 dx dy$$

where $e = (0, 1)$, $\Omega_{\mu} = \{-\mu < x < \infty, 0 < y < H\}$,

$$\lambda_2 = \frac{1}{h}, \lambda_1 = \frac{Q}{H-h}, \lambda = \lambda_2^2 - \lambda_1^2, \lambda_0 = \min(\lambda_1, \lambda_2)$$

and introduce the admissible set

$$K_{\mu} = \left\{ \begin{array}{l} v \in H^{1,2}(\Omega_{\mu}); v = -1 \text{ on } I_0, v = 0 \text{ on } N_1, -1 \leq v \leq 0 \text{ below } N_2, \\ 0 \leq v \leq Q \text{ above } N_2, v = h(y) \text{ on } x = -\mu \end{array} \right\}$$

with $h(y)$ suitable monotone increasing function. The problem

$$v \rightarrow \min J_{\lambda, \mu}(v), \quad v \in K_{\mu}$$

has a unique solution u with $u_y \geq 0$ (cf. Lemma 1.2). Thus the free boundary is a curve

$$\Gamma_{\lambda, \mu} : y = f_{\lambda, \mu}(x) .$$

Notice that h is the asymptotic height of the free boundary at $x = \infty$.

We now let h vary and show that for some value of h , say $h = h(\mu)$, $f_{\lambda, \mu}(0) = b$, that is, there is a continuous fit at A . We finally let $\mu \rightarrow \infty$ in order to complete the existence proof; for more details see ACF [6].

In principle, all the steps in the proof of Theorem 2.1 have their counterparts in the proof of Theorem 1.1, but they require a more refined analysis.

A significantly harder problem occurs when the upper fluid is not everywhere confined by $\{y = H\}$, that is, N_1 consists of $\{y = H, x \leq 0\}$ and one seeks, in addition to Γ , another free boundary Γ_1 initiating at $(0, H)$ such that

$$\frac{\partial u_1}{\partial \nu} = \bar{\lambda} \quad \text{on } \Gamma_1 .$$

This is a 2-parameter problem (with parameters $\lambda, \bar{\lambda}$) and they must be chosen in such a way as to achieve a continuous fit at both A and $(0,H)$.

This problem was solved in ACF [7]. The proof involves quantitative estimates on the location of the free boundaries Γ and Γ_1 , depending on the relations between λ and $\bar{\lambda}$. Some of these estimates are derived by comparison arguments; the flows used in the comparison are constructed by solving one-fluid jet problems.

Some of the techniques developed for 2-fluid jet problems can be applied to other 2-fluid problems. We briefly mention the 2-fluid dam problem (see Figure 2).

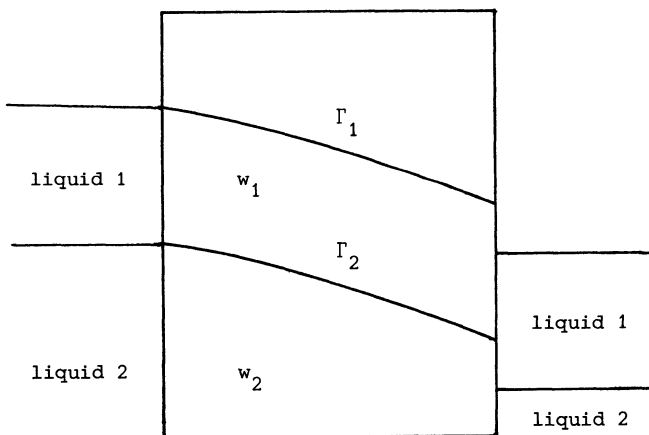


Figure 2

Denote by $\phi_i = p + \delta_i y$ the piezometric head of the i -th fluid and assume that $\delta_2 > \delta_1$. We introduce the stream function ψ (that is, $\phi - i\psi$ is holomorphic). Then in terms of ψ the problem reduces to

$$\Delta \psi = \frac{\partial}{\partial x} H(\psi) \quad \text{in the dam,}$$

FREE BOUNDARY PROBLEMS IN FLUID DYNAMICS

where

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ -\delta_2 + \delta_1 & \text{if } 0 < t < Q_1, \\ -\delta_2 & \text{if } t \geq Q_1; \end{cases}$$

here $\psi_2 = -Q_2$ at the bottom of the dam, and $\psi = Q_1$ at the top. On the top free boundary $\Gamma_{0,1}$:

$$\psi = Q_1 \quad \text{and} \quad \frac{\partial \psi_1}{\partial \nu} = \delta_1 \frac{\partial x}{\partial \nu},$$

and on the intermediate free boundary $\Gamma_{1,2}$:

$$\psi_1 = 0, \quad \frac{\partial \psi_1}{\partial \nu} - \frac{\partial \psi_2}{\partial \nu} = (\delta_1 - \delta_2) \frac{\partial x}{\partial \nu}.$$

ψ also satisfies suitable Dirichlet and Neumann conditions.

Theorem 2.2 (ACF [10]). *There exists a solution $(\psi, \Gamma_{0,1}, \Gamma_{1,2})$ of the 2-fluid dam problem with $\Gamma_{0,1}, \Gamma_{1,2}$ analytic and C^1 respectively, and*

$$\Gamma_{0,1} : y = f_1(x), \quad \Gamma_{1,2} : y = f_2(x)$$

where $f_i(x)$ are Lipschitz continuous and strictly monotone decreasing functions.

Partial results are obtained in ACF [10] for general 2-dimensional dams.

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FREE BOUNDARY PROBLEMS IN FLUID DYNAMICS

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