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THE PARAMETRIC PROBLEM OF CAPILLARITY: THE CASE OF TWO AND THREE FLUIDS

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I shall speak about the existence of equilibrium configurations in a container filled by two or three non-miscible, homogeneous fluids subjected to surface tension and gravitational energy.

If we denote by $\Omega \subset \mathbb{R}^n$ a bounded open set with Lipschitz-continuous boundary and by E_1 , E_2 subsets of Ω occupied by two non-miscible fluids with given densities ρ_1 and ρ_2 , we can write the global energy of the configuration in the following way:

$$\mathcal{E}(E_1,E_2) = \gamma_{12}\text{meas}_{n-1}(\partial E_1 \cap \partial E_2 \cap \Omega) + \beta_1\text{meas}_{n-1}(\partial E_1 \cap \partial\Omega) + \beta_2\text{meas}_{n-1}(\partial E_2 \cap \partial\Omega) +$$

$$+ g \sum_{i=1}^{2} \rho_i \int_\Omega x_n \phi_{E_i}(x)\,dx \quad .$$

We use the $(n-1)$-dimensional measure introduced by E. De Giorgi in 1954 (see [3]). More precisely, if E is a measurable subset of Ω , we define the perimeter of E in Ω as:

$$\text{meas}_{n-1}(\partial E \cap \Omega) = \int_\Omega |D\phi_E| = \sup\left\{\int_\Omega \text{div}g(x)\,dx;\ g \in C_0^1(\Omega, \mathbb{R}^n),\ |g| \le 1\right\} \quad .$$

We observe that the perimeter of E is the total variation on Ω of the vector valued measure $D\phi_E = (D_1\phi_E, D_2\phi_E, \ldots, D_n\phi_E)$ where $D_i\phi_E$ $i = 1,2,\ldots,n$ are the derivatives of the characteristic function of E in the distributional sense.

It is well-known that, if $\int_\Omega |D\phi_E| < +\infty$, then there exists the trace of ϕ_E on the Lipschitz-continuous surface $\partial\Omega$.

Using the perimeter and the trace of E, recalling that $E_2 = \Omega - E_1$, the global energy can be written in the form:

$$\mathcal{E}(E_1,E_2) = \gamma_{12}\int_\Omega |D\phi_{E_1}| + (\beta_1-\beta_2)\int_{\partial\Omega}\phi_{E_1}dH_{n-1} + g(\rho_1-\rho_2)\int_\Omega x_n\phi_{E_1}dx + H_{n-1}(\partial\Omega) + g\rho_2 H_n(\Omega)$$

Then, the problem is reduced to minimize the functional:

$$(1) \qquad \mathcal{F}(E) = \gamma\int_\Omega |D\phi_E| + \beta\int_{\partial\Omega}\phi_E dH_{n-1} + g\rho\int_\Omega x_n\phi_E(x)\,dx$$

in the class $\mathcal{H}$ of all subsets of Ω having prescribed volume $v \in (0, H_n(\Omega))$.

We observe the following:

a) if $\gamma \geq 0$, $\mathcal{F}$ has a finite lower bound;

b) if $\gamma > 0$, from the inequality

$$\int_\Omega |D\phi_E| \leq \frac{1}{\gamma}\left[\mathcal{F}(E) + |\beta| H_{n-1}(\partial\Omega) + g|\rho|\int_\Omega |x_n|\,dx\right] ,$$

if $\{E_h\}$ is a minimizing sequence, we have:

$$\int_\Omega |D\phi_{E_h}| \leq \text{cost} .$$

From a well-known compactness theorem, there exists a subsequence of $\{E_h\}$ converging in $L_1(\Omega)$ to a set E.

c) if $\gamma \geq |\beta|$, the functional $\mathcal{F}$ is lower semicontinuous with respect to $L_1(\Omega)$-convergence.

Then we can state the following

Theorem 1. *If* $\gamma \geq |\beta|$ $(\gamma > 0)$, *the functional* $\mathcal{F}$ *has a minimum* E *in the class* $\mathcal{H}$.

The regularity results of De Giorgi and M. Miranda can be applied to study the smoothness of ∂E and we obtain that there exists an open subset of $\partial E \cap \Omega$: $\partial^* E \cap \Omega$ that is an analytic manifold of dimension $n-1$ and moreover $H_s((\partial E - \partial^* E)\cap\Omega) = 0 \quad \forall\, s > n-8$. (See [3]).

Let us consider now a container Ω filled by three fluids: $(E_1,E_2,E_3) = E$.

If we denote by

$$|\Sigma_{ij}| = \text{meas}_{n-1}(\partial E_i \cap \partial E_j \cap \Omega) \qquad i,j = 1,2,3;\ i \neq j$$

the surface energy of the six interfaces, can be written as:

$$\mathcal{E}(E) = \mathcal{E}(E_1,E_2,E_3) = \gamma_{12}|\Sigma_{12}| + \gamma_{13}|\Sigma_{13}| + \gamma_{23}|\Sigma_{23}| + \sum_{i=1}^{3} \beta_i \int_{\partial\Omega} \phi_{E_i} dH_{n-1} \quad .$$

Now, if we suppose ∂E_i $(i = 1,2,3)$ Lipschitz continuous and $H_{n-1}(\partial E_1 \cap \partial E_2 \cap \partial E_3) = 0$, we have:

$$\int_\Omega |D\phi_{E_i}| = \sum_{\substack{j=1 \\ j\neq i}}^{3} |\Sigma_{ji}| \qquad i = 1,2,3$$

and then we can write:

$$\mathcal{E}(E) = \sum_{i=1}^{3} \gamma_i \int_\Omega |D\phi_{E_i}| + \sum_{i=1}^{3} \beta_i \int_{\partial\Omega} \phi_{E_i} dH_{n-1}$$

where:

$$2) \qquad \begin{cases} \gamma_1 = \dfrac{\gamma_{12}+\gamma_{13}-\gamma_{23}}{2} \\[2ex] \gamma_2 = \dfrac{\gamma_{23}+\gamma_{12}-\gamma_{13}}{2} \\[2ex] \gamma_3 = \dfrac{\gamma_{13}+\gamma_{23}-\gamma_{12}}{2} \end{cases}$$

Therefore the global energy of the configuration is given by:

$$3) \qquad \mathcal{F}(E) = \sum_{i=1}^{3} \left(\gamma_i \int_\Omega |D\phi_{E_i}| + \beta_i \int_{\partial\Omega} \phi_{E_i} dH_{n-1} + g\rho_i \int_\Omega x_n \phi_{E_i}(x)\,dx \right) \quad .$$

We have now to minimize the functional 3) in the class

$$\mathcal{K} = \left\{ E = (E_1,E_2,E_3);\ E_i \cap E_j = \emptyset\ i \neq j;\ H_n(E_i) = v_i,\ \sum_{i=1}^{3} v_i = H_n(\Omega) \right\}$$

It is easy to see that $\mathcal{F}$ has a finite lower bound if and only if

$$4) \qquad \gamma_i + \gamma_j \geq 0 \qquad i,j = 1,2,3 \quad i \neq j$$

In fact, if $\gamma_i > 0 \quad \forall\, i = 1,2,3$, we have

$$\mathcal{F}(E) \geq \sum_{i=1}^{3} \gamma_i \int_\Omega |D\phi_{E_i}| - c$$

where

$$c = \sum_{i=1}^{3} \left(|\beta_i| H_{n-1}(\partial\Omega) + g|\rho_i| \int_\Omega |x_n| dx \right) .$$

On the other hand, if $\gamma_1 \leq 0$, one has:

$$\mathcal{F}(E) \geq \gamma_1 \left(\int_\Omega |D\phi_{E_2}| + \int_\Omega |D\phi_{E_3}| \right) + \sum_{j=2}^{3} \gamma_j \int_\Omega |D\phi_{E_j}| - c = \sum_{j=2}^{3} (\gamma_j + \gamma_1) \int_\Omega |D\phi_{E_j}| - c$$

and then

$$\inf_{\mathbb{K}} \mathcal{F}(E) \geq -c \quad .$$

From the last two inequalities, if $\gamma_i + \gamma_j > 0 \quad i,j = 1,2,3 \quad i \neq j$, we obtain:

$$\int_\Omega |D\phi_{E_i}| \leq c_1 \mathcal{F}(E) + c_2 \quad \forall\, i = 1,2,3$$

and then one gets the compactness property we use to prove the existence of a minimum.

We note that 2) implies

$$\gamma_i + \gamma_j = 2\gamma_{ij} \quad ij = 1,2,3 \quad .$$

Physically, condition 4) means that the surface energies of the $i-j$ interfaces are non negative and the fluids do not mix up.

It is easy to see that the conditions

$$5) \quad \begin{cases} \gamma_i \geq 0 & i = 1,2,3 \\ \gamma_i + \gamma_j \geq |\beta_i - \beta_j| & i,j = 1,2,3 \quad i \neq j \end{cases}$$

are necessary for the lower semicontinuity of the functional $\mathcal{F}$. If they are sufficient it isn't clear yet.

We can prove the following:

Proposition A. If Ω has the interior sphere condition, $\gamma_i \geq 0$, $\gamma_i + \gamma_j > 0$ and $\gamma_i + \gamma_j \geq |\beta_i - \beta_j|$, then $\mathcal{F}$ is lower semicontinuous.

Proposition B. If we denote the Lipschitz constant of $\partial\Omega$ by L and $\gamma_i \geq 0$, $\gamma_i + \gamma_j > 0$,

6) $$\gamma_i + \gamma_j \geq \sqrt{1 + L^2}\,|\beta_i - \beta_j| \qquad i,j = 1,2,3 \quad ;$$

then $\mathcal{F}$ is lower semicontinuous.

Proposition C. Let us suppose $\beta_1 \leq \beta_2 \leq \beta_3$. If

7) $$\gamma_j \geq \beta_j - \beta_1 \qquad j = 2,3$$

then $\mathcal{F}$ is lower semicontinuous.

Outline of the proof.

A. We recall that interior sphere condition means that $\exists\, \rho > 0$ and $\forall\, x \in \Omega$ a ball of radius ρ with $x \in B_\rho \subset \Omega$. If Ω has the interior sphere condition, then $\forall\, \varepsilon > 0$ and $\forall\, E \subset \Omega$, the following inequality holds:

8) $$\int_{\partial\Omega} \phi_E dH_{n-1} \leq \int_{\Omega_\varepsilon} |D\phi_E| + c\int_{\Omega_\varepsilon} \phi_E dx$$

where $\Omega_\varepsilon = \{x \in \Omega,\ \mathrm{dist}(x,\partial\Omega) < \varepsilon\}$ and c is a constant depending on n,ρ,ε and Ω. (See [4]). Now, if we suppose $\beta_1 \leq \beta_2 \leq \beta_3$, from 8) we have:

$$\mathcal{F}(E) - \mathcal{F}(E^h) = \sum_{i=1}^{3} \gamma_i\left(\int_\Omega |D\phi_{E_i}| - \int_\Omega |D\phi_{E_i^h}|\right) + \sum_{i=1}^{3} g\rho_i \int_\Omega x_n\left(\phi_{E_i} - \phi_{E_i^h}\right)dx +$$

$$+ \sum_{i=1}^{3} \beta_i \int_{\partial\Omega}\left(\phi_{E_i} - \phi_{E_i^h}\right)dH_{n-1} \leq \sum_{i=1}^{3} \gamma_i\left(\int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i}| - \int_{\Omega-\Omega_\varepsilon} |D\phi_{E_i^h}|\right) +$$

$$+ \sum_{i=1}^{3} \gamma_i \int_{\Omega_\varepsilon} |D\phi_{E_i}| + \sum_{i=1}^{3} g\rho_i \int_\Omega x_n\left(\phi_{E_i} - \phi_{E_i^h}\right)dx + \sum_{j=1,3} |\beta_j - \beta_2| \int_{\Omega_\varepsilon} |D\phi_{E_j}| +$$

$$+ \sum_{j=1,3}\Big(|\beta_j - \beta_2| - \gamma_j\Big)\int_{\Omega_\varepsilon} |D\phi_{E_j^h}| - \gamma_2\int_{\Omega_\varepsilon} |D\phi_{E_j^h}| + c \sum_{j=1,3} |\beta_j - \beta_2| \int_{\Omega_\varepsilon} |\phi_{E_j} - \phi_{E_j^h}|dx \quad .$$

Now it is sufficient to prove

9) $$\limsup_h \left(\sum_{j=1,3} \left(|\beta_j - \beta_2| - \gamma_j \right) \int_{\Omega_\varepsilon} |D\phi_{E_j^h}| - \gamma_2 \int_{\Omega_\varepsilon} |D\phi_{E_j^h}| \right) = \limsup_h G(E^h) \le 0$$

when $E_j^h \to E_j$ in $L^1(\Omega)$.

In fact, if 9) is true, we have:

$$\limsup_h \left[\mathcal{F}(E) - \mathcal{F}(E^h) \right] \le \sum_{i=1}^{3} \gamma_i \int_{\Omega_\varepsilon} |D\phi_{E_i}| + \sum_{j=1,3} |\beta_j - \beta_2| \int_{\Omega_\varepsilon} |D\phi_{E_j}| \underset{\varepsilon \to 0}{\longrightarrow} 0 \quad .$$

Inequality 9) is trivial if $\gamma_j \ge |\beta_j - \beta_2|$ $j = 1,3$. On the other hand, if $\gamma_1 < |\beta_1 - \beta_2|$, we obtain

$$G(E^h) \le \left(|\beta_1 - \beta_2| - \gamma_1 \right) \left(\int_{\Omega_\varepsilon} |D\phi_{E_2^h}| + \int_{\Omega_\varepsilon} |D\phi_{E_3^h}| \right) + \left(\beta_3 - \beta_2 - \gamma_3 \right) \int_{\Omega_\varepsilon} |D\phi_{E_3^h}| - \gamma_2 \int_{\Omega_\varepsilon} |D\phi_{E^h}| =$$

$$= \left(\beta_2 - \beta_1 - \gamma_1 - \gamma_2 \right) \int_{\Omega_\varepsilon} |D\phi_{E_2^h}| + \left(\beta_3 - \beta_1 - \gamma_1 - \gamma_3 \right) \int_{\Omega_\varepsilon} |D\phi_{E_3^h}| \le 0 \quad .$$

Proposition B can be proved arguing in the same way. We now use the inequality

$$\int_{\partial\Omega} \phi_E dH_{n-1} \le \sqrt{1 + L^2} \int_{\Omega_\varepsilon} |D\phi_E| + c \int_{\Omega_\varepsilon} \phi_E dx$$

in the place of 8).

Finally, if 7) holds, using the identity

$$\int_{\mathbb{R}^n} |D\phi_E| = P(E) = \int_\Omega |D\phi_E| + \int_{\partial\Omega} \phi_E dH_{n-1}$$

we can write the functional $\mathcal{F}$ in the form

$$\mathcal{F}(E) = \gamma_1 \int_\Omega |D\phi_{E_1}| + \sum_{j=2}^{3} (\beta_j - \beta_1) P(E_j) + \sum_{j=2}^{3} \left(\gamma_j - (\beta_j - \beta_1) \right) \int_\Omega |D\phi_{E_j}| + \sum_{i=1}^{3} g\rho_i \int_\Omega x_n \phi_{E_i} dx$$

and all the functionals on the right side are lower semicontinuous.

The conditions $\gamma_i \ge 0$ $i = 1,2,3$ imply that

$$\gamma_{12} + \gamma_{13} - \gamma_{23} \ge 0$$

$$\gamma_{21} + \gamma_{23} - \gamma_{13} \ge 0$$

$$\gamma_{13} + \gamma_{23} - \gamma_{12} \geq 0 \quad .$$

Physically these conditions are necessary to have an equilibrium configuration. In fact if $\gamma_{12} + \gamma_{13} - \gamma_{23} < 0$ the liquid E_1 will spread on E_2 and equilibrium becomes impossiblé.

The same regularity results can be applied and we obtain that $\partial^* E_1, \partial^* E_2, \partial^* E_3$ are analytic $(n-1)$-dimensional manifolds in every ball B intersecting only two of the three sets E_1, E_2, E_3. Moreover $H_s((\partial E_i - \partial^* E_i) \cap B) = 0 \quad \forall\, i = 1,2,3$, $s > n-8$.

R E F E R E N C E S

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