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Astérisque, tome 118 (1984), p. 137-143

<http://www.numdam.org/item?id=AST_1984__118__137_0>

Société Mathématique de France
Astérisque 118 (1984) p.137 à 143.

UNIQUENESS OF MINIMAL POINT AND ITS LOCATION OF CAPILLARY FREE SURFACES OVER CONVEX DOMAIN

by J.-T. CHEN (National Taiwan University)

1. INTRODUCTION.

Let Ω be a convex domain in the plane, u be the solution of (1.1) or (1.2), we prove in section 2 that the minimal point is unique for all contact angle y, $0 \leq \gamma < \frac{\pi}{2}$ and estimate in section 3 for the location of the minimal point when y = 0.

$$(1.1) \qquad \begin{cases} \operatorname{div} Tu = H & \text{in } \Omega \\ Tu \cdot \nu = \cos \gamma & \text{on } \partial\Omega \end{cases}$$

$$(1.2) \qquad \begin{cases} \operatorname{div} Tu = Ku & \text{in } \Omega \\ Tu \cdot \nu = \cos \gamma & \text{on } \partial\Omega \end{cases}$$

where $\partial\Omega$ is the boundary of Ω, ν is the unit outer normal of $\partial\Omega$

$$Tu = \left\langle \frac{u_x}{\sqrt{1+u_x^2+u_y^2}}, \frac{u_y}{\sqrt{1+u_x^2+u_y^2}} \right\rangle$$

K and γ are constants, $0 \leq \gamma < \frac{\pi}{2}$.

In case of $\gamma = 0$ Chen and Huang [2] showed that the solution of (1.1) is strictly convex over a convex domain, Korevaar [8] showed that the solution of (1.2) is convex over a strictly convex domain. In case of $\gamma \neq 0$, solutions of (1.1) and (1.2) may fail to be convex, counter examples were given by Finn [5] and Korevaar [8].

2. UNIQUENESS OF THE MINIMAL POINT.

Lemma 2.1. *Let* v *be the one dimensional solution of* (1.1) *or* (1.2) *in a strip* $-a \leq x \leq a$, Γ *be any convex curve defined in* $-a \leq x \leq a$, ν *be the downward unit normal of* Γ. *Then for any* γ, $0 \leq \gamma < \frac{\pi}{2}$, *the set*

$$\{P \in \Gamma \,|\, Tv \cdot \nu < \cos\gamma \text{ at } P\}$$

is a connected subarc of Γ.

Lemma 2.2. *Let* u_0 *be the solution of* (1.1), *with the volume constraint*

$$\int_\Omega u_0 dxdy = 0 \quad ,$$

or the solution of (1.2) *corresponding to the contact angle* γ_0. *Let* $\gamma_n \downarrow \gamma_0$ *and* u_n *be the solutions of* (1.1), *with the same constraint, or* (1.2) *corresponding to the contact angle* γ_n. *Then there is a subsequence* u_{n_k} *converge uniformly to* u_{γ_0} *in* $C^2(K)$ *for every compact subset* K *of* Ω.

Sketch of the proof:

Step I. u_n is uniformly bounded on every compact subset K of Ω by using comparison principle for (1.2), or by using a theorem of Giusti [8] for (1.1).

Step II. $|\nabla u_n|$ is uniformly bounded on K by using a theorem of Gerhardt [6] for equation (1.2) or a theorem of Serrin [9] for equation (1.1).

Step III. It follows from step II that u_n is uniformly bounded in $C^{1,\alpha}(\Omega)$ and then apply the interior Schander estimate to obtain a uniformly $C^{2,\alpha}(\Omega)$ bound. The existence of convergent subsequence u_{n_k} in $C^2(K)$ then follows from Arzelà-Ascoli theorem.

Theorem 2.3. *Let* Ω *be a convex domain in the plane such that* (1.1) *or* (1.2) *has solution for* $\gamma = 0$. *Then for any* γ , $0 \leq \gamma < \frac{\pi}{2}$, *the solution of* (1.1) *and* (1.2) *corresponding to the contact angle* γ *cannot have more than one minimal point.*

Proof. Since $\gamma \geq 0$, u is increasing near the boundary along the normal direction of $\partial\Omega$, therefore the minimum point must happen in the interior of Ω.

Suppose there is a $\gamma_1 > 0$ such that u_{γ_1} has more than two minimal points in Ω, when γ is decreased the surfaces will change smoothly in $C^2(K)$ on any compact set $K \subset \Omega$ (Lemma 2.2). However, by the theorem of Chen and Huang [1] for (1.1) or Korevaar [2] for (1.2), the surface is convex for $\gamma = 0$, thus there exists a marginal number γ_0 such that u_γ is convex for all $0 \leq \gamma \leq \gamma_0$ and nonconvex for $\gamma_0 < \gamma \leq \gamma_1$. Since u_γ converges to u_{γ_0} uniformly in $C^2(K)$ for every compact subset $K \subset \Omega$ as $\gamma \downarrow \gamma_0$ and since each u_γ is nonconvex, there exists a point P in the graph of u_{γ_0} such that u_{γ_0} has zero Gaussian curvature at P and whose tangent plane at P is horizontal, let v be the one dimensional solution of (1.1) or (1.2) which is vertical on the defining strip and is tangent to u at P, we may adjust the direction of the strip so that the principle direction of v and u_{γ_0} at p are coincident. Since v and u_{γ_0} have the same Gaussian curvature and mean curvature at P and the principle direction of them are coincident, by Euler formula, they have the same curvature alone every direction, that is they are second order contact at P. In other words, $u_{\gamma_0} - v$ together with its first and second derivative vanish at P. However $u_{\gamma_0} - v$ satisfies an elliptic partial differential equation in Ω, the zero level curves divide Ω into at least six subregions such that $u_{\gamma_0} - v$ changes sign on each adjacent subregion, then by Lemma 2.1, comparison principle and the argument as in [2] we get a contradiction.

Corollary. *Let* Ω *be a convex domain and let* $k(x)$ *be the curvature of* $\partial\Omega$ *at* $x \in \partial\Omega$ *suppose* $k(x) > H$ *for all* $x \in \partial\Omega$. *Then the solutions of* (1.1) *can not have more than one minimal point.*

Proof. The conditions of Ω imply the solutions of (1.1) exists and bounded for $\gamma = 0$. (See Chen [3] and Giusti [7]).

3. LOCATION OF THE MINIMAL POINT.

We will give an estimation for the location of the minimal point for solution

(1.1) or (1.2) for $\gamma = 0$.

Let θ be a fixed angle, $0 \leq \theta < \pi$, $L(\theta)$ be the family of parallel chords in Ω of slope $\tan\theta$, $M(\theta)$ be the trace of the mid-points of these chords, and $S(\theta)$ be the smallest closed strip perpendicular to $L(\theta)$ which contains $M(\theta)$, with these notations we can state our theorem as follows.

Theorem 3.1. *Let* u *be the solution of* (1.1) *or* (1.2) *corresponding to* $\gamma = 0$. *Then* u *takes its minimal value in the set* $S = \bigcap_{0 \leq \theta < \pi} S(\theta)$.

Proof. Suppose that u takes its minimal value at $p \notin S$, the $p \notin S(\theta)$ for some θ. Let ℓ be the straight line pass through P and is parallel to the strip $S(\theta)$, let Ω' be the reflection of Ω with respect to ℓ. (Figure 1) and u' be the reflection of u with respect to the vertical plane contains ℓ, then u' is the solution of (1.1) or (1.2) in Ω' with $\gamma = 0$. Consider u and u' on the convex set $G = \Omega \cap \Omega'$ and let $\Gamma = \partial G \cap \Omega'$, $\Gamma' = \partial G \cap \Omega$.

Then we have

$$(3.1) \qquad \begin{cases} \operatorname{div} Tu = H & \text{in } G \\ Tu \cdot \nu = 1 & \text{on } \Gamma \end{cases}$$

$$(3.1)' \qquad \begin{cases} \operatorname{div} Tu' = H & \text{in } G \\ Tu' \cdot \nu = 1 & \text{on } \Gamma' \end{cases}$$

or

$$(3.2) \qquad \begin{cases} \operatorname{div} Tu = Ku & \text{in } G \\ Tu \cdot \nu = 1 & \text{on } \Gamma \end{cases}$$

$$(3.2)' \qquad \begin{cases} \operatorname{div} Tu' = Ku' & \text{in } G \\ Tu' \cdot \nu = 1 & \text{on } \Gamma' \end{cases}$$

Since u and u' take minimum at P, the tangent planes of u and u' at p are horizontal, and since u' is the reflection of u with respect to the vertical plane pass through p, we have $u'(p) = u(p)$. Hence u' contacts u

at p. Thus the zero level curves of the difference function $u - u'$ divides the neighborhood of P into at least four subregions on which $u - u'$ changes its sign on the adjacent subregions, say +, -, +, - on R_1, R_2, R_3, R_4 (Figure 2). By maximal principle, the zero level curves of $u - u'$ cannot meet in the interior of G, thus each R_i must contains ∂G, moreover by using the comparison principle to (3.1) and (3.1)', or (3.2) and (3.2)', one finds that the sub-boundaries $\overline{R}_1 \cap \partial G$ and $\overline{R}_3 \cap \partial G$ cannot lie inside Γ' completely, and similarly, $\overline{R}_2 \cap \partial G$ and $\overline{R}_3 \cap \partial G$ cannot lie inside Γ completely. But this is impossible since R_1, R_2, R_3 and R_4 are adjacent regions. This complete the proof of the theorem.

Remark. If there were more than four subregions, say $R_1, R_2, \ldots, R_{2n}$, $n > 2$, on which $u - v$ takes sign as +, -, ..., +, -. One will get more subregions contradict to the comparison principle.

The following Theorem is a consequence of the uniqueness theorem of minimal point.

Theorem 3.2. *If* Ω *has two axis of symmetry, then the minimal point lies on the intersection of the axis of the symmetry for any solution of* (1.1) *or* (1.2) *with any contact angle* γ, $0 \leq \gamma < \frac{\pi}{2}$.

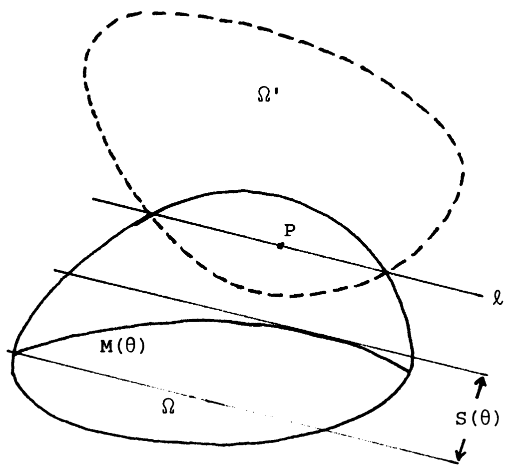

Figure 1

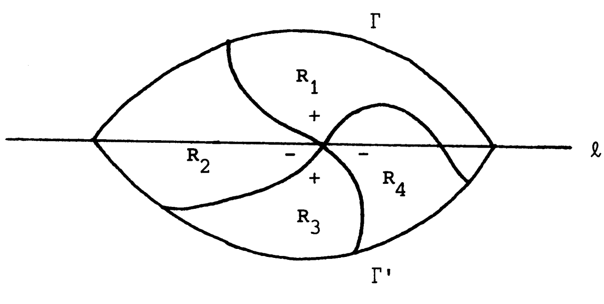

Figure 2

(contradiction happen on R_3)

REFERENCES

[1] A.D. ALEXANDROFF, Uniqueness theorems for surfaces in the large, vestnik Leningrad Univ. 11, no. 19 (1956); Amer. Math. Soc. transl. (2) 21 (1962), 341-352.

[2] J.-T. CHEN and W.-H. HUANG, Convexity of capillary surfaces in outer spaces, Invent. Math. 67 (1982), 253-259.

[3] J.-T. CHEN, On the existence of capillary free surfaces in the absence of gravity, Pacific J. Math. 88 (1980).

[4] P. CONCUS and R. FINN, On capillary free surfaces in the absence of gravity, Acta Math. 132 (1974), 177-198.

[5] R. FINN, Existence criteria for capillary free surfaces without gravity, preprint (1981).

[6] C. GERHARDT, Global regularity of the solutions to the capillary problem, Ann. Scuola Norm. Sup. Pisa IV.3 (1976), 157-175.

[7] E. GIUSTI, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math. 46 (1978), 111-137.

[8] ————, Boundary value problems for non-parametric surfaces of prescribed mean curvature, Ann. Scuola Norm. Sup. Pisa IV.3 (1976), 501-548.

[9] N. KOREVAAR, Capillary surface convexity above convex domain, preprint (1981).

[10] J. SERRIN, The Dirichlet problem for surfaces of constant mean curvature, Proc. London Math. Soc. (3) 21 (1970), 361-384.

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