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DUALITY THEOREMS FOR FOLIATIONS

by

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INTRODUCTION. In this paper we establish several results on the basic cohomology of foliations. The first main result, Theorem 1.4, establishes for any foliation \mathfrak{F} a canonical isomorphism between the cohomology of \mathfrak{F} -basic forms and the homology of transversal and holonomy-invariant currents with respect to \mathfrak{F} .

For the further statements we need the concepts of tense and taut foliations. A Riemannian foliation \mathfrak{F} on M is tense, if there exists a Riemannian metric on M such that the mean curvature of the leaves is covariant constant (i.e. parallel) along the leaves of \mathfrak{F} . If there exists a metric for which the mean curvature vanishes, then \mathfrak{F} is said to be taut [14] [15]. The harmonic foliations of [5] to [10] are foliations with minimal leaves on a Riemannian manifold (M, g_M) . Thus the harmonic foliations represent exactly the class of taut foliations.

The second main result, Theorem 2.14, establishes then for an oriented tense Riemannian foliation \mathfrak{F} on a closed oriented manifold the finite-dimensionality of the cohomology of \mathfrak{F} -basic forms, and an isomorphism of certain basic cohomology groups in complementary dimensions. This isomorphism is a precursor of the Poincaré Duality Theorem 3.1, valid for the basic cohomology of taut Riemannian foliations. Thus the original assertion of Poincaré duality in [12] is correct (exactly) in the case of taut foliations. An example of a Riemannian foliation violating Poincaré duality was found in [2]. The results of the present paper answer several questions raised during the conference in Toulouse.

1. For a foliation \mathfrak{F} on a manifold M , the complex of \mathfrak{F} -basic forms is the subcomplex $\Omega_{\mathfrak{B}}^{\cdot}(\mathfrak{F})$ of the DeRham complex $\Omega^{\cdot}(M)$ given by the forms ω satisfying $\theta(X)\omega = 0$, $i(X)\omega = 0$ for all $X \in \Gamma L$. Here $L \subset TM$ denotes the bundle of vectors tangent to \mathfrak{F} with normal bundle $Q = TM/L$. Similarly we have the complex of \mathfrak{F} -basic forms with compact support $\Omega_{c, \mathfrak{B}}^{\cdot}(\mathfrak{F})$. The differential $d_{\mathfrak{B}}$ is the restriction of d on $\Omega(M)$ to $\Omega_{\mathfrak{B}}^{\cdot}(\mathfrak{F})$, and on $\Omega_c(M)$ to $\Omega_{c, \mathfrak{B}}^{\cdot}(\mathfrak{F})$ respectively.

To explain the complex of transverse holonomy invariant currents with respect to \mathfrak{F} , we need the spectral sequence $E(\mathfrak{F})$ [5] [6]. It is associated to the following filtration of $\Omega^{\cdot}(M)$:

$$(1.1) \quad F^r \Omega^m = \{ \omega \in \Omega^m \mid i(v)\omega = 0, v = X_1 \wedge \dots \wedge X_{m-r+1}, X_i \in \Gamma L \}.$$

Then $E_1^{r, s} = H^s(M, \Lambda^r Q^{*L})$, where $\Lambda^r Q^{*L}$ is the sheaf complex of \mathfrak{F} -basic forms. Thus $E_1^{r, 0} \cong \Omega_{\mathfrak{B}}^r(\mathfrak{F})$, $d_1 = d_{\mathfrak{B}}$ and $E_2^{r, 0} \cong H^r(\Omega_{\mathfrak{B}}^{\cdot}(\mathfrak{F}))$. The spectral sequence converges to the DeRham cohomology $H_{DR}^{\cdot}(M)$ of M .

The same construction applied to $\Omega_c^{\cdot}(M)$ yields a spectral sequence $E_c(\mathfrak{F})$ converging to $H(\Omega_c^{\cdot}(M))$.

Now Haefliger [4] (and Ruelle-Sullivan [13] for $r = 0$) introduced the transversal holonomy invariant forms with compact support $\Omega_c^{\cdot}(\text{Tr}\mathfrak{F})$. This complex satisfies $E_{c, 1}^{\cdot, p} \cong \Omega_c^{\cdot}(\text{Tr}\mathfrak{F})$. For the dual space (of continuous linear functionals with respect to the C^{∞} -topology) we have canonically ($p = \dim \mathfrak{F}$)

$$(1.2) \quad E_{c, 1}^{\cdot, p} \cong C_{\cdot}(\text{Tr}\mathfrak{F}) \text{ (transversal, holonomy-invariant currents).}$$

The canonical differential $\partial_1 = d_1^T$ on the LHS corresponds precisely to the boundary on currents.

Throughout this paper we assume M to be oriented, and \mathfrak{F} a transversally oriented foliation of codimension q on M^n ($p+q = n$). We consider the duality map sending a form α to the current $c(\omega) = \int_M \alpha \wedge \omega$. This induces homomorphisms $D_1: E_1^{q-r, p-s} \rightarrow E_{c, 1}^{r, s^*}$ and in particular

$$(1.3) \quad D_1: \Omega_{\mathfrak{B}}^{q-r}(\mathfrak{F}) \rightarrow E_{c, 1}^{r, p^*}.$$

Our first result is then as follows.

1.4 DERHAM DUALITY THEOREM. The homomorphisms D_1 are compatible with differentials and induce isomorphisms

$$D_{1*}: H^{q-r}(\Omega_B^r(\mathcal{F})) \xrightarrow{\cong} H_r(E_{c,1}^{r,p^*}, \partial_1)$$

for $r = 0, \dots, q$. The canonical map

$$\text{can}: H_r(E_{c,1}^{r,p^*}, \partial_1) \rightarrow E_{c,2}^{r,p^*}$$

to continuous linear functionals on $E_{c,2}^{r,p}$ is surjective with kernel $(\bar{0})$.

For the case of the point foliation ($q = n$) this reduces to DeRham's theorem identifying $H_{DR}^r(M)$ with the homology of currents [3].

The proof consists in constructing diffusion operators

$$R_1: E_{c,1}^{r,p^*} \rightarrow \Omega_B^{q-r}(\mathcal{F})$$

and homotopy operators

$$A_1: E_{c,1}^{r,p^*} \rightarrow E_{c,1}^{r+1,p^*}$$

satisfying the following properties:

$$(1.5) \quad D_1 R_1 - \text{id}_{E_1} = \partial_1 A_1 + A_1 \partial_1 ;$$

$$(1.6) \quad R_1 \partial_1 = d_B R_1 \quad (d_B: \text{restriction of } d \text{ to } \Omega_B(\mathcal{F})) ;$$

(1.7) A_1 preserves diffuse currents (images under D_1) and induces

$$A_B: \Omega_B^{q-r}(\mathcal{F}) \rightarrow \Omega_B^{q-r-1}(\mathcal{F}) ;$$

$$(1.8) \quad R_1 D_1 - \text{id}_{\Omega_B} = d_B A_B + A_B d_B ;$$

The details of this construction will appear elsewhere. The surjectivity of the map can follow by an application of the Hahn-Banach theorem. ■

2. In the remainder of this paper we discuss the case of tense and taut Riemannian foliations (see Introduction). Let $\chi_{\mathfrak{F}}$ denote the characteristic p-form of \mathfrak{F} with respect to a bundle-like metric g_M on M [8] [10] [14]. Let ν be the transversal Riemannian volume on Q . Then $\nu \in \Omega_B^q(\mathfrak{F})$ and in fact $d\nu = 0$. We choose the orientation of $L \subset TM$ such that

$$(2.1) \quad * \nu = \chi_{\mathfrak{F}} ,$$

i.e. $\mu = \nu \wedge \chi_{\mathfrak{F}}$ represents the Riemannian volume form of M . The characteristic form $\chi_{\mathfrak{F}}$ satisfies then the formula

$$(2.2) \quad d\chi_{\mathfrak{F}} + \kappa \wedge \chi_{\mathfrak{F}} \equiv 0 \pmod{F^2 \Omega^{p+1}(M)} ,$$

i.e. modulo \mathfrak{F} -trivial forms. Here $\kappa \in \Gamma Q^*$ is the mean-curvature form of \mathfrak{F} with respect to g_M , given by $\kappa(s) = \text{Tr } W(s)$, $s \in \Gamma Q$, i.e. as the trace of the Weingarten operator of \mathfrak{F} [8],[10],[14]. \mathfrak{F} is tense, if $\kappa \in \Omega_B^1(\mathfrak{F})$ for a suitable bundle-like metric g_M on M , and in this case one has $d_B \kappa = 0$.

For a bundle-like metric there is further a star operator $\bar{*}$ in $\Omega_B^r(\mathfrak{F})$. The relationship between these operators is given by the formulas

$$(2.3) \quad \bar{*}\alpha = (-1)^{p(q-r)} *(\alpha \wedge \chi_{\mathfrak{F}}) = (-1)^{p(q-r)} i(\nu) * \alpha$$

and

$$(2.4) \quad * \alpha = \bar{*}\alpha \wedge \chi_{\mathfrak{F}} , \quad \text{for } \alpha \in \Omega_B^r(\mathfrak{F}) , \quad \nu = \xi_1 \wedge \dots \wedge \xi_p .$$

By (2.2) one has the following formula

$$(2.5) \quad d(\alpha \wedge \chi_{\mathfrak{F}}) \equiv (d_B \alpha - \kappa \wedge \alpha) \wedge \chi_{\mathfrak{F}} , \quad \alpha \in \Omega_B^r(\mathfrak{F}) ,$$

or by (2.4)

$$d * \alpha \equiv (d_B \bar{*}\alpha - \kappa \wedge \bar{*}\alpha) \wedge \chi_{\mathfrak{F}} , \quad \alpha \in \Omega_B^{q-r}(\mathfrak{F}) ,$$

where the congruence again has to be taken modulo the \mathfrak{F} -trivial forms $\mathfrak{F}^{r+1} \Omega^{p+r}$.

This motivates the following definitions.

For any $\gamma \in \Omega_B^1(\mathfrak{F})$ with $d_B \gamma = 0$ we define a new differential operator on $\Omega_B^l(\mathfrak{F})$ of degree l by

$$(2.6) \quad d_{\gamma} \alpha = d_B \alpha - \gamma \wedge \alpha, \quad d_{\gamma}^2 = 0$$

and an operator of degree -1 by

$$(2.7) \quad \bar{d}_{\gamma}^* \alpha = (-1)^{q(r-1)+1} \bar{*} d_{\gamma} \bar{*} \alpha \quad \text{for } \alpha \in \Omega_B^r(\mathfrak{F}).$$

We define further a pairing $\Psi_1: \Omega_B^r \otimes \Omega_B^{q-r} \rightarrow \mathbb{R}$ by $\Psi_1(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \chi_{\mathfrak{F}}$. This determines a scalar product on Ω_B^r by

$$(2.8) \quad \langle \alpha, \beta \rangle_B = \Psi_1(\alpha, \bar{*} \beta) \quad \text{for } \alpha, \beta \in \Omega_B^r.$$

In view of (2.4) this coincides with the canonical scalar product on $\Omega^r(M)$

$$(2.9) \quad \langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta,$$

and by (2.1) also

$$(2.10) \quad \Psi_1(\alpha, \beta) = \langle \alpha \wedge \beta, \nu \rangle_B.$$

The operators d_B and d_{κ} for a tense foliation are related by

$$(2.11) \quad \Psi_1(d_B \alpha, \beta) + (-1)^{r-1} \Psi_1(\alpha, d_{\kappa} \beta) = 0 \quad \text{for } \alpha \in \Omega_B^{r-1}, \beta \in \Omega_B^{q-r}.$$

2.12 PROPOSITION. For $\alpha \in \Omega_B^{r-1}, \beta \in \Omega_B^r$

$$\langle d_B \alpha, \beta \rangle_B = \langle \alpha, d_{\kappa}^* \beta \rangle_B$$

$$\langle d_{\kappa} \alpha, \beta \rangle_B = \langle \alpha, d_B^* \beta \rangle_B$$

i.e. \bar{d}_{κ}^* is the adjoint d_B^* of d_B , and d_B^* the adjoint d_{κ}^* of d_{κ} .

Note that this implies in particular for the transversal volume ν of \mathfrak{F}

$$(2.13) \quad \bar{d}_B^* \nu = \bar{*} \kappa.$$

We can now develop the harmonic theory for the transversally elliptic Laplacian $\Delta_B = (d_B + d_B^*)^2 = d_B d_B^* + d_B^* d_B$ as in [1] [12]. One obtains finite-dimensional spaces of harmonic forms $\mathcal{H}^r(\mathfrak{F})$ whose inclusions into $(\Omega_B^r(\mathfrak{F}), d_B)$ induce isomorphisms $\mathcal{H}^r(\mathfrak{F}) \xrightarrow{\cong} H^r(\Omega_B^r(\mathfrak{F}))$ for $r = 0, \dots, q$. The finite-dimensionality of

$H^r(\Omega_B^r, d_\kappa)$ follows in the same way by considering the Laplace operator $\Delta_\kappa = (d_\kappa + d_\kappa^*)^2$ on $\Omega_B^r(\mathfrak{F})$.

The second main result of this paper is then as follows.

2.14 THEOREM. Let \mathfrak{F} be an oriented Riemannian foliation on a closed, connected and oriented manifold M . Then the following statements are equivalent:

- (i) \mathfrak{F} is tense,
- (ii) there exists a bundle-like metric g_M , for which $d_B^* \nu \in \Omega_B^{q-1}(\mathfrak{F})$.

If \mathfrak{F} is tense, the cohomology spaces $H(\Omega_B^r, d_B)$ and $H(\Omega_B^r, d_\kappa)$ are finite-dimensional and Ψ_1 induces a non-degenerate pairing

$$(2.15) \quad \Psi_2: H^r(\Omega_B^r, d_B) \otimes H^{q-r}(\Omega_B^r, d_\kappa) \rightarrow \mathbb{R}$$

for $r = 0, \dots, q$.

The non-degeneracy of the pairing Ψ_2 gives rise to isomorphisms

$$(2.16) \quad D_{B*}: H^r(\Omega_B^r, d_B) \xrightarrow{\cong} H^{q-r}(\Omega_B^r, d_\kappa)^*$$

$$(2.17) \quad D_{B*}: H^{q-r}(\Omega_B^r, d_\kappa) \xrightarrow{\cong} H^r(\Omega_B^r, d_B)^* .$$

Let $\sigma(\alpha) = \alpha \wedge \chi_{\mathfrak{F}}$. Then σ induces a chain map $\sigma_1: (\Omega_B^r, d_\kappa) \rightarrow (E_1^{r,p}, d_1)$ by (2.5), and hence a map $\sigma_2: H(\Omega_B^r, d_\kappa) \rightarrow E_2^{r,p}$ with (continuous) dual $\sigma_2^*: E_2^{r,p*} \rightarrow H(\Omega_B^r, d_\kappa)^*$.

The map (2.16) above is then related to the duality map D_{1*} by the following commutative diagram

$$(2.18) \quad \begin{array}{ccc} H^{q-r}(\Omega_B^r, d_B) & \xrightarrow{D_{B*}} & H^r(\Omega_B^r, d_\kappa)^* \\ \cong \downarrow D_{1*} & \searrow D_2 & \uparrow \sigma_2^* \\ H_r(E_1^{r,p*}, d_1) & \xrightarrow{\text{can}} & E_2^{r,p*} \end{array}$$

in which all maps are isomorphisms as a consequence of our results. In particular D_2 is an isomorphism.

The map (2.17) appears similarly in the \pm commutative diagram

$$(2.19) \quad \begin{array}{ccc} E_2^{q-r,p} & \xrightarrow{D_1^*} & H_r(\Omega_B^*, \partial_B) \\ \uparrow \sigma_2 & \searrow D_2 & \downarrow \text{can} \\ H^{q-r}(\Omega_B, d_\kappa) & \xrightarrow[\cong]{D_B^*} & H^r(\Omega_B^*, d_B)^* \end{array}$$

Here (Ω_B^*, ∂_B) denotes the complex of basic currents.

2.20 PROPOSITION. All maps in (2.19) are isomorphisms. In particular, the duality map D_1 induces isomorphisms $D_2: E_2^{q-r,p} \cong E_2^{r,0^*}$, i.e.

$$(2.21) \quad D_2: H^{q-r}(\Omega_C(\text{Tr}\mathfrak{F})) \xrightarrow{\cong} H^r(\Omega_B, d_B)^*, \quad r = 0, \dots, q.$$

3. We consider finally the case of taut Riemannian foliations. In the notation of §2, they are characterized by $\kappa = 0$ and $d_\kappa = d_B$.

3.1 THEOREM. Let (M, \mathfrak{F}) be given as in Theorem 2.14. Then the following statements are equivalent:

- (i) \mathfrak{F} is taut;
- (ii) there exists a bundle-like metric Riemannian metric g_M such that the transversal invariant volume $\nu \in \Omega_B^q(\mathfrak{F})$ satisfies $d_B^* \nu = 0$ (or equivalently $\Delta_B \nu = 0$);
- (iii) $\dim H^q(\Omega_B^*) < \infty$, and there exists a volume form $\omega_0 \in \Gamma(\wedge^{P_1} L^*)$, such that the associated pairing $\Psi_{\omega_0}: \Omega_B^r \otimes \Omega_B^{q-r} \rightarrow \mathbb{R}$ induces a mapping $H^q(\Omega_B, d_B) \rightarrow H^0(\Omega_B, d_B)^* \cong \mathbb{R}$.

If \mathfrak{F} is taut, the basic cohomology spaces $H(\Omega_B^*, d_B)$ are finite-dimensional and Ψ_1 induces a non-degenerate pairing $H^r(\Omega_B^*) \otimes H^{q-r}(\Omega_B^*) \rightarrow \mathbb{R}$ for $r = 0, \dots, q$, i.e. the basic cohomology algebra satisfies Poincaré Duality.

A complete proof of this theorem will appear elsewhere. The pairing Ψ_{ω_0} in (iii) is defined by $\Psi_{\omega_0}(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \omega$, where $\omega \in \Omega^P(M)$ is a form representing $\omega_0 \in E_0^{0,p}$. Ψ_{ω_0} is independent of the choice of ω . We also note that the mapping $H^q(\Omega_B^*) \rightarrow H^0(\Omega_B^*)^* \cong \mathbb{R}$ is then necessarily surjective. Namely the cohomology class of the transversal volume $\nu \in \Omega_B^q$ is mapped to a non-zero number, i.e. the volume

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of M . As a consequence of the theorem, one has in fact $H^q(\Omega_B, d_B) \cong \mathbb{R}$. In a paper appearing in these proceedings, Y. Carrière asserts that the examples of (non-taut) Riemannian flows of [2] are actually tense. Thus they satisfy $H^q(\Omega_B, d_B) = 0$, but, by Theorem 2.14, $H^q(\Omega_B, d_K) \cong H^0(\Omega_B, d_B)^* \cong \mathbb{R}$ and $H^{q-1}(\Omega_B, d_K) \cong H^1(\Omega_B, d_B) \neq 0$.

Let \mathcal{F} be a compact foliation (all leaves compact). In view of Rummler's results [14] the conditions in Theorem 3.1 are satisfied exactly in the locally stable case. The leaf space B of \mathcal{F} is then a Satake manifold with a canonical submersion $f: M \rightarrow B$ and the DeRham complex of B is identified with the basic complex $\Omega_B^*(\mathcal{F})$ via f^* .

3.2 COROLLARY. If \mathcal{F} is a locally stable compact foliation on M , then the DeRham cohomology of the leaf space B satisfies Poincaré duality.

The isomorphism $D_2: E_2^{0,p} \xrightarrow{\cong} H^q(\Omega_B^*)^*$ of (2.21) for $r = q$ was established by Rummler [14] for this particular case.

Combination with the preceding results now also yields Poincaré duality for the homology of transversal holonomy invariant currents. Some geometric applications were discussed in [9] [10].

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