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Loop Spaces and Differential Forms [1]

by Kuo-Tsai Chen

Let M be a differentiable space, and $\Lambda(M)$ its de Rham complex. For example, if M is a C^∞ manifold, then $\Lambda(M)$ is the usual de Rham complex of M. If M is the differentiable space of a simplicial complex K, then $\Lambda(M)$ is the C^∞ de Rham complex of K.

Let P(M) denote the free path space of the differentiable space M. Then P(M) is also a differentiable space. Hereafter differentiable spaces and differentiable maps will be referred to simply as spaces and maps. The de Rham complex $\Lambda(P(M))$ is extremely complicated. We narrow our attention to a subcomplex that, roughly speaking, consists of those path space differential forms invariant under reparametrizations and reductions. A reduction means the replacement of a path of the type $\alpha\gamma\gamma^{-1}\beta$ by $\alpha\beta$ or vice versa. Even this complex is still very complicated. We further restrict our attention to the subcomplex $\Lambda(M)'$, which can be constructed from the de Rham complex $\Lambda(M)$ in the following manner:

Let p_0, p_1: P(M) ——> M be the end point maps given by $p_0(\gamma) = \gamma(0)$ and $p_1(\gamma) = \gamma(1)$. Then p_0 and p_1 induce respecitively maps

$$p_0^*,\ p_1^*:\ \Lambda(M) \longrightarrow \Lambda(P(M)).$$

Let η: M ——> P(M) be the map sending $x \in M$ to the constant path at x. There is a standard homotopy F from the map p_0 to the identity map of P(M). This homotopy is carried out by contracting each path along itself to tis initial point. The induced cochain homotopy

$$\int_F :\ \Lambda(P(M)) \longrightarrow \Lambda(P(M))$$

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which is a graded map of degree -1 such that $\int_F du + d\int_F u = u - (\eta p_0)^* u$, $u \in \Lambda(P(M))$. We define $\Lambda(M)'$ to be the smallest subalgebra of $\Lambda(P(M))$ generated by $p_0^*\Lambda(M)$ and $p_1^*\Lambda(M)$ and stable under $\int_F$. It turns out that $\Lambda(M)'$ is a subcomplex of $\Lambda(P(M))$ and is spanned by elements of the type $p_0^* w' \wedge (\int w_1 \cdots w_r) \wedge p_1^* w''$, where $w', w'', w_1, \ldots, w_r$ are forms on M and, inductively, $\int w_1 = \int_F p_1^* w_1$ and, for $r > 1$,

$$\int w_1 \cdots w_r = \pm \int_F [(\int w_1 \cdots w_{r-1}) \wedge p_1^* w_r] .$$

Remark 1. In the past, we have used two alternative choices of the definition of paths in M. A path means either a C^∞ path or a piecewise C^∞ path. Since differential forms belonging to $\Lambda(M)'$ are invariant under reparametrization of paths, either choice of the definition of paths will yield the same complex $\Lambda(M)'$.

Remark 2. If M is a C^∞ manifold, then $(\Lambda(M)')^0$ contians sufficiently many path space functions to distinguish two piecewise regular paths in M up to a reparametrization and a finite number of reductions.

Path space differential forms of the type $\int w_1 \cdots w_r$ are called iterated (path) integrals. The complex $\Lambda(M)'$ has an ascending filtration by length of iterated integrals. Let $\{E_r, d_r\}_{r \geqq 0}$ denote the resulting spectral sequence. Assume that the de Rham cohomology group $H_{DR}^*(M)$ is of finite type. We have

$$E_1^p = H_{DR}^*(M) \otimes [\otimes^p H_{DR}^{*-1}(M)] \otimes H_{DR}^*(M) ,$$

where $(H_{DR}^{*-1}(M))^{q-1} = H_{DR}^q(M)$, $q \geqq 1$. Moreover the differential d_1 can be explicitly expressed in terms of the cup product. Each term E_r is of finite type and represents a contravariant functor on the homotopy category of differentiable spaces.

We shall now give an account of developments of this aspect of path space differential forms beyond what has already been described in [Ch 1]. (See also [Ce] and [L].)

If $f: N \longrightarrow M\times M$ is a map, then there is a pullback diagram

$$\begin{array}{ccc} E_f & \xrightarrow{\tilde{f}} & P(M) \\ \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle (p_0,p_1)} \\ N & \xrightarrow{f} & M\times M \end{array} .$$

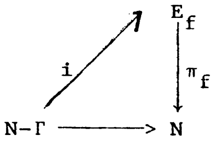
The de Rham complex $\Lambda(E_f)$ of the pullback space E_f contains a subcomplex

$$\Lambda(M)'_f = \pi_f^*\Lambda(N)\wedge\tilde{f}^*\Lambda(M)'$$

with ascending filtration induced by that of $\Lambda(M)'$. The resulting spectral sequence will be denoted by $\{E(f)_r, d_r\}_{r\geqq 0}$.

We mention here some interesting cases of the pullback space E_f. (a) If $f_i: N_i \longrightarrow M$, $i=0,1$, are inclusions, then $E_{f_0\times f_1}$ is the space of paths from N_0 to N_1. (b) If $f: M \longrightarrow M\times M$ is the diagonal map, then E_f is the free loop space. (c) If $f_0: N_0 \longrightarrow M$ is a fibration and if $E \longrightarrow N_1$ is the pullback of f_0 of the Eilenberg-Moore type through a map $f_1: N_1 \longrightarrow M$, then $E_{f_0\times f_1}$ is of the homotopy type as E .

Individual differential forms in $\Lambda(M)'_f$ may provide geometrical information regarding the map f through the Stokes formula. We illustrate with a simple but rather restrictive situation as follows: Let N be a compact oriented C^∞ n-manofold and let $f: N \longrightarrow M\times M$ be a map. Let Γ be a set of zero measure in N so that, for sufficiently small $\varepsilon > 0$, an ε-tubular neighborhood $T(\varepsilon)$ of Γ can be constructed. Let there be a map $N-\Gamma \longrightarrow P(M)$ so that there is a commutative diagram



If $v \in \Lambda^{n-1}(M)'_f$ is such that $dv = \pi_f^* w$ for some closed n-form w on N, then

(1) $$\int_N w = \int_{N-\Gamma} i^* dv = \lim_{\varepsilon\to 0} \int_{\partial(N-T(\varepsilon))} i^* v = -\lim_{\varepsilon\to 0} \int_{\partial T(\varepsilon)} i^* v ,$$

which is eseentially a residue and has possible geometrical significance.

Example 1. If ω is a closed 2-form on $M = CP^m$ representing the generator of $H^2(CP^m;Z)$, and let $f = (f_0, f_1)$, where $f_0, f_1 \colon N \longrightarrow CP^m$ are two C^∞ maps. If Γ consists of those points of N so that $f_0 y$ and $f_1 y$ are orthogonal in CP^m, then there is a map $N-\Gamma \longrightarrow P(CP^m)$ sending y to the geodesic from $f_0 y$ to $f_1 y$. Under mild conditions, the formula (1) leads to the conclusion that, with multiplicities counted, Γ is Poincaré dual to $f_1^*\omega - f_0^*\omega$. (See [Ch 6].)

Example 2. Let τ be the normalized volume element of $M = S^n$, and let $f = (f_0, f_1) \colon N = S^{2n+1} \longrightarrow S^n \times S^n$, where f_0 is the constant map to a given point $x_0 \in S^n$, and f_1 is a C^∞ map. Let $x_1 \in S^n$ be antipodal to x_0, and let $\Gamma = f_1^{-1}\{x_1\}$. Then there is a map $N-\Gamma \longrightarrow P(S^n)$ sending y to the geodesic from x_0 to $f_1 y$. Let $du = f_1^* \tau$ and

$$v = -\tilde{f}^* \int \tau\tau + \tilde{f}^* \int \tau \wedge \pi_f^* u .$$

Then

$$dv = \tilde{f}^*((-1)^n \int \tau \wedge p_1^* \tau) + \tilde{f}^* p_1^* \tau \wedge \pi_f^* u - (-1)^n \tilde{f}^* (\int \tau \wedge p_1^* \tau) = \pi_f^*(f_1^* \tau \wedge u) ,$$

and the formula (1) becomes

(2) $$\int_{S^{2n+1}} f_1^* \tau \wedge u = -\lim_{\varepsilon\to 0} \int_{\partial T(\varepsilon)} i^* v .$$

A calculation shows that the r.h.s. is equal to the linking number of $f_1^{-1}\{x_0\}$ and Γ as cycles. This is precisely the Hopf invariant of f_1 as originally discovered by Hopf. The integral formula (2) of Whitehead is now in a rather general perspective.

From a structural theoretical point of view, it is important to determine the correctness of the cohomology of $\Lambda(M)_f'$. The first step is the case of the point map $\varepsilon: \{0\} \longrightarrow M\times M$ given by $0 \longmapsto (x_o, x_o)$. Then E_ε is the loop space $\Omega(M)$. If M is simply connected, then the answer is affirmative under reasonable conditions. In addition, Gugenheim [G] shows a general theorem which implies that the isomorphism $H^*(\Lambda(M)_\varepsilon') \simeq H^*(\Omega(M);R)$ is multiplicative under usual circumstances.

If M and N are C^∞ manifolds with finite Betti numbers and if M is simply connected, then there is an isomorphism

$$H(\Lambda(M)_f') \simeq H^*(E_f;R)$$

via integration [Ch 3].

In the case of pullback fibrations of the Eilenberg-Moore type as mentioned in (c), the complex $\Lambda(M)_f'$ is isomorphic to that of two sided bar construction. It seems that the correctness of the cohomology of $\Lambda(M)_f'$ can be established through the work of Wu [W] and Givel [G] at least in the simplicial situation.

A discussion of the algebraic structure of $\Lambda(M)_f'$ can be found in [Ch4], and some applications are given in [Ch 5].

In order to compute the spectral sequence $\{E(\varepsilon)_r\}$ of $\Lambda(M)_\varepsilon'$, it is advantageous to consider the dual aspect. Dualizing the $E(\varepsilon)_1$ term, we have

$$(E(\varepsilon)_1^p)^* = \otimes^p H_{*-1}(M;R)$$

provided the de Rham theorem holds. Thus $E(\varepsilon)_1^*$ has the structure of the graded tensor algebra $T(V)$ based on the graded vector space $V = H_{*+1}(M;R)$. It turns out that we can equip $T(V)$ with a differential ∂ of degree -1 so that $(T(V),\partial)$ is a d.g. algebra having a descending filtration. The resulting spectral sequence can be identified with the dual spectral sequence $\{E(\varepsilon)_r^*, d_r^*\}$. The differential ∂ is a perturbation of d_1^* determined by a twisting cochain condition

$$(d\otimes 1 + 1\otimes\partial)\omega = (J\otimes 1)\omega\wedge\omega$$

where ω is an element of $\prod \Lambda^p(M)\otimes T(V)^{p-1} \subset \Lambda(M)\hat{\otimes} T(V)$, and $J: \Lambda(M) \longrightarrow \Lambda(M)$ is given by $Jw = (-1)^{\deg w}w$.

The significance of this dual approach is geometrical. The element ω is acturally a connection form in a differential geometrical sense, and there is an associated transport or a holonomy map defined through iterated integrals. We call ω a homology connection. The transport is a chain map preserving the path multiplication and induces an isomorphism of algebras

$$H_*(\Omega(M);R) \simeq H_*(T(V),\partial)$$

under mild condition including simply-connectedness of M. (See [Ch 1,2], [Cu], [H1,2].)

It has been sensed that the theories of minimal models and iterated integrals should be related to each other in the frame work of Quillen's rational homotopy theory. Precise relations between the two theories are established by Hain and Tanré. Their works are independent and overlap each other in part. Both of them make essential use of results on minimal Lie algebras and related constructions due to Baues and Lemaire. In [T1,2], Tanré gives and explicit account of interrelations of the two theories including homology connections and works of Adams -Hilton, Eilenberg-Moore, Felix and Halperin-Stasheff.

In the context of this talk, Hain's main results [H1,2] is briefly as follows: Let $\rho: \underline{M} \longrightarrow \Lambda(M)$ be the minimal model, where $\underline{M}$ is freely generated by $\{x_i\}$. Let $\{X_\lambda\}$ denote a basis for $V = H_{*-1}(M;R)$. Then there exist an element an element $\tilde{\omega} \in \underline{M}\hat{\otimes}L(V)$ and a decomposible differential of the graded free Lie algebra $L(V)$ satisfying the twisting cochain condition $(d\times 1+1\times\partial)\tilde{\omega} = J\tilde{\omega}\wedge\tilde{\omega}$ and also one of the following two equivalent conditions:

(a) $\tilde{\omega} = \sum w_\lambda X_\lambda + \sum w_{\lambda\mu} X_\lambda X_\mu + \ldots$

where $\{w_\lambda\}$ are closed in $\underline{M}$ and represent a basis of $H^+(\underline{M})$, which is dual to $\{X_\lambda\}$.

(b) $\tilde{\omega} = \sum x_i W_i + \sum x_i x_j W_{ij} + \ldots$

where $\{W_i\}$ are closed in $L(V)$ and represent a basis of $H(L(V))$.

As methods for computation, minimal models and homology connections have differential advantages and seem to complement each other. Hain's dual approach combines both methods in one and provides effective computations in rational homotopy. Moreover $\tilde{\omega}$ descends to a homology connection, where iterated integrals are available for possible further information [H3].

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