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TOPOLOGICAL MINIMAL ALGEBRAS AND SULLIVAN - DE RHAM EQUIVALENCE

by Hans Michael UNSÖLD

1. Statement of theorem.

Let $\mathcal{QS}_1$ denote the category of 1-reduced rational simplicial sets and let $\mathcal{M}in_1$ denote the category of 1-connected minimal topological algebras over $\mathcal{Q}$ (see section 3 below).

Theorem: There exist functors $\mathcal{M}$ and $\mathcal{G}$ inducing an equivalence of homotopy categories

$$\mathcal{M} : Ho(\mathcal{QS}_1) \rightleftarrows Ho(\mathcal{M}in_1) : \mathcal{G}$$

For $M \in \mathcal{M}in_1$ and $X \in \mathcal{QS}_1$ we have :

(i) $H^*(M) \cong H^*(\mathcal{G}M;\mathcal{Q})$

(ii) $(QM)' \cong \pi_*(\mathcal{G}M)$

(i)' $H^*(X;\mathcal{Q}) \cong H^*(\mathcal{M}X)$

(ii)' $\pi_*(X) \cong (Q\mathcal{M}X)'$

Here $QM := M^+ / \text{closure}(M^+.M^+)$, and $'$ denotes the topological dual.

Remark: If $(M,d) = (\Lambda Z,d)$ is an "ordinary" minimal algebra then we can never have properties (i) and (ii) together (whatever $\mathcal{G}$ may be !). Look at the following

Example: Fix an integer $r \geq 2$ and a $\mathcal{Q}$-vectorspace V. Let $Z = \bigoplus_i Z^i$, $Z^i = 0$ ($i \neq r$) , $Z^r = V$, $d = 0$. Assuming (i) we get $H^r(\mathcal{G}M;\mathcal{Q}) \cong V$, and assuming (ii)

we get : $H^r(\mathcal{G}M;\mathbb{Q}) \cong \mathrm{Hom}(H_r(\mathcal{G}M),\mathbb{Q}) \cong \mathrm{Hom}(\pi_r(\mathcal{G}M),\mathbb{Q}) \cong V''$.
But in general $V \ncong V''$, so this example suggests to look out for a topology guaranteeing $V \cong V''$.

2. Linear topologies.

We regard $\mathbb{Q}$, the field of rationals , as a discrete space.

Definition (see [L]): A topological vectorspace V over $\mathbb{Q}$ is said to be linearly topological (abbrev: l.t.) iff it is hausdorff and there is a fundamental system of neighbourhoods of 0 consisting of nuclear (i.e. open - closed) subspaces .
V is called linearly compact (l.c.) iff any filterbase $\mathcal{F}$ consisting of affine subspaces has a clusterpoint , i.e. $\bigcap\{\mathrm{closure}(F) : F \in \mathcal{F}\} \neq \emptyset$.

Let V be a l.t.space. We topologize the topological dual V' by requiring that for any l.c. subspace K of V the annihilator $K^0 := \{\psi \in V' : \psi(K) = 0\}$ is nuclear in V' .

Theorem (S.Lefschetz)

(a) A l.t. space V is l.c. iff V' is discrete.
(b) If V is l.c. or discrete then $V \cong V''$.
(c) If V,W are discrete (resp. l.c.) then $L(V,W) \cong L(W',V')$.

Remark: There is another interisting link between l.c. spaces and discrete spaces (see [L]):
Any two of the following properties imply the third:

(1) V is discrete.
(2) V is l.c.
(3) $\dim V < \infty$.

Proposition: Given two l.t.spaces V, W there is a linear topology on $V \otimes W$ such that the canonical map $V \times W \to V \otimes W$ is universal with respect to uniformly continuous bilinear maps.

We denote by $V \hat{\otimes} W$ the completion of $V \otimes W$.
If V, W are l.c. (resp. discrete) then $V \hat{\otimes} W$ is l.c. (resp. discrete) and $(V \hat{\otimes} W)' \cong V' \hat{\otimes} W'$.

Now let $A = \bigoplus_{p \geq 0} A^p$ be a (differential) graded algebra.

(We assume all our algebras to be augmented an commutative in the graded sense.)

A is called a complete (D)GA iff:

(i) A^p is a complete l.t.space for all $p \geq 0$.
(ii) Multiplication $A^p \times A^q \to A^{p+q}$ is uniformly continuous.

Call A linearly compact if A^p is l.c. for all p.

3. Topological minimal algebras.

Let $V = \bigoplus_{i \geq 1} V^i$ be a connected graded l.t.space.

Proposition: There is a linear topology on ΛV such that the usual universal mapping property holds with respect to continuous maps.

We denote by FV the completion of ΛV . FV is a complete graded algebra in the sense of our definition.
If V is l.c. in each degree then FV is l.c. and $(FV)'$ is the symmetric coalgebra over V' . If V is discrete then ΛV is discrete, hence $\Lambda V = FV$.

Definition: A complete 1-connected DGA (M,d) is called minimal (in the topological sense) iff:

(1) Disregarding differentials $M \cong FV$ for some graded 1-connected l.c.space V .

(2) $d(V^n) \subset F(V^2 \oplus V^3 \oplus \ldots \oplus V^{n-1})$ for all $n \geq 2$.

If M happens to be of finite type then M is a (discrete) minimal algebra in the usual sense.

4. Minimal models for simplicial sets.

Let X be a simplicial set and let $A^*(X)$ be the algebra of $\mathbb{Q}$-polynomial forms on X (as in [BG]).
Topologize $A^*(X)$ as follows : for any simplicial map $\tilde{x}:\Delta^q \to X$ the subspace $\ker(A^p(\tilde{x}):A^p(X) \to A^p(\Delta^q))$ is nuclear in $A^p(X)$.
Then $A^*(X)$ is a complete DGA.
Now assume X,Y to be 1-reduced simplicial sets.

Proposition:

(a) There exist $\mathcal{M}X \in \mathcal{M}in_1$ and a weak equivalence $e_X:\mathcal{M}X \to A^*(X)$.

(b) Given a simplicial map $f:X \to Y$ there exists $\mathcal{M}f:\mathcal{M}Y \to \mathcal{M}X$, unique up to algebraic homotopy , such that $e_X o \mathcal{M}f$ is homotopic to $A^*(f) o e_Y$. Furthermore if f is homotopic to g then $\mathcal{M}f$ is homotopic to $\mathcal{M}g$.

The rule $X \rightsquigarrow \mathcal{M}X$, $f \rightsquigarrow \mathcal{M}f$ defines a functor $\mathcal{M} : Ho(QS_1) \longrightarrow Ho(\mathcal{M}in_1)$.

5. An adjoint for $\mathcal{M}$.

For any complete DGA M let $\mathcal{G}M$ be the simplicial set given by $(\mathcal{G}M)_q = \mathrm{Hom}(M, A^*(\Delta^q))$. It is easy to prove that A^* and $\mathcal{G}$ are adjoint functors

$$A^* : \text{simplicial sets} \rightleftarrows \text{complete DGA's} : \mathcal{G} .$$

Let $\rho_M : M \to A^*\mathcal{G}M$, $\tau_X : X \to \mathcal{G}A^*X$ denote the adjunction maps.
Using a little abstract homotopy theory (as in [B]) it can be shown that $\mathcal{M}$ and $\mathcal{G}$ induce adjoint functors

$$\mathcal{M} : Ho(QS_1) \rightleftarrows Ho(\mathcal{M}in_1) : \mathcal{G}$$

It remains to show that :

(1) The adjunction $\tilde{\rho}_M : M \to \mathcal{MG}(M)$ $(M \in \mathcal{M}in_1)$ is a weak equivalence. (In fact $\tilde{\rho}_M$ can be shown to be an isomorphism.)

(2) The adjunction $\tilde{\tau}_X : X \to \mathcal{GM}X$ $(X \in \mathcal{QS}_1)$ is a homotopy equivalence.

If $e_{\mathcal{MG}M} : \mathcal{MG}M \to A^*(\mathcal{G}M)$ is the minimal model of $\mathcal{G}M$ then the adjunction map $\tilde{\rho}_M$ is defined up to homotopy by requiring that $e_{\mathcal{MG}M} \circ \tilde{\rho}_M$ is homotopic to ρ_M.
In order to prove (1) it suffices to show that ρ_M is a weak equivalence.
First assume that $M = FV$, $d=0$, where V is concentrated in some degree $r \geq 2$ (i.e. $V^i = 0$ if $i \neq r$).
There exists an inverse system $\{W_\alpha\}_\alpha$ of finite dimensional spaces such that $V \cong \text{inv lim } W_\alpha$.
The maps $\rho_{\Lambda W_\alpha} : \Lambda W_\alpha \to A^*\mathcal{G}\Lambda W_\alpha$ are weak equivalences (see [BG]).
It can be checked that $FV \cong \text{inv lim } \Lambda W_\alpha$, $H^*A^*\mathcal{G}FV \cong \text{inv lim } H^*A^*\mathcal{G}\Lambda W_\alpha$ and that ρ is compatible with inverse limits.

Definition: Let (M,d) be a complete DGA, W a l.c.space (not graded). Denote by (W,r) the graded space given by W in degree r and 0 otherwise. Let $t : W \to Z^{r+1}(M)$ be a linear map and define a differential d_t on $M \mathbin{\hat{\otimes}} F(W,r)$ by $d_t|M = d$, $d_t|W = t$.
The algebra $(M \mathbin{\hat{\otimes}} F(W,r), d_t)$ is denoted by $M \mathbin{\hat{\otimes}_t} F(W,r)$ and is called an elementary extension of M.
The homology class $[t] \in H^{r+1}(L(W,M)) \cong L(W,H^{r+1}(M))$ is called the structure class.

Proof of (1), general M : We proceed by induction over the elementary extensions $F(V^{<n}) \to F(V^{\leq n})$.
The inductive step is achieved with the help of the following propositions.

Proposition: Assume that $\rho_M : M \to A^*(\mathcal{G}M)$ is a weak equivalence. Let $M \to N = M \mathbin{\hat{\otimes}_t} F(W,r)$ be an elementary extension.
Then $\mathcal{G}N \to \mathcal{G}M$ is a principal simplicial fibre bundle with fibre $F = K(W',r)$ and the transgression $T : W \cong H^r(F;\mathbb{Q}) \to H^{r+1}(\mathcal{G}M;\mathbb{Q}) \cong H^{r+1}(M)$ is given by the structure class $[t]$.

Proposition (Hirsch - lemma) : Let $F \to E \to B$ be a principal simplicial fibre bundle with fibre $F = K(\pi,r)$, π a rational vectorspace (discrete).
Let $e_B : \mathcal{M}B \to A^*(B)$ be a minimal model of B and suppose that the transgression $T : H^r(F;\mathbb{Q}) \to H^{r+1}(B;\mathbb{Q})$ is represented by some map $t : \pi' \to Z^{r+1}(\mathcal{M}B)$.
Then there is a weak equivalence

$$\mathcal{M}B \mathbin{\hat{\otimes}_t} F(\pi',r) \xrightarrow{\sim} A^*(E) .$$

The proof of (2) is now very easy. $\tilde{\tau}_X$ is defined by $\tilde{\tau}_X = \mathcal{G}(e_X) \circ \tau_X$ where $e_X : \mathcal{M}X \to A^*(X)$ is the model of X. A straightforward computation shows that $e_X = A^*(\tilde{\tau}_X) \circ \rho_{\mathcal{M}X}$. e_X and $\rho_{\mathcal{M}X}$ are weak equivalences hence $A^*(\tilde{\tau}_X)$ is a weak equivalence and by duality of $H^*(-;\mathbb{Q})$ and $H_*(-;\mathbb{Q})$ we get $\tilde{\tau}_{X*} : H_*(X;\mathbb{Q}) \cong H_*(\mathcal{G}\mathcal{M}X;\mathbb{Q})$. Since X and $\mathcal{G}\mathcal{M}X$ are 1-connected rational simplicial sets an application of Whitehead's theorem gives the desired result.

References

[B] K.S. Brown : Abstract homotopy theory and generalized sheaf cohomology. Transact.Amer.Math.Soc. 186(1973) 419-458.

[BG] A.K. Bousfield , V.K.A.M. Gugenheim : On PL de Rham theory and rational homotopy type. Memoirs of the Amer.Math.Soc. 179(1976)

[L] S. Lefschetz : Algebraic topology . Amer.Math. Soc. Colloq. Publ. Vol. XXVII , 1942 .

[U] H.M. Unsöld : Über die Sullivan - de Rham Theorie einfach zusammenhängender simplizialer Mengen. Diplomarbeit , Freie Universität Berlin , 1982.