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RATIONAL HOMOTOPY TYPES OF COFIBRATIONS

Hiroo SHIGA

1. Main results.

In this note we study the rational homotopy types of cofibrations. Spaces are always assumed to be connected and simply connected CW-complex and cohomologies are those with rational coefficient. The rational homotopy type of a space is described by Sullivan's theory of minimal model. A space is called a formal space if its minimal model can be constructed by its cohomology ring ($[1]$). Let X be a formal space and $m(X)$ be its minimal model. Then there is a D.G.A. homomorphism $\rho_X : m(X) \rightarrow H(X)$ which induces an isomorphism on the cohomology. There is also a decomposition in each degree i , $m^i(X) = \bigoplus_{j \geq 0} W_j^i$ such that $W_j^i \cdot W_k^m \subset W_{j+k}^{i+m}$,

$$dW_j^i \subset W_{j-1}^{i+1} \quad ([2], [3]).$$

Let Y a finite complex and formal and $f : Y \rightarrow X$ be a map. In general the cofiber of f is not a formal space, so we introduce the following condition :

(P) There is a decomposition $m^i(X) = \bigoplus W_j^i$ and ρ_Y as above such that $\rho_Y f : m(X) \rightarrow H(Y)$ satisfies $\rho_Y f|W_j^i = 0$ for $j \geq 2$. Here we use the same notation f for the induced map between minimal models. Then we have

THEOREM : The cofiber $X \bigcup_f CY$ is a formal space if the condition (P) is satisfied.

If there are ρ_X and ρ_Y such that $\rho_Y \circ f = f^* \circ \rho_X$ then f is called a formal map. A formal map satisfies condition (P). Hence

COROLLARY 1 : The cofiber of f is a formal space if f is a formal map.

This answers to a problem listed in [4] under the assumption that Y is a finite complex.

Theorem is proved by reducing to the case where f^* is surjective.

2. Proof of Theorem.

The proof of Theorem consists of two steps :

A. Reduction to the case where $\rho_Y \circ f : m(X) \rightarrow H(Y)$ is surjective.

B. Reduction to the case where f^* is surjective.

First we prove step A. Let I_1 be an ideal of $H(Y)$ generated by the image of $\rho_X \circ f$ and J_1 be a quotientring of $H(Y)$ by I_1 .

Let Y_1 and Z_1 be formal spaces, whose cohomology rings are isomorphic to I_1 and J_1 respectively. Then we have a cofibration

$$Z_1 \xrightarrow{q_1} Y \xrightarrow{p_1} Y_1$$

where q_1 and p_1 are formal maps which induce the quotient map and the inclusion map on cohomologies. Since $q_1^* \circ \rho_Y \circ f : m(X) \rightarrow H(Z_1)$ is null homomorphism $q_1^* f$ is null homotopic by Theorem (1.2) in [1]. Hence f factors through Y_1 as follows :

$$\begin{array}{ccc} & Y_1 & \\ & \uparrow p_1 & \searrow f_1 \\ & Y & \nearrow f_2 \\ & & X \end{array}$$

Then we have a homotopy commutative diagram of cofibrations.

$$\begin{array}{ccc} Y_1 & \xrightarrow{\quad} & X \vee SZ_1 \\ \uparrow & & \uparrow \\ Y & \xrightarrow{\quad} & X \\ \uparrow & & \uparrow \text{0-map} \\ Z_1 & = & Z_1 \end{array}$$

By considering the horizontal cofibers we have a rational homotopy equivalence :

$$\begin{array}{ccc} X \cup CY & & (X \cup CY_1) \vee SZ_1 \\ f & & f_1 \end{array}$$

where SZ_1 is a suspension of Z_1 and it has a same rational homotopy type of wedge of spheres.

Next we take an ideal I_2 of I_1 and do the same process.

Proceeding this process we arrived at the situation where image of $\rho_{Y_{n-1}} \circ f_{n-1}$

is itself an ideal in I_{n-1} since $\dim H(Y) < \infty$. Then we have a space Y_n and a map $f_n : Y_n \rightarrow X$ such that $\rho_{Y_n} \circ f_n : m(X) \rightarrow H(Y_n)$ is surjective and rational homotopy equivalence : $\begin{array}{ccc} X \cup CY & \rightarrow & (X \cup CY_n) \vee S^k_i \\ f & & f_n \end{array}$.

Now we prove step B. We take an ideal I_{n+1} of $H(Y_n)$ generated by the image of f_n^* and J_{n+1} be a quotient ring of $H(Y_n)$ by I_{n+1} . Let Y_{n+1} and Z_{n+1} be a corresponding formal spaces. Then we have a diagram :

$$\begin{array}{ccccccc} & & Y_{n+1} & & & & \\ & & \uparrow p_{n+1} & \searrow f_{n+1} & & & \\ Z_{n+1} & \xrightarrow{q_{n+1}} & Y_n & \xrightarrow{f_n} & X & \xrightarrow{c_{n+1}} & X \cup_{f_n q_{n+1}} CZ_{n+1} \end{array}$$

where c_{n+1} is a natural map and we have a map f_{n+1} such that $p_{n+1} \circ f_{n+1}$ is homotopic to $c_{n+1} \circ f_n$. By similar way as in step A we have a rational homotopy equivalence

$$\begin{array}{ccc} X \cup CY_n & \longrightarrow & (X \cup CZ_{n+1}) \cup_{f_{n+1}} CY_{n+1} \\ f_n & & \end{array}$$

Proceeding this process we have a cofibration

$$Y_{n+k} \rightarrow ((X \cup CZ_{n+1}) \dots) \cup CZ_{n+k} \rightarrow ((X \cup CZ_{n+1}) \dots) \cup CY_{n+k}$$

such that f_{n+k} and rational homotopy equivalence.

$$X \cup CY_n \longrightarrow (X \bigcup_{n+1}^{n+k} CZ) \cup CY_{n+k}$$

where $X \bigcup_{n+1}^{n+k} CZ$ denotes $(\dots(X \cup CZ_{n+1})\dots) \cup CZ_{n+k}$.

Now we need to show

LEMMA : Each Z_i , $n+1 \leq i \leq n+k$, is rational homotopy equivalent to a wedge of spheres and $X \bigcup_{n+1}^{n+k} CZ$ is a formal space.

PROOF : By condition (P) any element in $H(Z_i)$ is in the image of $q_i^* \circ \rho_{Y_{i-1}} \circ f_{i-1}(W_1^*)$. Hence any product of elements in $H(Z_i)$ is zero.

Therefore by formality Z_i is a wedge of spheres. Next we show the formality of

$X \bigcup_{n+1}^{n+k} CZ$. There is an a selfmap $P_t : Z_{n+1} \rightarrow Z_{n+1}$ such that $P_t^* = t^{i+1} :$

$H^i(Z_{n+1}) \rightarrow H^i(Z_{n+1})$ and grading automorphism $G_t : m(X) \rightarrow m(X)$ such that

$G_t|_{W_j^i} = t^{i+j} \text{Id}$. Then by condition (P) we have a homotopy commutative diagram :

$$\begin{array}{ccccc} Z_{n+1} & \xrightarrow{\quad} & X & \xrightarrow{\quad} & X \cup CZ_{n+1} \\ \downarrow P_t & & \downarrow G_t & & \downarrow \\ Z_{n+1} & \xrightarrow{\quad} & X & \xrightarrow{\quad} & X \cup CZ_{n+1} \end{array}$$

and a map G_t^{n+1} such that $(G_t^{n+1})^* = t^i \text{Id}$ on $H^i(X \cup CZ_{n+1})$.

Hence $X \cup CZ_{n+1}$ is a formal space. By induction we can show that $X \bigcup_{n+1}^{n+k} CZ$ is a formal space.

This ends the step B.

Now we consider the case where f^* is surjective.

Let $Y \xrightarrow{f} X \xrightarrow{g} Z$ be a cofibration such that f^* is surjective.

Let $A(Z)$ be a kernel of $f : m(X) \rightarrow m(Y)$. Then $A(Z)$ is a free D.G.A.

quasi-isomorphic to $m(Z)$. Since c^* is injective the map $\rho_X|_{A(Z)} : A(Z) \rightarrow H(X)$ induce isomorphism on $H(Z)$. Hence Z is a formal space and this completes the proof of Theorem.

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