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SULLIVAN-QUILLEN MIXED TYPE MODEL FOR FIBRATIONS
AND THE HAEFLIGER MODEL FOR THE GELFAND-FUKS COHOMOLOGY

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1. Introduction (The Bott "Conjecture").

Let M be a paracompact Hausdorff C^∞ -manifold of dimension $n \geq 1$ and L_M be the topological Lie algebra of C^∞ -vector fields on M . Gelfand-Fuks [1] considered the differential graded algebra (DGA for brevity) $C_c^*(L_M)$ of continuous cochains of L_M , and its cohomology $H^*(C_c^*(L_M))$ is called the Gelfand-Fuks cohomology of M .

On the other hand, let $EU_n^{(2n)} \rightarrow BU_n^{(2n)}$ be the universal U_n -bundle restricted over the (homotopical) $2n$ -skelton of the base and

$$(1.1.) \quad \hat{\gamma}_n : EU_n^{(2n)} \rightarrow EU_n^{(2n)} \times_{U_n} EU_n \rightarrow BU_n$$

be the associated fiber bundle over BU_n with fiber $EU_n^{(2n)}$. And let τ_M^C be the complexification of the tangent bundle of M classified by a map $f_M^C : M \rightarrow BU_n$. Consider the cross-section space $\Gamma((f_M^C)^* (\hat{\gamma}_n))$ of the induced bundle $(f_M^C)^* (\hat{\gamma}_n)$ equipped with the compact open topology. Then the Bott "Conjecture" asserts ;

$$(1.2.) \quad H^*(C_c^*(L_M)) \cong H^*(\Gamma((f_M^C)^* (\hat{\gamma}_n)); \mathbb{R}).$$

A. Haefliger [3], [4] affirmatively solved this conjecture by constructing a Sullivan-Quillen mixed type model for the fibration $(f_M^C)^* (\hat{\gamma}_n)$. Here, by a Sullivan-Quillen mixed type model for a fibration, we mean a DG Lie algebra

$L = A^* \otimes \bar{L}$ over a DGA A^* with a differential d , whose restriction $(A^*, d, A^* = d_A)$ is a model for the base space in the sense of Sullivan and whose quotient $(\bar{L} = R \otimes_{A^*} L, l \otimes_{A^*} d)$ is a model for the fiber in the sense of Quillen.

The superiority of the mixed type model lies in the following fact. The cochain complex $C_A^*(L)$ over A^* of L is a Sullivan model for the total space of the fibration while the cochain complex $C_R^*(L)$ over R of L is a Sullivan model for the cross-section space of the fibration.

But if we want to construct a mixed type model on the universal level, i.e. a model for $\hat{\gamma}_n$ itself instead of the induced one $(f_M^C)^*(\hat{\gamma}_n)$, we have no longer a differential on L but a pair (D, χ) of a derivation D on L and the Euler element χ in L_{-2} , χ being the obstruction for D to be a differential and, at the same time, being a representative for the obstruction class to the existence of a cross-section of the fibration.

In this note we give a sketch of the following two subjects, the details of which will appear elsewhere. First we present a general view of the Sullivan-Quillen mixed type model in section 2, generalizing the Haefliger-Silveira theory of mixed type model for fibrations with a given cross-section [7]. In section 3, we exhibit a very explicit description of the mixed type model for the fibration (1.1.), and thus give a complete answer to the algebraic computational problem posed by Haefliger [3]. We remark that partial results to this problem permitted us to deduce the following result.

THEOREM (1.3) ([6]) : A closed connected orientable manifold M of dimension ≥ 1 has finitely generated Gelfand-Fuks cohomology (as an R -algebra) if and only if $M = S^1$.

I am greatly indebted to S. Hurder's suggestion for accomplishing my computations of the differential in Haefliger's model. I also owe a great deal to A. Haefliger for suggesting me to generalize the mixed type model theory to fibrations without cross-section. Finally the discussions with H. Sliga clarified me the role of the Euler element in the mixed type model.

2. Sullivan-Quillen mixed type model for fibrations.

Let $A^*(= A_{-*})$ be a positively graded DGA with a differential d_A and L_* be a graded Lie algebra over A^* with the grading $\deg(a \cdot y) = \deg(y) - \deg(a)$ for $a \in A^*$ and $y \in L$.

DEFINITION 2.1. : A graded Lie algebra L_* over A^* is an algebraic fibration of mixed type over A^* if $(R \otimes_A L_*)_p = 0$ for $p \leq 0$ and if it is equipped with a pair (χ, D) , where χ is an element of L_{-2} called an Euler element and $D : L_* \rightarrow L_*$ is an A^* -Lie derivation of degree -1 , i.e.

$$(2.2) \quad D(a \cdot [y_1, y_2]) = d_A(a) \cdot [y_1, y_2] + (-1)^{\deg(a)} a \cdot \{ [D(y_1), y_2] + (-1)^{\deg(y_1)} [y_1, D(y_2)] \},$$

satisfying the following trace formulas ;

$$(2.3) \quad D(\chi) = 0, \text{ and}$$

$$(2.4) \quad (D)^2(y) = [\chi, Y] \text{ for every } y \in L_*.$$

The quotient DG Lie algebra $(R \otimes_A L_*, l \otimes D)$ is called the fiber of this algebraic fibration.

DEFINITION 2.5. : Let (L_*, χ, D) be an algebraic fibration of mixed type over A^* . Its chain complex $C_*^{A, \chi}(L_*)$ over A^* is the DG coalgebra $(S_*^A(\sigma L_*), d = d_L + D + d_\chi)$, where σL_* is the suspension of L_* (the shift of degree by $+1$), $S_*^A(\sigma L_*)$ denotes the symmetric coalgebra of σL_* taken over A^* , d_L is the usual differential on $S_*^A(\sigma L_*)$ arising from the Lie bracket of L_* , D is the coderivation on $S_*^A(\sigma L_*)$ induced by the derivation on L_* denoted by the same symbol, and d_χ is the differential which is nothing but the multiplication by the suspension $\sigma\chi$ of χ , i.e. $d_\chi(x) = \sigma\chi \cdot x$. We call d_χ the Euler differential in $C_*^{A, \chi}(L_*)$. The trace formulas (2.3) and (2.4) are equivalent to ; $d^2 = 0$ in $S_*^A(\sigma L_*)$. The cochain complex $C_{A, \chi}^*(L_*)$ over A^* of L_* is the A^* -dual of $C_*^{A, \chi}(L_*)$, namely

$$(2.6) \quad \text{Hom}_A(S_*^A(\sigma L_*), A^*) \cong A^* \otimes S_R^*(R \otimes_A \sigma L_*) ; \text{Hom}_A(d, l).$$

This is an algebraic fibration over A^* in the sense of Sullivan.

Conversely, starting from an algebraic fibration $A^* \rightarrow E^*$ in the sense of Sullivan, we can construct a mixed type fibration (L_*, D, χ) over A^* with χ being a representative of the obstruction class to the existence for a cross-section in the minimal model for the fibration above.

3. The Haefliger model.

Now we return to the fibration $\hat{\gamma}_n$ of (1.1). The minimal model for the base

space BU_n is given by

$$(3.1) \quad I^n = R[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n] ; \deg \bar{c}_i = 2i, \quad d(\bar{c}_i) = 0.$$

A model for the fiber $EU_n^{(2n)}$ is given by

$$(3.2) \quad \hat{W}_n = E(h_1, h_2, \dots, h_n) \otimes (R[c_1, c_2, \dots, c_n] / (\deg > 2n))$$

with $\deg h_i = 2i-1$, $\deg c_i = 2i$, $d(h_i) = c_i$, $d(c_i) = 0$.

A model (in the sense of Sullivan) for the total space is given by

$$(3.3) \quad I^n \otimes \hat{W}_n ; \quad d(h_i) = c_i - \bar{c}_i, \quad d(c_i) = d(\bar{c}_i) = 0.$$

The fiber $EU_n^{(2n)}$ has the rational homotopy type of a bouquet of spheres and its minimal model (in the sense of Quillen) is

$$(3.4) \quad L(\sigma^{-1} \tilde{H}^*(\hat{W}_n)') ; \quad d \equiv 0.$$

A convenient basis $\{[h_I c_J]\}$; partitions I and J satisfy certain inequalities} for $\tilde{H}^*(\hat{W}_n)$ was found by J. Vey [2].

Now $I^n \otimes L(\sigma^{-1} \tilde{H}^*(\hat{W}_n)')$ has the natural graded Lie algebra structure over I^n . We define the Euler element χ in it by

$$(3.6) \quad \chi = \sum_{\omega} \bar{c}_{\omega} \otimes \sigma^{-1} [h_{\omega_1} c_{\omega_2} c_{\omega_3} \dots]'$$

where the summation runs over all the partitions $\omega = (\omega_1, \omega_2, \omega_3, \dots)$ such that $1 \leq \omega_1 \leq \omega_2 \leq \dots$, $\omega_2 + \omega_3 + \dots \leq n$, and that $\omega_1 + \omega_2 + \omega_3 + \dots > n$. And we define I^n -Lie derivation D as a sum of two differentials d_1 and d_2 ; (c.f. [6], p. 398 for the notations)

$$(3.7) \quad D = d_1 + d_2 : I^n \otimes L(\sigma^{-1} \tilde{H}^*(\hat{W}_n)') \rightarrow I^n \otimes L(\sigma^{-1} \tilde{H}^*(\hat{W}_n)'),$$

$$(3.8) \quad d_1(1 \otimes y(I, J)) = - \sum_{(I)} \text{sign} \prod_{l \leq v} (\omega_l - i_v) \bar{c}_{\omega} \otimes y(\omega_1 + I; \omega - \omega_1 + J) \\ + \sum_{(J)} \text{sign} \prod_{l \leq v} (j_t - i) \bar{c}_{\omega} \otimes y(\omega_1 + I - i_1 + j_t; \omega - \omega_1 + i_1 + J - j_t),$$

where $y(I; J) = \sigma^{-1} [h_I c_J]'$, and

$$\begin{aligned}
 (3.9) \quad & d_2(1 \otimes y(I;J)) \\
 &= \sum_{(1)} (-1)^{|I(1)|} \text{sign} \sum_{1 \leq \mu, \nu} (i_\mu^{(1)} - i_\nu^{(2)}) \bar{c}_\omega \otimes [y(I(1);J(1)), \\
 &\quad y(\omega_1 + I(2) - i_1^{(2)}; \omega - \omega_1 + i_1^{(2)} + J(2))] \\
 &+ \sum_{(2)} (-1)^{|I(1)|} \text{sign} \prod_{1 \leq \mu, \nu} (i_\mu^{(1)} - i_\nu^{(2)}) \bar{c}_{\omega(1)} \bar{c}_{\omega(2)} \otimes \left[y(\omega_1^{(1)} + I(1) \right. \\
 &\quad \left. - i_1^{(1)}; \omega(1) - \omega_1^{(1)} + i_1^{(1)} + J(1)), y(\omega_1^{(2)} + I(2) - i_1^{(2)}; \omega(2) - \omega_1^{(2)} + i_1^{(2)} + J(2)) \right].
 \end{aligned}$$

One checks by direct computations that χ and D defined above satisfy the trace formulas (2.3) and (2.4). Thus $(I^n \otimes L(\sigma^{-1}H^*(\hat{W}_n)'), \chi, D)$ is an algebraic fibration of mixed type. Its cochain complex $C_{I^n, \chi}^*(I^n \otimes L(\sigma^{-1}H^*(\hat{W}_n)'), \chi)$ is proved to be the minimal model for the fibration (3.3).

Now let M be an n -dimensional manifold as stated in the introduction, and $\Omega^*(M)$ be its de Rham algebra. A choice of Pontrjagin forms $\tilde{p}_i \in \Omega^{4i}(M)$ makes $\Omega^*(M)$ an I^n -algebra via the homomorphism defined by $\bar{c}_{2i} \rightarrow \tilde{p}_i, \bar{c}_{2i-1} \rightarrow 0$. By the scalar extension, we obtain a DG Lie algebra over $\Omega^*(M)$

$$(3.10) \quad (\Omega^*(M) \otimes_{I^n} (I^n \otimes L(\sigma^{-1}H^*(\hat{W}_n)'), \chi) \cong \Omega^*(M) \otimes L(\sigma^{-1}H^*(\hat{W}_n)'), \chi; 1 \otimes_{I^n} D)$$

whose cochain complex $C_R^*(\Omega^*(M) \otimes L(\sigma^{-1}H^*(\hat{W}_n)'), \chi)$ over R is a model for the cross-section space $\Gamma((f_M^C) * (\hat{\gamma}_n))$. This is the Haeffliger model for the Gelfand-Fuks cochain complex $C_c^*(L_M)$. Notice that $(f_M^C) * (\hat{\gamma}_n)$ admits a unique homotopy class of cross-sections since the fiber $EU_n^{(2n)}$ is $2n$ -connected.

REMARK 3.11. : The minimal model for the algebraic fibration (3.3) is isomorphic to that of DGA $I_{(n)} = I^n / (\deg > 2n)$. So the minimal model above can also be regarded as the minimal model $M_{I_{(n)}}$. In fact, the modulo $(\bar{M}_{I_{(n)}})^3$ -reduction of the formulas (3.6)-(3.9) gives rise to formulas (2.15)-(2.19) of Hurder-Kamber [5].

Since $R[p_1, p_2, \dots, p_{n/2}] \cong I^n / (\bar{c}_{2i-1})$, we obtain the minimal model (= the Postnikov decomposition) for the algebraic fibration $P_n \rightarrow P_n \otimes \hat{W}_n$ by putting $\bar{c}_{2i-1}$ in the model above. This is a complete answer to the computational problem posed in [3].

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