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ON THE CENTRE OF GRADED LIE ALGEBRAS

by

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If $\underline{a}$ is a graded Lie algebra over a field k in general, its centre may of course be any abelian graded Lie algebra. But if some restrictions are imposed on $\underline{a}$ such as

1) $cd(\underline{a})$ ($= \text{gldim } U(\underline{a})$) $< \infty$
2) $U(\underline{a}) = \text{Ext}_R(k,k)$ R local noetherian ring
3) $\underline{a} = \pi_* S \otimes \underline{Q}$, $cat_0(S) < \infty$

what can be said about the centre?

Notation. For a graded Lie algebra $\underline{a}$, let $Z(\underline{a})$ denote its centre.

Felix, Halperin and Thomas have results in case 3) (cf [1]): Suppose $\dim_{\underline{Q}}(\underline{a}) = \infty$ then for each $k \geq 1$, $\sum_{n=k}^{2k-1} \dim_{\underline{Q}}(Z_{2n}(\underline{a})) < cat_0(S)$.

In case 1) we have the following result.

Theorem 1 Suppose $cd(\underline{a}) = n < \infty$. Then $\dim_k Z(\underline{a}) \leq n$ and $Z_{odd}(\underline{a}) = 0$. Moreover if $\dim_k Z(\underline{a}) = n$, then $\underline{a}$ is an extension of an abelian Lie algebra on odd generators by its centre $Z(\underline{a})$.

Proof. We have that $U(Z(\underline{a}))$ is a sub Hopf algebra of $U(\underline{a})$ and hence $U(\underline{a})$ is free over $U(Z(\underline{a}))$ (cf [5] , th 4.4) and from this it follows that $cd(Z(\underline{a})) \leq cd(\underline{a}) = n$. But $U(Z(\underline{a}))$ is a tensor product of a polynomial algebra on the even generators of $Z(\underline{a})$ and an exterior algebra on the odd generators of $Z(\underline{a})$. Since the global dimension of an exterior algebra is infinite, we must have $Z_{odd}(\underline{a}) = 0$ and since the global dimension of a polynomial algebra is the number of variables,we also get $\dim_k Z(\underline{a}) \leq n$. Suppose now $\dim_k Z(\underline{a}) = n$. If $x \in \underline{a}$ and $\deg(x)$ is even and $x \notin Z(\underline{a})$, then $Z(\underline{a}) \oplus k \cdot x$ is a sub Lie algebra of $\underline{a}$ of cohomological dimension $n+1$ which is impossible. Hence

$x \notin Z(\underline{a})$ implies $\deg(x)$ odd, and then it follows that $\underline{a}/Z(\underline{a})$ is abelian on odd generators.

Problem. Characterise those graded Lie algebras $\underline{a}$ having $cd(\underline{a}) < \infty$ and which are an extension of an abelian finite dimensional Lie algebra on odd generators by its centre $Z(\underline{a})$.

In case 2) we have the following result.

Theorem 2. Suppose (R,m) is an equi-characteristic local noetherian ring with $m^3 = 0$. Let $\underline{g}$ be the underlying Lie algebra of $\mathrm{Ext}_R(k,k)$ $(k = R/m)$. Then $Z(\underline{g}) = 0$ or $R = k[x]/(x^3)$ or $cd(\underline{g}) = 2$ (the last case is equivalent to saying that $\underline{g}$ is generated by its one-dimensional elements).

This theorem is a consequence of the following one.

Theorem 3. Suppose $\underline{a}$ is a graded Lie algebra and $V \neq 0$ is a syzygy in a (not necessarily minimal) resolution of k over $U(\underline{a})$. Let $\underline{g} = \underline{a} \ltimes F(V)$ be the semi-direct product of $\underline{a}$ by $F(V)$ = the free Lie algebra on V. Then $Z(\underline{g}) = 0$ or $\underline{a}$ is abelian on one single odd generator.

Notations. If A is an augmented ring, we will use $I(A)$ as a notation for the augmentation ideal. If I is an ideal in a ring A we denote by $\mathrm{Ann}(I)$ the ideal $\{x \in A \; ; \; x \cdot I = 0\}$.

We will use the following lemma (cf e.g. [6]).

Lemma 1. Suppose $\underline{a}$ is a graded Lie algebra and $\mathrm{Ann}(IU(\underline{a})) \neq 0$, then $\underline{a}$ is abelian and generated by finitely many odd elements.

We also have the following lemma.

Lemma 2. Suppose A is a graded ring, $A = \bigoplus_{n \geq 0} A_n$, $I(A) = \bigoplus_{n \geq 1} A_n$ and $a \in A$ is a homogeneous element satisfying $a^2 = 0$ and $\{x \; ; \; xa = 0\} = Aa$. Suppose further $F_n \xrightarrow{d} F_{n-1} \xrightarrow{d} \cdots \longrightarrow F_0 \xrightarrow{\varepsilon} A_0 \longrightarrow 0$ is the beginning of a graded free resolution of A_0 as a right A-module, and let $V = \ker(F_n \xrightarrow{d} F_{n-1})$.

Suppose also $V \cdot a = 0$. Then $V \cdot I(A) = 0$. In particular, if $V \neq 0$ then $\mathrm{Ann}(I(A)) \neq 0$.

Proof. Take a homogeneous element v of V . Since v is also an element of the free A-module F_n and $v \cdot a = 0$, we can use the assumption $\{x \in A \; ; \; xa = 0\} = Aa$ to get an element x_n of F_n such that $v = x_n a$. But $0 = dv = (dx_n)a$ so in the same way we have an element x_{n-1} of F_{n-1} such that $dx_n = x_{n-1}a$. Finally we get $dx_1 = x_0 a$ where $x_0 \in F_0$. If x_0 has positive degree, there is $y_1 \in F_1$ with $dy_1 = x_0$. The equality $d(x_1 - y_1 a) = 0$ implies that there is $y_2 \in F_2$ such that $x_1 - y_1 a = dy_2$. From $dx_2 = x_1 a = (dy_2)a$ it follows that there is $y_3 \in F_3$ such that $x_2 - y_2 a = dy_3$ etc. At last $d(x_n - y_n a) = 0$, hence $x_n - y_n a \in V$ and since $V \cdot a = 0$ it follows that $v = x_n a = 0$. Suppose now x_0 is of degree zero. Then $\deg(x_1) = \deg(a)$, ..., $\deg(x_n) = n \cdot \deg(a)$ and $\deg(v) = (n+1) \cdot \deg(a)$. Hence V is concentrated in one single degree and therefore $V \cdot I(A) = 0$.

Remark. Lemma 2 is valid also for a local commutative ring A (with A_0 equal to the residue field).

Lemma 3. Suppose $\underline{a}$ is a graded Lie algebra. Suppose $V \neq 0$ is a syzygy in a free (not necessarily minimal) resolution of k over $U(\underline{a})$ such that $V \cdot I(U(\underline{a})) = 0$. Then $\underline{a}$ is abelian generated by one single odd element.

Proof. Since V is contained in a free module, the assumption $V \cdot I(U(\underline{a})) = 0$ implies that $\mathrm{Ann}\, I(U(\underline{a})) \neq 0$. Hence by lemma 1 $\underline{a}$ is abelian on finitely many odd generators. If V is a n^{th} syzygy it follows that

$$\mathrm{Tor}_{n+1+i}^{U(\underline{a})}(k,k) = \mathrm{Tor}_i^{U(\underline{a})}(k,k) \otimes_k V \quad \text{for } i \geq 1 .$$

Hence $P_{U(\underline{a})}(z) = \mathrm{Pol}(z)/(1 - \dim(V) z^{n+1})$ where $\mathrm{Pol}(z)$ is a polynomial in z . But we also know that there are numbers $e_1, \ldots, e_r$ such that

$$P_{U(\underline{a})}(z) = \prod_{i=1}^{r} (1 - z^{e_i})^{-1} .$$

The first expression shows that $z=1$ is a pole of order at most one, while the second expression shows that $z=1$ is a pole of order r . Hence $r=1$.

Proof of Theorem 3. Suppose $z \neq 0$ is an element of $Z(\underline{g})$ and $z = x + a$ where $x \in F(V)$ and $a \in \underline{a}$. 1) Assume first that $x \neq 0$. For each $y \in V$, $[x,y] + [a,y] = 0$. Now $F(V)$ is graded by the Lie degree "deg" defined by letting the elements of V have degree one. Since $\deg([a,y]) = \deg(y) = 1$ and $\deg([x,y]) = \deg(x) + 1$, it follows that $[x,y] = 0$. Since $F(V)$ is free, V must be one-dimensional and hence $V \cdot I(U(\underline{a})) = 0$. Since V is contained in a free $U(\underline{a})$-module, it follows that $\operatorname{Ann} I(U(\underline{a})) \neq 0$ and by lemma 1 and 3 $\underline{a}$ is abelian on one odd generator. 2) Assume now that $x=0$, i.e. $z \in \underline{a}$. Then $y \cdot z = [y,z] = 0$ for all $y \in V$. Since V is non-zero and contained in a free $U(\underline{a})$-module, z must be a zero-divisor on $U(\underline{a})$. But then z must be of odd degree and $z^2 = 0$. This follows from the Poincaré-Birkhoff-Witt theorem. Also from this theorem we get that $\{b \in U(\underline{a}) \ ; \ bz = 0\} = U(\underline{a}) \cdot z$. Since also $V \cdot z = 0$, lemma 2 may be applied to get $\operatorname{Ann} I(U(\underline{a})) \neq 0$ and $V \cdot I(U(\underline{a})) = 0$ and then also in this case lemma 1 and 3 may be applied to get the result.∎

Finally, Theorem 2 follows from Theorem 3 since we know the structure of $\operatorname{Ext}_R(k,k)$ if (R,m) is an equi-characteristic local ring with $m^3 = 0$. This may be deduced (with some effort) from [4] , and hopefully it will appear in a forth-coming paper by the author. The structure of the underlying Lie algebra $\underline{g}$ of $\operatorname{Ext}_R(k,k)$ is given as follows. Let $\underline{a}$ be the underlying Lie algebra of $\operatorname{Ext}_R^{(1)}(k,k)$ = the sub algebra of $\operatorname{Ext}_R(k,k)$ generated by its one-dimensional elements. Let V be the third syzygy in a minimal resolution of k over $U(\underline{a})$. Then $\underline{g} = \underline{a} \ltimes F(V)$.

An application.

Notation. For a local ring R , let $e_i(R)$ denote $\dim(\underline{g}_i)$ where $\underline{g}$ is the Lie algebra of R .

Theorem 4. If (R,m) is local with $m^3 = 0$, then $e_i(R) > 0$ for all $i \geq 1$ or R is a complete intersection (which is possible only if $\dim(m/m^2) \leq 2$).

Proof. We may assume that R is equi-characteristic since $P_R = P_{gr(R)}$ ($gr(R) = \bigoplus_{i\geq 0} m^i/m^{i+1}$) (a result by Levin, cf [4]) and $e_i(R)$ may be computed from P_R. During this conference I learned from Yves Felix that if $e_i(R) = 0$ for some i, then a "special variable" in the sense of André is defined (or, if you prefer, there exists a "Gottlieb element"). But according to Jacobsson [3] this defines an element in the centre of $\underline{g}$ = the Lie algebra of R. And by Theorem 2 in this paper the centre of $\underline{g}$ is trivial, unless $\underline{g}$ is generated by $\underline{g}_1$ (or $R = k[x]/(x^3)$ which is a complete intersection). In this case we have the following. If $e_i(R) = 0$ for some i then $e_j(R) = 0$ for all $j \geq i$. Hence by Gulliksens theorem [2], R is a complete intersection.

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