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NON-ABELIAN COHOMOLOGY AND THE HOMOTOPY CLASSIFICATION OF MAPS (\*)

by Ronald Brown

To a filtered space

$$X : X_0 \subset X_1 \subset \dots \subset X_n \subset \dots \subset X$$

we can associate the *homotopy crossed complex*  $\pi X$ , which consists for  $n = 1$  of the fundamental groupoid  $\pi_1 \underline{X} = \pi_1(X_1, X_0)$ , and for  $n \geq 2$  of the family  $\pi_n \underline{X}$  of relative homotopy groups  $\pi_n(X_n, X_{n-1}, v)$ ,  $v \in X_0$ , with the usual boundaries  $\delta : \pi_n \underline{X} \rightarrow \pi_{n-1} \underline{X}$  and action of  $\pi_1 \underline{X}$  on  $\pi_n \underline{X}$ . The formal properties satisfied by  $\pi \underline{X}$  define the notion of *crossed complex*, and we have a category  $XC$  of crossed complexes. Note that crossed complexes generalise chain complexes  $C$  (with  $C_i = 0$  for  $i < 1$ ), and they also generalise groups, groupoids, and crossed modules. A brief survey of their use in topology and algebra is given in [6]. See also [4, 5, 7].

The category  $XC$  of crossed complexes has a convenient notion of homotopy [10, 6, 7]. So for crossed complexes  $D, C$  we can define the set

$$[D, C]$$

of homotopy classes of morphisms  $D \rightarrow C$ .

The object of this talk is to advertise the definition (suggested in §.5 of [6])

$$H^0(X ; C) = [\pi \underline{X}, C]$$

for CW-complex  $X$  with skeletal filtration  $\underline{X}$ , and for a crossed complex  $C$ . That is, we take  $[\pi \underline{X}, C]$  as the *cohomology of  $X$  with coefficients in  $C$* .

The definition makes sense, because  $\pi \underline{X}$  is a homotopy invariant of  $X$ . The proof of this is not entirely trivial. One proof is given by J.H.C. Whitehead in [10] another is given in [7]. (Here we mean  $X \simeq Y$  implies  $\pi \underline{X} \simeq \pi \underline{Y}$ ).

The point of the definition is that we expect cohomology to have something to do with the sets  $[X, Y]$  of homotopy classes of maps of spaces. From [7] we take :

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Theorem 1. *There is a functor  $B : XC \rightarrow \text{Top}$  assigning to a crossed complex  $C$  a CW-complex  $BC$  with the property that there is a natural bijection*

$$[X, BC] \cong H^0(X; C)$$

for CW-complexes  $X$ .

Two special cases are of interest :

(i) If  $C$  is a group  $G$  in dimension  $n$  (where  $G$  is abelian if  $n \geq 2$ ) and zero otherwise, then  $BC = K(G, n)$ , and Theorem 1 generalise a classical result of Eilenberg-MacLane. Note that the non-abelian case  $n = 1$  is also included.

(ii) If  $C_1$  is a group  $G$ ,  $C_n$  is a  $G$ -module  $M$ ,  $C_i = 0$  for  $i \neq 1, n$  and all boundaries are zero then  $H^0(X; C)$  is a kind of twisted cohomology of  $X$  with coefficients in the  $G$ -module  $M$ , and so we have a twisted homotopy classification theorem.

There are three obvious questions about Theorem 1,

- Q1. How do you prove it ?
- Q2. What use is it in tackling the *general* problem of listing the elements of the set  $[X, Y]$  of homotopy classes of maps  $X \rightarrow Y$  ?
- Q3. How do you compute  $H^0(X; C)$  ?

All these have interesting answers which we can only outline here. More details are given in [4,5,7].

The construction of the "classifying space"  $BC$  is done dublically. So we construct a cubical complex  $NC$ , the *nerve* of  $C$ , by setting

$$(NC)_n = XC(\pi_{\sim}^n I^n, C)$$

where  $I^n$  is the standard skeletal filtration of the  $n$ -cube. We then set  $BC = |NC|$ , the geometric realisation of the cubical complex  $NC$ . (There is also a simplicial, and homotopy equivalent, version  $B^{\Delta}C$ ; see the Introduction to 3, which includes the relevant theses [1,8].)

The first part of the proof of Theorem 1 is to note that it is sufficient to restrict to the case when  $X$  is the realisation  $|K|$  of a cubical complex  $K$ , and then to use an equivalence of homotopy categories to obtain

$$[|K|, BC] \cong [K, NC] .$$

For this we need to know  $NC$  is a Kan complex. In fact,  $NC$  has a lot of extra structure, since it turns out to be an example of an  $\omega$ -groupoid, which is a complicated algebraic structure defined in [4]. Any  $\omega$ -groupoid is a Kan complex,

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and hence  $NC$  is a Kan complex. We write (as in [4,5])  $\lambda C$  for  $NC$  with its structure of  $\omega$ -groupoid.

Because  $\lambda C$  is an  $\omega$ -groupoid, we have a bijection

$$[K, NC] \cong [\rho K, \lambda C]$$

where the latter set of homotopy classes is taken in the category of  $\omega$ -groupoids, and  $\rho K$  denotes the free  $\omega$ -groupoid on  $K$ . But it also turns out that there is an equivalence, of categories with homotopy, between  $\omega$ -groupoids and crossed complexes, and that this equivalence takes  $\rho K$  to  $\pi[\underline{K}]$ , and  $\lambda C$  to  $C$ . So

$$[\rho K, \lambda C] \cong [\pi[\underline{K}], C]$$

and we are done.

Unfortunately, the details of the above are strenuous. However, the pattern of argument parallels the case  $BC = K(G,n)$  ( $n \geq 2$ ), which uses the simplicial abelian group structure on  $K(G,n)$ . We are using  $\omega$ -groupoid structures instead, and this is what allows for non-abelian results.

Something needs to be said about the homotopy type of  $BC$ . For convenience we restrict to the reduced case, i.e. when  $C_0$  is a point. Then  $\pi_1(BC, v)$  is the quotient group  $G = C_2/\delta C_1$ , while for  $n \geq 2$   $\pi_n(BC, v)$  is the homology of  $C$ , i.e.  $\text{Ker}\delta/\text{Im}\delta$ , together with the action of  $G$ . Further, there is a fibration  $BC \rightarrow K(G,1)$  whose fibre is 1-connected and is of the homotopy type of a product of Eilenberg-MacLane spaces. (This observation is due to J.L. Loday. I am not too clear about the classification of such non-principal fibrations.)

Now let  $Y$  be a reduced CW-complex with cellular filtration  $\underline{Y}$ . We can form the homotopy crossed complex  $\pi \underline{Y}$  and the classifying space  $B\pi \underline{Y}$ . In this case  $\pi_1(B\pi \underline{Y}, v) \cong \pi_1(Y, v)$  and for  $n \geq 2$   $\pi_n(B\pi \underline{Y}, v)$  is isomorphic to  $H_n(\hat{Y})$ , the homology of the universal cover  $\hat{Y}$  of  $Y$ . Further there is a map  $q : Y \rightarrow B\pi \underline{Y}$  which induces, on homotopy groups  $\pi_n$ , an isomorphism for  $n = 1$ , and for  $n \geq 2$  a morphism equivalent to the Hurewicz  $\pi_n(Y, v) \xrightarrow{\omega} H_n(\hat{Y})$ .

These facts are deducible from results of §.8, 9 of [5], but are not explicit there, so it should prove useful to explain the procedure.

For any filtered space  $Y$  there are cubical complexes and maps

$$\begin{array}{ccc} R\underline{Y} & \xrightarrow{i} & KY \\ p \downarrow & & \\ \rho \underline{X} & & \end{array}$$

where  $KY$  is the cubical singular complex of  $Y$ , and  $i$  is the inclusion of the *filtered singular complex*  $RY$  of  $Y$ ; that is  $RY$  consists in dimension  $n$  of all filtered maps  $\underline{I}^n \rightarrow Y$ . The mapping  $p$  is a quotient mapping. It identifies two filtered maps  $\underline{I}^n \rightarrow Y$  if and only if they are homotopic, relative to the vertices of  $\underline{I}^n$ , and through filtered maps. (This definition is not exactly the same as that given in [5], but the two definitions agree if  $\pi_0 Y_0 = Y_0$ , which is sufficient for our purposes.)

The cubical complex  $\rho Y$  has the structure of  $\omega$ -groupoid, and its associated crossed complex is  $\pi Y$ . That is,  $\rho Y$  is isomorphic as  $\omega$ -groupoid to  $\lambda \pi Y$ .

In [5] it was shown that  $p : RY \rightarrow \rho Y$  is a fibration in the sense of Kan. This result was found to be an important technical tool in the proofs of the main results of [5], since it helped in proving  $\rho Y \simeq \lambda \pi$ , and in establishing a crucial property of "thin elements" in  $\rho Y$ . We can now give this fibration property of  $p$  another rôle.

The cubical complexes  $RY$  and  $KY$  are known to be Kan complexes. (The corresponding property for  $\rho Y$  is not so simple to prove.) The inclusion  $i : RY \rightarrow KY$  is a homotopy equivalence if the functions induced by inclusion  $\pi_0 Y_r \rightarrow \pi_0 Y$  are surjective for  $r = 0$  and bijective for  $r > 0$ , and the based pairs  $(Y, Y_m, v)$  are  $m$ -connected for all  $m \geq 1$  and  $v \in Y_0$ . In particular,  $i$  is a homotopy equivalence if  $Y$  is the skeletal filtration of a CW-complex  $Y$ . For such a  $Y$ , the realisation  $|KY|$  has the same homotopy type as  $Y$ , and in this way we obtain the map  $q : Y \rightarrow B\pi Y$  with the properties set out above.

Let  $X$  be a CW-complex. We have an induced function

$$q_* : [X, Y] \rightarrow [X, B\pi Y].$$

This function is bijective if  $\dim X \leq m$  and  $q : Y \rightarrow B\pi Y$  has  $m$ -connected homotopy fibre. This will be true if, for example,  $\pi_i Y = 0$  for  $1 < i < m$ . In these circumstances we obtain a bijection

$$[X, Y] \rightarrow H^0(X; \pi Y).$$

So we can see the relevance of this non-abelian cohomology to some general homotopy classification problems, particularly in the non-simply connected case.

How do we compute  $H^0(X; C)$ ? For this we generalise some ideas of Whitehead in [10].

For simplicity, we restrict to the reduced case. Let  $GC_*$  be the category with objects the triples  $(K, G, v)$  in which  $G$  is a group,  $K$  is a chain complex of  $G$ -modules (with  $K_i = 0$  for  $i < 0$ ), and  $K_0$  is a free  $G$ -module with basis the

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element  $v \in K_0$ . The morphisms of  $GC_*$  are to be pairs  $(f, \theta) : (K, G, v) \rightarrow (K', G', v')$  where  $\theta : G \rightarrow G'$  is a morphism of groups,  $f : K \rightarrow K'$  is a chain map and an operator morphism over  $\theta$ , and  $f(v) = v'$ .

Let  $XC_*$  be the category of reduced crossed complexes. There is a functor  $\Delta : XC_* \rightarrow GC_*$  in which if  $(K, G, v) = \Delta C$ , then  $G = C_1 / \delta C_2$ ;  $K_n = C_n$  as a  $G$ -module for  $n \geq 3$ ;  $K_2$  is  $C_2$  made abelian;  $K_1$  is the  $C$ -module induced from the augmentation ideal  $IC_1$  by the quotient morphism  $C_1 \rightarrow G$ ; and  $K_0$  is the free  $G$ -module on the element  $v \in C_0$ . (This construction is given in [7] and extends a construction given in [10] for the case  $C_1$  is free. A further result proved in [7] is that  $\Delta$  has a right adjoint, and so preserves colimits.) This functor  $\Delta$  transforms homotopies to homotopies, for a suitable definition of homotopy in  $GC_*$ . So for reduced crossed complexes  $C, D$  we have a function

$$\Delta_* : [D, C] \rightarrow [\Delta D, \Delta C].$$

Now Whitehead proves (but does not state) that if  $C_1$  and  $D_1$  are free groups and  $D_2$  is a free crossed  $D_1$ -module, then  $\Delta_*$  is a bijection. Also, he notes that if  $\underline{X}$  is the skeletal filtration of a reduced CW-complex  $X$ , then  $\Delta \pi \underline{X}$  consists of the cellular chains  $C_*(\tilde{X})$  of the universal cover  $\tilde{X}$  of  $X$ , these chains being taken as modules over the fundamental group of  $X$ . That is, we have a bijection

$$H^0(X; C) \cong [C_*(\tilde{X}), \Delta C].$$

This gives a reasonable computational description of  $H^0(X; C)$ , and so of  $[X, BC]$ . For example, it leads to the homotopy classification of maps from a surface to the projective plane [2].

Consider again the bijection

$$[X, Y] \cong [C_*(\tilde{X}), C_*(\tilde{Y})]$$

given when  $\dim X \leq m$  and  $\pi_1 Y = 0$  for  $1 < i < m$ . If also  $\pi_1 X = 0$ , then  $\tilde{Y} = Y$  and the definition of morphism and chain homotopy in  $GC_*$  implies that

$$[C_*(\tilde{X}), C_*(\tilde{Y})] \cong [C_*(X), C_*(Y)]$$

where  $C_*(X)$  is the usual cellular chain complex of  $X$ . Since  $C_*(Y)$  is a chain complex of free abelian groups there is a chain map  $\phi : C_*(Y) \rightarrow H_*(Y)$  (where the latter has zero differential) inducing an isomorphism in homology. So we obtain

$$\begin{aligned} [X, Y] &\cong [C_*(X), H_*(Y)] \\ &\cong H^0(X; H_*(Y)) \\ &\cong H^m(X; H_m(Y)). \end{aligned}$$

This result includes the Hopf classification theorem (which is the case  $Y = S^m$ ). Thus the non-abelian results reduce to classical abelian results.

All these results give point to a remark of Whitehead in the Introduction to [10], which reads in our terminology :

*The crossed complex  $\pi_{\underline{X}}$  appears to be more useful than the chain complex  $C_*^{\vee}(\tilde{X})$  in problems concerning geometric realisability. On the other hand, the chain complex  $C_*^{\vee}(\tilde{X})$  is useful in studying concrete problems.*

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