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On the structure of the homotopy Lie algebra of a local ring.

by

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In this note R denotes a commutative noetherian local ring R with (unique) maximal ideal $\underline{m}$ and residue field $R/\underline{m} = k$. There is a functorially attached to R graded Lie k-algebra $\pi^*(R)$, which we call the homotopy Lie algebra of R. For the definition of this functor, in a considerably larger setup, cf. [3]. The dimensions $e_i = \dim_k \pi^i(R)$ appear in the well known expression

$$P_R(t) = \frac{(1+t)^{e_1}(1+t^3)^{e_3}\ldots\ldots}{(1-t^2)^{e_2}(1-t^4)^{e_4}\ldots\ldots}$$

for the Poincaré series

$$P_R(t) = \sum_{i\geq 0} \dim_k \mathrm{Tor}_i^R(k,k)t^i$$

The rings for which $\pi^*(R)$ is finite-dimensional have been characterized by Gulliksen [7] as being the complete intersections (the definition of this class of rings is recalled in [3, §4]). In fact, it is known that when R is a complete intersection, $\pi^i(R) = 0$ for all $i \geq 3$, and a question raised in [8, p 154] and taken up in [1] as conjecture C_3, asks whether the vanishing of a single e_i $(i \geq 1)$ characterizes complete intersections. This is known to be true for small values of i : $e_1 = 0 \iff$ R is a field; $e_2 = 0 \iff$ R is regular; $e_3 = 0 \iff e_4 = 0 \iff$ R is a complete intersection (cf. e.g. [8]).

The following result settles the conjecture for i large enough; in the context of graded augmented (skew-commutative) algebras over a field of characteristic 0, it is already given by Felix and Thomas in [6].

Theorem 1. If R is not a complete intersection, there exists an integer i(R), such that for $i \geq i(R)$ one has $\pi^i(R) \neq 0$.

Few classes of rings for which the non-vanishing of all the e_i's is known have been exhibited so far. In these Proceedings Löfwall shows this is the case when $\underline{m}^3 = 0$ (and R is not a complete intersection). We add to the list:

Proposition 2. Assume $\dim_k(\underline{m}/\underline{m}^2)$ - depth $R \leq 3$, or R is Gorenstein with $\dim_k(\underline{m}/\underline{m}^2)$ - depth R = 4. Then either R is a complete intersection, or

$e_i \neq 0$ for all i.

(Note that the existence of infinite arithmetic sequences of indices for which $e_i \neq 0$ have been obtained in [1]).

The proof of Theorem 1 makes essential use of a result on the Lie algebra structure of $\pi^*(R)$, which can be formulated as follows:

Theorem 3. If R is not a complete intersection, there exist elements $\alpha \in \pi^2(R)$, $\beta \in \pi(R)$ such that for all $n \geq 1$:

$$(\mathrm{ad}\alpha)^n \beta \neq 0$$

where $$(\mathrm{ad}\alpha)\gamma = [\alpha,\gamma].$$

The proof of the second theorem depends on the use of the minimal models for DG algebras, introduced in [2, 3], and parallels an argument of [4]. Note also that in the context of rational homotopy groups of finite CW complexes, a stronger non-vanishing result for iterated Whitehead products is available [5].

As an immediate consequence we have several characterizations of complete intersections in terms of the Lie algebra structure:

Corollary. The following are equivalent:

(1) R is a complete intersection;

(2) $\pi^{\geq 2}(R)$ is abelian;

(3) $\pi^*(R)$ is nilpotent;

(4) $\pi^*(R)$ is Engel (i.e. $(\mathrm{ad}\alpha)^{n(\alpha)} = 0$ for each $\alpha \in \pi^*(R)$ and some integer $n(\alpha) \geq 1$, depending on α).

Note that going down is trivial; in the opposite direction only (2) => (1) was known earlier [2].

Proofs will be published elsewhere.

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