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Loop space decompositions in the theory of exponents

by

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The theory of exponents deals with the order of the identity map on H-spaces, especially loop spaces, in the group of all homotopy classes of maps of the space into itself. We consider spaces localized at a prime p, and there is usually some indication that the order is finite, for example, the homotopy groups are all p-torsion and kill by some power p^r of p. During the period 1977-1981 considerable progress was made by Cohen, Moore, and Neisendorfer in the theory of exponents using decompositions of loop spaces. These decompositions were constructed by Lie algebra and Hurewicz homomorphism techniques. A very striking result proved by the three authors is that the identity map on $\Omega^2 S^m(p^r)$ has finite order when p is odd for a mod p^r sphere $S^m(p^r) = S^{m-1} \cup_{p^r} e^m$. The result is unknown in the case of the prime p = 2. Neisendorfer refined the techniques considerably and was able to show that the order of the identity was exactly p^{r+1} which is the best possible since ther are elements of order p^{r+1} in the homotopy groups.

In this article we formulate the problems connected with exponents and develop some of the main ideas used to decompose loop spaces in order to get at a determination of their exponents. Our presentation here is inspired by the series of papers of Cohen, Moore, and Neisendorfer and of Neisendorfer and is in large part a survey of their more recent work.

§1. Ω^i- exponents and S^i- exponents.

All spaces are localized at a prime p which will usually be taken to be odd. The question of exponents is whether or not certain algebraic invariants are annihilated by some power of p, i.e. are p torsion. Further, if they are p-torsion, then determine the minimal r such that p^r annihilates the invariant. In the following definitions we make this more precise.

(1.1) Definition. A space X has an Ω^i-exponent r provided $p^r.id$ is null homotopic and $p^{r-1}.id$ is not null homotopic on the loop space $\Omega^i X$.

Observe that if an i-connected space X has an Ω^i-exponent r, then $p^r \pi_*(X) = 0$. For an H-space we have the notion of Ω^0-exponent, or simply, exponent included in the above definition since the property is independent of which H-space structure on $\Omega^i(X)$. A space X has homotopy exponent r provided $p^r.\mathrm{Tors}\pi_*(X) = 0$ and $p^{r-1}.\mathrm{Tors}\pi_*(X) \neq 0$. If an H-space has an exponent, then observe that all the homotopy is torsion and annihilated by the power of p to the exponent.

(1.2) Definition. A space X has an S^i-exponent r provided $p^r.id$ is null homotopic and $p^{r-1}.id$ is not null homotopic on the suspension $S^i(X)$.

Observe that if a space X has an S^i-exponent r, then $p^r\ \bar{H}_*(X) = 0$. For a coH-space we have the notion of S^0-exponent, or simply, exponent included in the above definition since the property is independent of which coH-space structure on $S^i(X)$. A space X has homology exponent r provided $p^r.\mathrm{Tors}\bar{H}_*(X) = 0$ and $p^{r-1}.\mathrm{Tors}\bar{H}_*(X) = 0$. If a coH-space has an exponent, then observe that all the reduced homology is torsion and annihilated by the power of p to the exponent.

(1.3) Examples (1). The map $p^r : S^m \longrightarrow S^m$ of degree p^r has a (homotopy) cofibre denoted $S^{m+1}(p^r)$, called a mod p^r sphere and a (homotopy) fibre denoted $S^m\{p^r\}$. The coH-space $S^{m+1}(p^r)$ has coH-space exponent p^r. For an odd prime p the localized odd spheres S^{2m+1} are H-spaces and $p^r : S^{2m+1} \longrightarrow S^{2m+1}$ is an H-map so that the fibre $S^{2m+1}\{p^r\}$ has an H-space structure. The exponent of this H-space is p^r.

These examples are elementary. The next ones are the result of the theory of Cohen, Moore, and Neisendorfer (Theorem I) and finally of Neisendorfer (Theorem II). In the remainder of this section p is an odd prime.

(1.4) Theorem I. The double suspension map $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ has a partial inverse $\pi : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ in the sense that $\pi E^2 = p$ and $E^2 \pi = \Omega^2 p$ where p refers to a map of degree p.

For the proof of this theorem see the two papers of Cohen, Moore, and Neisendorfer in 1979, especially [3].

(1.5) Examples (2). Starting with S^1 and using (1.4), we see that $p^n \pi_{2n+1+i}(S^{2n+1}) = 0$ for $i > 0$. Since $\pi_m(S^m) = \mathbb{Z}_{(p)}$ for spaces localized at p, we can form $S^{2n+1}\langle 2n+1\rangle$ the (2n+1) - connected cover where just the bottom homotopy group is killed and deduce that $p^n \pi_*(S^{2n+1}\langle 2n+1\rangle) = 0$ for this p torsion space $S^{2n+1}\langle 2n+1\rangle$. By a result of B. Gray [5] there are elements of order p^n in these homotopy groups so that $S^{2n+1}\langle 2n+1\rangle$ has homotopy exponent n at p. By [3] and [10] the space $S^{2n+1}\langle 2n+1\rangle$ has an Ω^{2n}-exponet n and by Neisendorfer and Selick [13] the space does not have an Ω^k-exponent for $k \leqq 2n-2$.

(1.6) Example (3). Let C(n) denote the H-space which is the fibre of the H-map double suspension $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$. Then by (1.4) the exponent of C(n) is one.

(1.7) Conjecture of Moore. All finite simply connected simplicial complexes X such that $\pi_*(X) \otimes Q$ is of total finite dimension have a homotopy exponent when localized at any prime p.

The conjecture is true for spheres as observed above, and an application of the Hilton-Milnor theorem to the wedge of two spheres shows why a hypothesis like total finite dimensionality of rational homotopy is necessary for the assertion to be true.

The following result is contained in [4] and is a proved using the fact that $H_*(\Omega S(X))$ is the tensor algebra on $\bar{H}_*(X)$ over a field.

(1.8) Proposition. If X is a space with $\bar{H}_*(X,\mathbb{F}_p) \neq 0$, then S(X) does not have an Ω^1-exponent.

This leads to the fundamental result of Neisendorfer [11].

(1.9) Theorem II. The mod p^r sphere $S^m(p^r)$ has an Ω^2-exponent equal to r+1.

(1.10) Conjecture of Barratt. If a double suspension $S^2(X)$ has coH-space exponent r at p, then it has an Ω^2-exponent of r+1.

A final problem is to determine what exponents exist at the prime 2 and in particular what part of the work at the odd primes extends to the prime 2.

§2. Loop space decompositions I. Lie algebra methods.

The notation $S^m(p^r)$ for a mod p^r sphere is extended to $S^\phi(p^r)$ or $S(\phi(t);p^r)$ for a wedge of mod p^r spheres indexed by the coefficients of $\phi(t) = a_2t^2 + \ldots + a_nt^n + \ldots$ in $t^2\mathbb{N}[[t]]$ where

$$S(\phi(t);p^r) \;=\; \varinjlim_{>n}\{S^2(p^r)\vee \overset{a_2}{\ldots}\vee S^2(p^r)\vee\ldots\vee S^n(p^r)\vee \overset{a_n}{\ldots}\vee S^n(p^r)\}$$

The decomposition of an even sphere, localized at p an odd prime, given by Serre [16]

$$\Omega S^{2n+2} \xrightarrow{\simeq} S^{2n+1} \times \Omega S^{4n+3}$$

including a left homotopy inverse of the suspension map $S^{2n+1} \longrightarrow \Omega S^{2n+2}$ has a version for mod p^r spheres proved by Cohen, Moore and Neisendorfer [2].

(2.1) Theorem. There is a homotopy equivalence

$$\Omega S^{2n+2}(p^r) \xrightarrow{\simeq} S^{2n+1}\{p^r\} \times \Omega S(\frac{t^{4n+3}}{1-t^{2n}};p^r)$$

including a left homotopy inverse of the map defined $S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+2}(p^r)$ which is a factor in the factorization of the suspension map $S^{2n+1}(p^r) \longrightarrow S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+2}(p^r)$.

The result of Serre comes from the study of the free Lie algebra on one odd generator in degree 2n+1 and the integral Hurewicz homomorphism and theorem (2.1) comes from the study of the free Lie algebra on two genearators in degrees 2n and 2n+1 and the mod p^r Hurewicz homomorphism.

The decomposition of $S^{2n+1}(p^r)$ is much more complicated, and it begins with making the fibre sequences for each map in the composite $S^{2n+1}(p^r) \longrightarrow S^{2n+1}\{p^r\} \longrightarrow S^{2n+1}$ and using the fact that the fibres fit into a fibre sequence also which is written vertically.

$$\begin{tikzcd}[column sep=small]
 & E^{2n+1}\{p^r\} \arrow[d] \arrow[dr] & & & \\
\Omega S^{2n+1} \arrow[r] & F^{2n+1}\{p^r\} \arrow[r] \arrow[d] & S^{2n+1}\{p^r\} \arrow[dr] \arrow[drr] & & \\
 & \Omega S^{2n+1} \arrow[rr] & & S^{2n+1}\{p^r\} \arrow[r] & S^{2n+1}.
\end{tikzcd}$$

If $C(n) \longrightarrow S^{2n-1}$ is the fibre of the double suspension map $S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$, then for the corresponding diagram of loop spaces we have a commutative square induced by the double suspension map

$$\begin{tikzcd}
C(n) \arrow[r] \arrow[d] & E^{2n+1}\{p^r\} \arrow[d] \\
S^{2n-1} \arrow[r] & F^{2n+1}\{p^r\}.
\end{tikzcd}$$

Both horizontal maps have a left homotopy inverse decomposing $\Omega E^{2n+1}\{p^r\} \longrightarrow \Omega F^{2n+1}\{p^r\}$ each into a product of $C(n) \longrightarrow S^{2n-1}$ with a single space. This is the main result of [2] and [3].

(2.2) <u>Theorem</u>. There is a commutative diagram with horizontal arrows being homotopy equivalences

$$\begin{tikzcd}
C(n) \times \prod_{i \geq 1} S^{2p^i n-1}\{p^{r+1}\} \times \Omega S(\phi_{2n+1}(t);p^r) \arrow[r] \arrow[d] & \Omega E^{2n+1}\{p^r\} \arrow[d] \\
S^{2n-1} \times \prod_{i \geq 1} S^{2p^i n-1}\{p^{r+1}\} \times \Omega S(\phi_{2n+1}(t);p^r) \arrow[r] & \Omega F^{2n+1}\{p^r\}
\end{tikzcd}$$

where $\phi_{2n+1}(t) = \dfrac{t^2 - t^2\psi(t)}{1+t}$ and

$$\psi(t) = \frac{1 - t^{2n} - t^{4n-2} - t^{4n-1}}{1 - t^{2n}} \prod_{i \geq 1} \frac{1 + t^{2p^i n-1}}{1 - t^{2p^i n-2}} .$$

This theorem is deeper than (2.1) in that it involves the Bockstein spectral sequence in mod p homotopy and homology and properties of differential Lie algebras. The Lie product in homotopy is the Samelson product, and in order to choose the correct maps for the splittings, the notion of a relative Samelson product is introduced and studied for the induced map of a fibre into the total space of a fibration.

(2.3) Remark. The splitting of S^{2n-1} off of the H-space $\Omega F^{2n+1}\{p^r\}$ for $r = 1$ gives mappings $S^{2n-1} \longrightarrow \Omega F^{2n+1}\{p\} \longrightarrow S^{2n-1}$ which compose to the identity up to homotopy. This retraction onto S^{2n-1} composed with $\Omega^2 S^{2n+1} \longrightarrow \Omega F^{2n+1}\{p\}$ in the following commutative diagram

$$\begin{array}{ccc} & \Omega F^{2n+1}\{p\} & \\ \nearrow & & \searrow \\ \Omega^2 S^{2n+1} & \xrightarrow{\quad \Omega^2 p \quad} & \Omega^2 S^{2n+1} \end{array}$$

yields the quasi-inverse $\pi : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ where the composites $\pi E^2 = p$ and $E^2\pi = \Omega^2 p$.

(2.4) Remark. From the defining diagram for E^{2n+1} and F^{2n+1} we have natural mappings

$$\Omega E^{2n+1}\{p^r\} \longrightarrow \Omega F^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}(p^r).$$

Composing this second map with one of the maps in the splitting (2.2), we obtain a basic map

$$\Omega S[n] = \Omega S(\phi_{2n+1}(t);p^r) \longrightarrow \Omega S^{2n+1}(p^r)$$

This map of loops on the wedge $S[n]$ of mod p^r spheres has the property that it induces a monomorphism on homology over $\mathbb{F}_p$. The next section is devoted to show that $\Omega S[n]$ is a direct summand of $\Omega S^{2n+1}(p^r)$.

§3. Loop space decompositions II. Hurewicz homomorphism methods.

The ideas which are used to split $\Omega S^{2n+1}(p^r)$ go beyond the results outlined in the previous section and are related to a characterization of the category $\underline{\underline{s}}(p^r)$ of wedges of mod p^r spheres $S^{\phi}(p^r)$ within the category of spaces $\underline{\underline{t}}(p^r)$ of p-torsion spaces where the Bockstein spectra sequence $E^s\bar{H}_*(X,\mathbb{F}_p)$ satisfies

$$\bar{H}_*(X,\mathbb{F}_p) = E^1\bar{H}_*(X) = \ldots = E^r\bar{H}_*(X) \text{ and } E^{r+1}\bar{H}_*(X) = 0.$$

(3.1) Proposition. A simply connected CW-type space X in $\underline{\underline{t}}(p^r)$ is a wedge of mod p^r spheres, i.e. is in $\underline{\underline{s}}(p^r)$ if and only if the Hurewicz homomorphism followed by reduction mod p is an epimorphism $\pi_*(X,\mathbb{Z}/p^r) \longrightarrow \bar{H}_*(X,\mathbb{Z}/p^r) \longrightarrow \bar{H}_*(X,\mathbb{F}_p)$.

This proposition is proved by mapping mod p^r spheres $S^m(p^r)$ into X to realize homology classes in X, taking a wedge of these spheres $S^{\phi}(p^r)$, and mapping $f : S^{\phi}(p^r) \longrightarrow X$ such that $H_*(f)$ is an isomorphism Then the Whitehead theorem implies that f is a homotopy equivalence.

Let $\underline{\underline{C}}$ denote the category of simply connected, acyclic complexes over $\mathbb{F}_p$, and let $\underline{\underline{s}}'(p^r)$ denote the category of simply connected wedges of mod p^r spheres and homotopy classes of mappings. Then $\bar{H}_* = E^r\bar{H}_* : \underline{\underline{s}}'(p^r) \longrightarrow \underline{\underline{C}}$ is a functor where $\bar{H}_*$ has the rth Bockstein as a differential.

(3.2) Proposition. The functor $\bar{H}_* : \underline{\underline{s}}'(p^r) \longrightarrow \underline{\underline{C}}$ induces a bijection between isomorphism classes of objects. A map $f : X \longrightarrow Y$ in $\underline{\underline{s}}'(p^r)$ has a left (resp. right) homotopy inverse if and only if $\bar{H}_*(f)$ has a left (resp. right) inverse.

The proof of this proposition follows from techniques similar to those used in (3.1). We need an extension of (3.2) for the splitting of $S^{2n+1}(p^r)$ which we use for the study of the exponents of mod p^r spheres.

(3.3) Remarks. Let $f : \Omega X \longrightarrow Z$ be a map such that $Sf : S\Omega X \longrightarrow SZ$ has a left homotopy inverse h, then so does $f : \Omega X \longrightarrow Z$. In effect, the left homotopy inverse of f is the composite $Z \longrightarrow \Omega SZ \xrightarrow{\Omega h} \Omega S\Omega X \longrightarrow \Omega X$. Further, if a space X is a wedge of simply connected mod p^r spheres, then $S\Omega X$ has the homotopy type of a wedge of simply connected mod p^r spheres. In effect, we use the relation

$$S^m(p^r) \wedge S^n(p^r) \simeq S^{m+n}(p^r) \vee S^{m+n-1}(p^r)$$

and a result of the Hilton-Milnor theorem

$$S\Omega SY = S\Omega X \simeq S(Y) \vee S(Y \wedge Y) \vee \ \ldots$$

With these two remarks and (3.2) we deduce the following result.

(3.4) Proposition. For X and Y simply connected wedges of mod p^r spheres a map $f : \Omega X \longrightarrow \Omega Y = Z$ has a left homotopy inverse if and only if $\bar{H}_*(f)$ has a left inverse.

This proposition will be applied to the inclusion map of the fibre into the total space of a fibration, and the following splitting result, which easily verified, is of use.

(3.5) Proposition. Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a fibre sequence where X is an H-space. Then the following are equivalent:

(i) The projection p has a right homotopy inverse.

(ii) The inclusion i has a left homotopy inverse.

(iii) There is a space T and a map $u : T \longrightarrow X$ such that the composite $T \times F \xrightarrow{(u,i)} X \times X \longrightarrow X$ is a homotopy inverse and $p(f|T \times *)$ is a homotopy equivalence $T \longrightarrow B$.

The above is a version of a discussion in Cohen, Moore, and Neisendorfer which is designed to split $\Omega S^{2n+1}(p^r)$. There are several extensions of these results in this form which should lead to further results which will be considered in a later publication. One of the wedges of mod p^r spheres $S[n] = S(\phi_{2n+1}(t);p^r)$ in (2.2) had the additional property, see (2.4), that when mapped further from $\Omega F^{2n+1}\{p^r\}$ into $\Omega S^{2n+1}(p^r)$ by $\Omega S[n] \longrightarrow \Omega S^{2n+1}(p^r)$ induced a split homomorphism on homology viewed with the Bochstein, as an acyclic complex over $\mathbb{F}_p$. The map $\Omega S[n] \longrightarrow \Omega S^{2n+1}(p^r)$ is the loop of a map $S[n] \longrightarrow S^{2n+1}(p^r)$.

(3.6) Definition. Let $T^{2n+1}\{p^r\}$ denote the homotopy fibre of $S[n] \longrightarrow S^{2n+1}(p^r)$.

This gives rise to a fibre sequence

$$\Omega S[n] \longrightarrow \Omega S^{2n+1}(p^r) \longrightarrow T^{2n+1}\{p^r\} \longrightarrow S[n] \longrightarrow S^{2n+1}(p^r),$$

and applying (3.4) and (3.5) to the first three terms of this fibre sequence, we obtain the following assertion.

(3.7) Proposition. There is a homotopy equivalence

$$T^{2n+1}\{p^r\} \times \Omega S[n] \longrightarrow \Omega S^{2n+1}(p^r).$$

The importance of $T^{2n+1}\{p^r\}$ is contained in the following proposition which is an inductive use of the Hilton-Milnor theorem for the decomposition of $\Omega S(X \vee Y)$.

(3.8) Proposition. For $\phi(t) \in t^3\mathbb{N}[[t]]$ the space $S^{\phi}(p^r)$ is an inductive limit of finite products of spaces $S^{2n+1}\{p^r\}$ and $T^{2m+1}\{p^r\}$ for n, $m \geq k$ where order $\phi(t) = 2k+1$ or $2k+2$ and only a finite number of given n and m occur in the limit.

§4. Analysis of the exponent of $\Omega T^{2n+1}\{p^r\}$.

In this section we outline the key steps in the paper of Neisendorfer [11] which lead to the sharp exponent of p^{r+1} for the H-space structure on $\Omega T^{2n+1}\{p^r\}$, and hence also on $\Omega^2 S^m(p^r)$.

(4.1) Remark. We have the fibrations

$$S^{2n+1}\{p^s\} \longrightarrow S^{2n+1}\{p^{r+s}\} \xrightarrow{\ \phi\ } S^{2n+1}\{p^r\}$$

and cofibrations

$$S^{2n+1}(p^s) \xrightarrow{\ \psi\ } S^{2n+1}(p^{r+s}) \longrightarrow S^{2n+1}(p^r)$$

from the properties of the fibre and cofibre of a composite, see for example [8,1.5].

(4.2) Main construction. There are maps w_k defined $T^{2n+1}\{p^r\} \longrightarrow S^{2p^k n-1}\{p^r\}$ such that the following composites have the stated form:

(1)
$$\begin{tikzcd} S^{2p^j n-1}\{p^{r+1}\} \arrow[r] \arrow[dr, swap, "\substack{\text{null homotopic if } j \neq k \\ \text{fibration } \phi \text{ in (4.1) for} \\ s = 1 \text{ if } j = k.}"] & T^{2n+1}\{p^r\} \arrow[d, "w_k"] \\ & S^{2p^k n-1}\{p^r\} \end{tikzcd}$$

(2)
$$\begin{tikzcd} C(n) \arrow[r] \arrow[d] \arrow[dr, dashed] & T^{2n+1}\{p^r\} \arrow[d, "w_k"] \\ S^{2n-1} \arrow[r] & S^{2p^k n-1}\{p^r\} \end{tikzcd}$$
null homotopic composite

Observe that the composite

$$\Omega\prod S^{2p^k n-1}\{p^{r+1}\} \longrightarrow T^{2n+1}\{p^r\} \longrightarrow \Omega\prod S^{2p^i n-1}\{p^r\}$$

is the loop of a product of maps ϕ described in (4.1), and hence the fibre of the composite is $\Omega(\prod S^{2p^{k-1}}\{p\})$. These mappings w_k together with the map $\Omega T^{2n+1}\{p^r\} \xrightarrow{\ w\ } \Omega S^{2n+1}\{p^r\}$ are enough to find the order of the identity on the H-space $\Omega T^{2n+1}\{p^r\}$. Recall that w is the restriction of the natural $\Omega S^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}\{p^r\}$.

(4.3) Proposition. Let $F \longrightarrow E \longrightarrow B$ be a fibration of H-spaces. If $p^r.\mathrm{id}_B$ is null homotopic and if $p^s.\mathrm{id}_F$ is null homotopic, then $p^{r+s}\mathrm{id}_E$ is null homotopic.

Proof. Since $p^r.\mathrm{id}_E$ projects to $p^r\mathrm{id}_B$ which is null homotopic, there is a homotopy of $p^r.\mathrm{id}_E$ with a map $E \longrightarrow E$ whose image is contained in F. Composing with $p^s.\mathrm{id}_F$, we have a map which is null homotopic and homotopic to $p^{r+s}.\mathrm{id}_E$. This proves the proposition.

Now consider the maps
$\Omega C(n) \times \Omega(\prod_k S^{2p^k n-1}\{p^{r+1}\}) \longrightarrow \Omega T^{2n+1}\{p^r\}$ and $\Omega T^{2n+1}\{p^r\} \longrightarrow \Omega\prod S^{2p^k n-1}\{p^r\} \times \Omega S^{2n+1}\{p^r\}$. This leads to the following result.

(4.4) Assertion. The homotopy fibre of

$$\Omega T^{2n+1}\{p^r\} \longrightarrow \Omega\prod S^{2p^k n-1}\{p^r\} \times \Omega S^{2n+1}\{p^r\}$$

is $\Omega C(n) \times \Omega(\prod_k S^{2p^k n-1}\{p\}) \longrightarrow \Omega T^{2n+1}\{p^r\}$.

Now we apply (4.3) to this fibration, and we use the fact that p.id is null homotopic on the fibre and p^r.id is null homotopic on the base to deduce the following result.

(4.5) Theorem. (Neisendorfer) On the H-spaces $\Omega T^{2n+1}\{p^r\}$ and $\Omega^2 S^m(p^r)$ the map p^{r+1}.id is null homotopic.

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