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Lie Dynamical Systems : Dedicated in Friendship to Georges REEB

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LIE DYNAMICAL SYSTEMS

by

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TABLE

1. Motivations and examples of Lie Dynamical Systems (1)

A Lie dynamical system will be later defined as an action of a certain kind of Lie group G on a space M that is a differentiable manifold. Our goal is to develop various generalizations of the classical results of the qualitative theory of ordinary differential equations, or classical flows where the Lie group is the real time line R. In order to maintain the concepts of past and future trajectories, we shall impose the demand that the group G will be the product of R with some connected compact Lie group K and the classical results obtain when K collapses to a single point. Furthermore in order to extend the classical theories of flows on surfaces to comparable dynamical results on n-dimensional manifolds M, we shall be primarily concerned with Lie dynamical systems of co-dimension 1, mainly the case where K has orbits of highest dimension (n-2) and some G-orbits are of dimension (n-1). We always take $n \geqslant 2$.

One of the most important classical results for smooth flows of R on surfaces is the Poincaré-Bendixson Theorem |8,13| ; a future recurrent orbit of the sphere S^2, containing no singular point in its future limit set, must be a periodic orbit.

Another phrasing of this same classical result asserts that a minimal set Σ for a flow on S^2 must be either a singular orbit (critical or stationary point) or else a periodic orbit (topological circle). In this form the theorem has been shown to hold on every compact surface M^2, excepting the torus T^2, see |13,38|. For smooth flows on T^2 the famous theorem of Denjoy |8,10,13| asserts ; a minimal set on T^2 must be either a singular point, a periodic orbit, or the entire surface T^2, which is then filled by almost periodic trajectories so the flow is topologically equivalent to a familiar "linear irrational flow" on the torus.

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(1) This article is an enlargement and extension of the invited address on Lie Dynamical Systems presented by this author before the AMS meeting in St. Louis in April 1972. The author was encouraged to return to these ideas and to complete this paper by his participation in two Symposia |41,42| at the University of Warwick during the Summers of 1979 and 1980. A will-o'the-wisp fragment of our main Theorem 3 was glimpsed by the author while chasing a Queen Bee with Georges Reeb, and later was indicated in a Harvard Student Mathematics Club talk in 1951 ; thus this struggle has something in common with the Thirty Years' War.
This research was partially supported by the NSF Grant MCS 79 - 01998.

Our principal theorems will constitute natural generalizations of the theorems of Poincaré-Bendixson and of Denjoy for Lie dynamical systems of co-dimension 1 on compact n-manifolds. In order to relate these generalizations to the classical theory of ordinary differential equations, we shall briefly review some of the definitions and concepts of Lie transformation groups, and then present a few interesting examples.

In our theory of Lie dynamical systems the "phase space", that is, the ambient space containing the dynamical orbits, will be an n-dimensional differentiable manifold M without boundary (that is, M is a separable metrizable connected space with a maximal atlas of local charts or coordinate systems that are inter-related by C^{∞}-differentiable transformations). For instance M could be the real number space R^n, the standard n-sphere S^n, or the n-torus T^n. The "generalized time" or dynamical group G will be a Lie group (that is, G is a topological group whose identity component is open in G, and this component has the structure of a differentiable manifold on which all group operations are C^{∞}-differentiable). For instance G could be the real line R, the vector group R^m, the toral group $T^m = R^m/Z^m$, the special orthogonal group SO(m,R), or perhaps some product of these. The action Φ of G on M,

$$\Phi : GxM \to M$$

will be C^{∞}-differentiable- a phrase that we shall usually abbreviate as "differentiable" or "smooth".

We recall that a differentiable transformation group $\{G,M,\Phi\}$ (that is, M is a G-space in the terminology we usually follow |6|) consists of a smooth action of the Lie group G on the differentiable manifold M ; that is,

$$\Phi : GxM \to M \quad (g,x) \to \Phi(g,x)$$

and the related map

$$\Phi_g : M \to M \quad x \to \Phi(g,x) = \Phi_g(x)$$

are differentiable. Furthermore Φ_g is required to be a diffeomorphism of M onto M, for each given element $g \in G$, and the usual group homomorphism axioms hold :

$$G \to \text{Diffeom}\ (M) \quad g \to \Phi_g$$

can be viewed as a group homomorphism.

If the action Φ is defined only on some neighborhood of (e,M) in $G \times M$, wherein the required axioms hold, then we obtain a <u>local transformation group</u> |6,30|.

<u>Définition</u> - A <u>Lie dynamical system</u> is a differentiable transformation group $\{G,M,\Phi\}$ with the additional requirement that $G = K \times R$ where K is a connected compact Lie group.

<u>Example</u>

1. Take $G = SO(1,R) \times R$ acting on any differentiable manifold M. The trivial factor $SO(1,R)$, which is merely a single point, has no geometric significance and so the Lie dynamical system $\{G,M,\Phi\}$ reduces to an action of R on M. In this way every action of R on M can be interpreted as a Lie dynamical system. Thus, the theory of Lie dynamical systems includes the classical theory of flows of ordinary differential systems.

2. Consider the Newtonian gravitational problem in astronomy with m mutually attracting stars or planets interacting as point masses in the physical 3 - dimensional inertial space. The dynamical flow, according to the corresponding first-order differential system in the position - velocity phase space of 6m-dimensions , is specified by the force vector field (after fixing the center of mass at the origin so the 6 position-velocity coordinates of this centroid are held at zero). Then the determination of the system energy and the angular momentum 3-vector complete the specification of the 10 classical integrals, and the dynamical flow is thus described by an action of the time line R on the phase space M (which is, in general, a $(6m-10)$ - manifold).

In astronomical literature there is a further simplification known as the "elimination of the nodes", which takes into account the circular symmetry of the physical problem under rotations about the axis through the centroid, as designated by the angular momentum vector. That is, if the entire astronomical configuration were revolved through any specified angle about this axis in Euclidean 3-space, then the geometry of the orbit configuration would remain unchanged. In this manner the circle group S^1 yields an additional symmetry of the dynamical problem (that is, S^1 acts <u>equivariantly</u> on the dynamical system) and this is customarily incorporated into the astronomical analysis in various ways. In particular, we could describe the problem as a Lie dynamical system with $G = S^1 \times R$ acting on the phase space M.

This approach can be illustrated in more detail with reference to the basic Kepler problem of a single planet orbiting a fixed force center in the plane. In this case the position-velocity phase space is R^4 (excising the singularity at the force center) and a determination of the energy E of the planet specifies a 3-manifold M_E on which the dynamical flow of the time R acts. Then the elimination of the nodes defines a Lie dynamical system with group $G = S^1 x R$ acting on M_E. Here the orbits are of co-dimension 1 and generally consist of 2-tori in M_E (excepting some singular orbits corresponding to circular orbits in the physical plane). Then further specification of the angular momentum provides a selection of certain of these 2-tori which are moreover filled by periodic trajectories under the action of R.

3. Consider the oscillations of a particle attracted to a fixed center in Euclidean 3-space, by an attractive force that varies linearly with the radial distance to the center and is independent of the direction from the center. If we fix the mechanical energy, the resulting dynamical system is described by a flow of R on the phase space S^5.

Since the physical system has rotational symmetry, the special orthogonal group SO(3,R) acts equivariantly on this dynamical flow. Keeping in mind such symmetries, we can study the physical oscillator as a Lie dynamical system with group $G = SO(3,R) x R$ acting on S^5. Since each oscillation has the same period (independent of the energy level), the orbits of the Lie dynamical system are generically 3-manifolds, although some singular orbits that are diffeomorphs of S^2 also arise (corresponding to circular orbits in the physical Euclidean space).

4. Let M be a sympletic 2n-manifold with canonical charts $(x^1,\ldots,x^n,$ $y_1,\ldots,y_n)$, see |1,23,24|. Let H be a real differentiable function on M ; traditionally, H is called a Hamiltonian function with corresponding gradient dH and Hamiltonian vector field $dH^{\#}$ whose components in each canonical chart describe the Hamiltonian differential system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

Then $dH^{\#}$ generates a dynamical flow of R acting on M (say M is compact-otherwise we must deal with local flows).

A real differentiable function F on M is an integral for $dH^{\#}$ in case

$$dH^{\#}(F) = (F_x)\dot{x} + (F_y)\dot{y} = (F_x)(H_y) + (F_y)(-H_x) = 0 ,$$

so F is constant along each trajectory of $dH^{\#}$. Each such integral F specifies a Hamiltonian vector field $dF^{\#}$ that commutes with $dH^{\#}$, relative to the Lie bracket product of vector fields on M. In this manner the Hamiltonian vector fields $dF^{\#}$ that commute with $dH^{\#}$ are considered to be symmetries or "vector integrals" of H.

The set of all such vector integrals $dF^{\#}$ of H constitute a Lie algebra $\mathfrak{J}(H)$, which is a subalgebra of $\mathcal{L}(M)$, the Lie algebra of all differentable vector fields on M with the Lie bracket product, as usual. Each finite dimensional subalgebra $\mathfrak{J}_1 \subset \mathfrak{J}(H)$ then generates a unique effective transformation group $\{G(\mathfrak{J}_1), M, \Phi\}$, according to the definitions and concepts indicated later.

It is plausible that a study of the Lie dynamical system with group $G = G(\mathfrak{J}_1) \times R$ (generated by $\mathfrak{J}_1$ and $dH^{\#}$) may well be of interest in such Hamiltonian dynamical problems. In the particular case where $G(\mathfrak{J}_1)$ is the n-toral group T^n acting freely on a region of M, it is known that $dH^{\#}$ is then completely integrable in the sense of classical analytical dynamics |1,23|.

5. On a given differentiable n-manifold M consider a control dynamical system specified by r tangent vector fields $f_1, f_2, \ldots, f_r$ and r scalar controllers $u_1(t), u_2(t), \ldots, u_r(t)$ that can be chosen as arbitrary piecewise continuous control functions. Then the control dynamics are defined on M by the collection of all time - varying vector fields of the form $u_1(t)f_1 + \ldots + u_r(t)f_r$, or the ordinary differential system

$$\frac{dx^i}{dt} = u_1(t)f_1{}^i(x) + \ldots + u_r(t)f_r{}^i(x) \quad \text{for} \quad i = 1,\ldots,n,$$

as indicated in each local chart $(x^1, x^2, \ldots, x^n)$.

If the vector fields $f_1, \ldots, f_r$ generate a finite dimensional Lie subalgebra of $\mathcal{L}(M)$, then the corresponding Lie group G acts on M (technically, as a local transformation group) to produce orbits that are precisely the accessibility sets for the control problem, see |7,19|.

2. Generalities of Differentiable Transformation Groups

In order to make this presentation more in accord with the methods and concepts of classical dynamical systems, and to make this investigation more readily accessible to researchers in the qualitative theory of differential systems, we review in this section some general properties of transformation groups, in particular, the foliations by orbits of tubular neighborhoods, see |6,34,36|.

Let $\{G,M,\Phi\}$ be a differentiable transformation group with the Lie group G having the action Φ on the differentiable n-manifold M. For each point $x \in M$ we define the orbit of x to be the set

$$G(x) = \{\Phi_g(x) | g \in G\},$$

and we somotimes refer to the map

$$G \to G(x) \qquad g \to \Phi_g(x)$$

as a trajectory tracing the orbit G(x). The stability or isotropy subgroup G_x of x is the closed subgroup of G leaving x fixed,

$$G_x = \{g \in G | \Phi_g(x) = x\}.$$

It is easy to see that each point y on the orbit G(x) has a stability subgroup G_y conjugate to G_x in G ; and every such conjugate subgroup so arises - for this reason we denote the G-isotropy type of G(x) by the conjugacy class (G_x).

The homogeneous space of left cosets G/G_x is diffeomorphic with the orbit G(x), which is thereby recognized as an embedded (not necessarily as a topological subspace) submanifold of M. Moreover the natural action of G on G/G_x is equivariantly diffeomorphic with the action of G on G(x). Furthermore the action of G on some orbit G(z) projects equivariantly onto the orbit G(x) if and only if

$$(G_z) \leqslant (G_x) \ ,$$

that is, G_z is conjugate to a subgroup of G_x ; and we also denote this relation by

$$\underline{\text{orbit type}}\ G(z) \geqslant \underline{\text{orbit type}}\ G(x).$$

If $\bigcap_{x \in M} G_x = G_M$ is discrete in G, then the transformation group is <u>locally (or almost) effective</u> on M ; and if $G_M = e$ then the action is <u>effective</u> on M. Otherwise, G_M is necessarily a closed normal subgroup and we can define an effective action of the quotient group G/G_M on M. For this reason we frequently study transformation groups that are effective ; or possible only locally effective so G_M is discrete and each element in some deleted neighborhood of e in G moves some point of M.

The orbit G(x) for $x \in M$ is <u>nonsingular (or regular)</u> in case G_x is discrete, in which case each isotropy group of the conjugacy class (G_x) is also discrete, and hence dim G(x) = dim G. Otherwise, when dim G(x) < dim G the orbit G(x) is called <u>singular</u>. In the extreme case where $G_x = G$ the orbit G(x) consists of a single point $x \in M$, which is called a <u>singular or stationary point</u> for the transformation group.

In other terminology $\{G,M,\Phi\}$ is <u>free</u> at x when $G_x = e$ and is <u>locally free</u> at x when G_x is discrete in G. In such cases the same condition holds at each point of the orbit G(x). Hence the group G acts locally freely on G(x) if and only if the orbit G(x) is nonsingular. Of course, nonsingular orbits can exist only when dim G $\leq$ dim M.

<u>Example</u> - In the case of a <u>classical dynamical flow</u> of the group R on a differentiable manifold M, the map

$$R \to M \quad t \to \Phi_t(x), \text{ for each point } x \in M,$$

describes the trajectory initiating at the point $x \in M$, when $t = 0$. The tangent vectors to all such trajectories in M constitute a vector field

$$f(x) = \left.\frac{\partial \Phi_t(x)}{\partial t}\right|_{t=0},$$

which defines the <u>infinitesimal generator</u> of the dynamical flow on M. On the other hand each differentiable tangent vector field on M generates, by means of its integral curves, a unique (local) flow on M-in fact, a flow for all $t \in R$ provided each trajectory is complete for all times, which is certainly the case when M is compact.

We now interpret the prior concepts for transformation groups as applied to this example of a classical flow. The flow Φ_t on M is locally effective just in case f(x) is somewhere nonzero. In addition Φ_t is then effective on M, unless each trajectory is periodic with the same common period -in which case the circle group $S^1 = R/Z$ acts effectively on M.

The flow Φ_t is locally free at a point $x_o \in M$ in case $f(x_o) \neq 0$. In the contrary case where $f(x_o) = 0$, the point x_o constitutes a singular orbit. In addition Φ_t is free at x_o, unless the trajectory through x_o is periodic and thus traces an orbit that is a diffeomorph of a circle.

We next recall the concept of the <u>infinitesimal generator</u> Λ of a (local) transformation group $\{G,M,\Phi\}$, and we note how Λ consists of a collection of vector fields whose flows generate the action of G on M. We have previously mentioned that the set of all differentiable tangent vector fields on M form an infinite dimensional Lie algebra $\mathcal{L}(M)$. That is, $\mathcal{L}(M)$ is a real linear space under pointwise operations, and an algebra under the usual Lie bracket product on vector fields

$$[v,w]^i = \frac{\partial v^i}{\partial x^j} w^j - \frac{\partial w^i}{\partial x^j} v^j$$

for tangent vector fields v and w as expressed in terms of components in any local chart $(x^1,\ldots,x^n)$ on M. In the same manner the right-invariant vector fields on G form a finite dimensional Lie algebra g, which can also be described as the tangent space at e with the appropriate product introduced. The infinitesimal generator Λ of $\{G,M,\Phi\}$ will be defined by a Lie algebra homomorphism of g into $\mathcal{L}(M)$.

Let $\{G,M,\Phi\}$ be a local transformation group on the differentiable manifold M. For each vector u at the identity e of G there is a corresponding 1-parameter subgroup, namely exp(tu), at least for t near the zero of R. Thus $\{R,M,\Phi^{(u)}\}$, with $\Phi^{(u)}(t,x) = \Phi(\exp(t\,u),x)$, is a local flow of R on M. We denote the infinitesimal generator of the flow corresponding to u by $\Lambda(u)$; so $\Lambda(u)$ is a differentiable tangent vector field on M. Then the correspondence

$$g \to \mathcal{L}(M) \qquad u \to \Lambda(u)$$

is known to be a Lie algebra homomorphism of g into $\mathcal{L}(M)$. The image $\Lambda \subset \mathcal{L}(M)$ is called the infinitesimal generator of $\{G,M,\Phi\}$.

Clearly two local transformation groups $\{G,M,\Phi\}$ and $\{G_1,M_1,\Phi_1\}$ which are equivariantly differentiably isomorphic (allowing a diffeomorphism Ψ of M onto M_1), must have infinitesimal generators that are isomorphic (induced by the diffeomorphism Ψ). On the other hand, the recovery of the local transformation group $\{G,M,\Phi\}$ from its infinitesimal generator Λ is also possible.

Let Λ be a finite dimensional Lie subalgebra of $\mathcal{L}(M)$ on a differentiable manifold M. Then there exists a local transformation group $\{G,M,\Phi\}$ having the given infinitesimal generator Λ. Moreover this local transformation group is unique, up to differentiable equivariant isomorphism, provided we require that $\{G,M,\Phi\}$ is locally effective |6,30|. Furthermore if M is compact (or if each trajectory of the vector fields of Λ is complete for all times $t \in R$), then Λ generates a global transformation group $\{G,M,\Phi\}$ which is locally effective. If we further demand that G is simply-connected, or else that G acts effectively on M, then $\{G,M,\Phi\}$ exists as the unique locally effective transformation group with the infinitesimal generator Λ on M, see |6,29, 30|.

In terms of this infinitesimal generator Λ for a given transformation group $\{G,M,\Phi\}$ we can assert that the action is locally effective just in case the homomorphism $g \to \Lambda$ is an isomorphic injection into the Lie algebra $\mathcal{L}(M)$. In the same spirit let Λ_{x_o} be the set of tangent vectors at a point x_o, belonging to the various vector fields of Λ. Then there is a linear map $g \to \Lambda_{x_o} \subset T_{x_o}M$ into the tangent space at x_o, and we can assert that the action is locally free at x_o just in case this map is an isomorphic injection onto Λ_{x_o}.

Let $\{G,M,\Phi\}$ be a differentiable transformation group acting on the n-manifold M. Let $G(x)$ be the orbit of a point $x \in M$. Then the limit set λ for x is the intersection of the closure of "orbit tails", that is

$$\lambda(x) = \bigcap_{C} C\ell\ \{\Phi_g(x) \mid g \notin C\},$$

where C runs over the collection of all compact subsets of G. Clearly $\lambda(x)$ is a closed invariant set in M, under the action of the transformation group. Also $\lambda(x) = \lambda(y)$ for each point y on the same orbit $G(x)$, and so $\lambda(x)$ is called the limit set of the orbit $G(x)$.

The point $x \in M$, and also the orbit $G(x)$, is called recurrent in case $x \in \lambda(x)$, (so then $G(x) \subset \lambda(x)$).

A compact invariant set $\Sigma \subset M$ is called minimal in case Σ contains no compact invariant proper subset ; hence each orbit in Σ must be recurrent. Clearly each compact invariant set in M must contain a minimal set for the transformation group. If Σ has a nonempty interior in M, then $\Sigma = M$.

Finally let us turn to some special, but nevertheless fundamental, properties of the orbits for $\{\overline{G},M,\Phi\}$, under the assumption that $\overline{G}$ is a compact Lie group acting on the n-manifold M |6|. In this case each orbit $\overline{G}(x)$ is a compact submanifold of M, and the action of $\overline{G}$ near $\overline{G}(x)$ in M can be understood in terms of the structure of tubular neighborhoods. To explain this concept, we consider the normal bundle N (with respect to some auxiliary Riemann metric on M) over $\overline{G}(x)$, and we note the natural $\overline{G}$-action on N with the corresponding action of the compact isotropy group $\overline{G}_x$ on the slice N_x of normal vectors at the point $x \in \overline{G}(x)$ (in fact, we can take a Euclidean metric on N_x so $\overline{G}_x$ acts as linear isometries), for details see |6,25,26|. Then a disk-bundle neighborhood of $\overline{G}(x)$ in N (with $\overline{G}$ acting by fiber-preserving transformations on the bundle) is equivariantly diffeomorphic with an invariant tubular neighborhood U of $\overline{G}(x)$ in M. As is proved in |6, p. 46, 82, 306-308|, the action of the compact group $\overline{G}$ on a tubular neighborhood U of the orbit $\overline{G}(x)$ is entirely determined (up to $\overline{G}$-equivariant diffeomorphism) by i) the group $\overline{G}$ and stability subgroup $\overline{G}_x$, and ii) the slice representation of $\overline{G}_x$ in the group of Euclidean isometries of the vector space N_x.

As a consequence of this construction of a tubular neighborhood around any selected orbit, we can conclude that the isotropy subgroup (and hence the orbit-type) varies in a semi-continuous manner on M. That is, for each point z near x in M we find that $(\overline{G}_z) \lesssim (\overline{G}_x)$, and only a finite number of distinct orbit types can meet a prescribed compact subset of M when $\overline{G}$ is compact.

From these considerations it follows that there exists a unique largest orbit-type (smallest $\overline{G}$-isotropy type), denoted as the principal orbit-type, and the principal orbits fill an open dense subset in M. If M is orientable, then it is known that each principal orbit is also orientable. For a principal orbit $\overline{G}(x)$ the isotropy subgroup $\overline{G}_x$ acts trivially on the slice N_x, defined by the normal vectors to $\overline{G}(x)$ at the point x. Hence the invariant tubular neighborhood of a principal orbit $\overline{G}(x)$ is just the product manifold $\overline{G}(x) \times N_x$ (or rather some disk neighborhood of $\overline{G}(x) \times 0$ therein).

Example 1 - For each integer $1 \leqslant \ell \leqslant n-1$ we define an effective action of the compact Lie group $K_\ell = SO(\ell,R) \times SO(n-\ell,R)$ on the sphere S^n, and such that the principal orbits have codimension 2.

Consider R^{n+1} with the usual Euclidean metric so that the unit vectors comprise the sphere S^n. Let $SO(\ell,R)$ act as usual on the first ℓ-coordinate of R^{n+1} (that is, as the rotation group on the subspace R^ℓ), and let $SO(n-\ell,R)$ act as rotations on the Euclidean subspace $R^{n-\ell}$ spanned in R^{n+1} by the next $(n-\ell)$-coordinates - and let the last coordinate of R^{n+1} remain constant. Note that K_ℓ acts on R^{n+1} so as to preserve the Euclidean norm (and with the final coordinate fixed), and thus K_ℓ acts on S^n (and with the poles fixed at $(0,0,\dots 0,\pm 1)$). This action of K_ℓ on S^n is effective, and the principal orbits each have dimension $(\ell-1) + (n-\ell-1) = n-2$.

Moreover the point $(0,0,\dots,0,1)$ in S^n is left fixed under K_ℓ, and so we can delete this point to obtain the space R^n on which K_ℓ acts effectively, with principal orbits of codimension 2.

Example 2 - Each compact connected Lie group K acts effectively, in a trivial way, on the product $M^n = K \times T^2$ (where T^2 is the 2-torus group). Here each orbit $K(x)$ is diffeomorphic to K and has codimension 2 in the n-manifold M^n.

3. Generalities of Lie Dynamical Systems

In the preceeding two sections we reviewed aspects of the general theory of transformation groups, as appropriate for our theory of Lie dynamical systems ; and now we turn to the development proper for this new approach to dynamical theory. Let us examine the general concepts of the prior section as related to a differentiable transformation group $\{G,M,\Phi\}$ that forms a Lie dynamical system on a given n-manifold M. That is, in this section we hereafter assume that $G = K \times R$, where K is a connected compact Lie group, acts on M. Since the group G then has two topological ends, namely the past and the future according to the time coordinate $t \in R$, we can examine the particular behavior of an orbit $G(x)$ at each end individually.

For the Lie dynamical system with $G = K \times R$ each point $x \in M$ has an orbit

$$G(x) = \{\Phi_{kt}(x) \mid (k,t) \in K \times R\},$$

and also the sub-orbits $K(x)$ and $R(x)$ under the corresponding subgroups $(K,0)$ and $(e_K xR)$ of $G = KxR$. The transformation group $\{K,M,\Phi_K\}$ refers to the actions of the compact group K on M, and accordingly the theory of the preceeding section dealing with "compact transformation groups" is applicable. The transformation group $\{R,M,\Phi_K\}$ is a classical flow or differential system on M. Of course, each action of K commutes with each action of R within $\{G,M,\Phi\}$. Also, the corresponding stability subgroups are G_x, together with K_x and R_x. Note that the product group $K_x x R_x \subset G_x$, but the inclusion may be proper.

The orbit $G(x)$ can be decomposed into a past (when $t < 0$), a future (when $t > 0$), and a present at $t = 0$. Note that if

$$\Phi_{k_1 t_1}(x) = \Phi_{k_2 t_2}(x) ,$$

then the sections of the orbit $G(x)$ at times t_1 and t_2 coincide,

$$\{\Phi_{kt_1}(x) | k \in K\} = \{\Phi_{kt_2}(x) | k \in K\}.$$

If the past and future half-orbits of $G(x)$ meet, then as shown later, the orbit $G(x)$ is compact so the past and future half-orbits actually coincide.

As usual, we define the positive or future ω-limit set of the orbit $G(x)$ by the set

$$\omega(x) = \bigcap_{\tau>0} C\ell \, \{\Phi_{kt}(x) | k \in K, \; t>\tau\} \quad ,$$

and we note that $\omega(x)$ is independent of the choice of point x on the orbit $G(x)$. It is easy to verify that a point y lies in $\omega(x)$ if and only if - there exists sequences $k_i \in K$ and $t_i \to +\infty$ with $\Phi_{k_i t_i}(x) \to y$. In the case $y \in \omega(x)$, we remark that $G(y) \subset \omega(x)$.

The negative or past α-limit set $\alpha(x)$ of the orbit $G(x)$ is defined in the same manner for $t \to -\infty$. Both $\alpha(x)$ and $\omega(x)$ are closed invariant sets ; and, when M is compact they each are necessarily nonempty compact connected invariant sets.

In accord with the usual terminology of topological dynamics we say that x, or the orbit $G(x)$, is future recurrent in case $G(x) \subset \omega(x)$; past recurrent in case $G(x) \subset \alpha(x)$; and recurrent in case $G(x)$ is both past and future recurrent. These concepts are developed in the treatise |12| . Also the point x, or the orbit $G(x)$, is future topologically transitive in case $\omega(x) = M$; past topologically transitive in case $\alpha(x) = M$; and topologically transitive in case $\alpha(x) \bigcap \omega(x) = M$.

Let $\Sigma \subset M$ be a minimal set for the Lie dynamical system $\{G,M,\Phi\}$ where $G = K \times R$, as stated earlier. Then each point $x \in \Sigma$ has an orbit that is recurrent, and topologically transitive in Σ. From this transitivity we can conclude that each G-orbit in Σ has the same dimension and, in fact, the same orbit-type as specified by the conjugacy class (G_x). Hence if one G-orbit in Σ is nonsingular (or even K-principal), then so is every G-orbit in Σ, and we call Σ a nonsingular (or a K-principal) minimal set. As remarked earlier, if a minimal set Σ has a nonempty interior in the connected space M, then $\Sigma = M$.

Theorem 1 - Consider a Lie dynamical system $\{G,M,\Phi\}$ with group $G = K \times R$, for a connected compact Lie group K, acting on the n-manifold M. Then each orbit $G(x)$ must be of exactly one of the following three kinds

1) stationary orbit $G(x) = H$, where $H = K(x) = K/K_x$ is a compact homogeneous space ; and G_x projects onto all R so $K(x)$ is stationary as a set, for time-actions of R.

2) periodic orbit $G(x) = H \rtimes S^1$ is a compact fiber bundle over the base circle with the compact homogeneous space $H = K/K_x$ as fiber. If K_x is discrete, then the orbit $G(x)$ is an orientable manifold with $\dim G(x) = \dim K + 1$.

3) line-manifold orbit $G(x) = H \times R$ is a differentiable injective immersion (submanifold topology may not coincide with subspace topology) of the product $H \times R$ into M.

Proof

For the point x on the orbit $G(x)$ define the return times by the set

$$T = \{t \in R \mid \exists k \in K \text{ with } \Phi_{kt}(x) = x\}.$$

Note that $t_1 \in T$ if and only if the entire K-orbit returns to $K(x)$, that is,

$$K(\Phi_{e_K t_1}(x)) = K(x).$$

From this we easily find that T is a closed subgroup of R.
Hence there are three distinct cases for analysis

1) $T = R$; so $K(x)$ is stationary (as a set) for all times $t \in R$,

2) T is a discrete subgroup of R ; and thus T consists of all integral multiples of some smallest positive period τ,

3) $T = 0$; so $K(\Phi_{e_K t}(x))$ never returns exactly to $K(x)$ for any $t \neq 0$.
Case 1) yields the stationary orbit, and case 3) yields the line-manifold orbit. In both these cases the K-stability subgroup K_x determines the homogeneous space $H = K(x) = K/K_x$; and for the line-manifold $G(x) = HxR$ since the K-actions and R-actions commute with each other.

We henceforth concentrate attention on case 2) where $T = \{m\tau | m \in Z\}$, for the smallest positive period τ for the return times within the orbit $G(x)$. Then for each positive $\varepsilon < \tau$ for the orbit-segment

$$G_\varepsilon(x) = \{\Phi_{kt}(x) | k \in K,\ 0 \leqslant t \leqslant \varepsilon\}$$

is a differentiable product $Hx[o,\varepsilon]$ that is topologically embedded in M. Since $K(\Phi_{e_K \varepsilon}(x)) \to K(x)$ as $\varepsilon \to \tau$, we conclude that the manifold $G(x)$ coincides with the compact set that is the orbit-segment for $o \leqslant t \leqslant \tau$.

Thus $G(x)$ is a compact submanifold of M, and further the projection map

$$G(x) \to S^1 \qquad \Phi_{kt}(x) \to t$$

(where the circle S^1 is recognized as the interval $[0,\tau]$ with endpoints identified), displays $G(x)$ as a fiber bundle over S^1 with the fiber $H = K/K_x$. We designate this fiber dundle by $G(x) = H \rtimes\!\!\!\ltimes S^1$, which can be either the product bundle or a twisted bundle over the base S^1, depending on the homotopy type of the identification map of the fiber H upon encircling the base S^1. $\Box$

Remark 1. In the case of a periodic orbit $G(x) = H \rtimes\!\!\!\ltimes S^1$ the compact manifold $G(x)$ can be obtained as a projection of the product HxR correspoinding to the identification, upon following the R-flow for duration $\tau > o$,

$$(h,0) \approx (k_\tau h,\tau)$$

for some appropriate element $k_\tau \in K$. Here $h = kK_x$ is an arbitrary left coset of the homogeneous space $H = K/K_x$. Moreover the "identification multipliers" k_τ and $\hat{k}_\tau$ yield the same diffeomorphism of H (the fiber over $t = 0$) if and only if, for all $k \in K$, we have

$$k_\tau(kK_x) = \hat{k}_\tau(kK_x) \text{ or } \hat{k}_\tau^{-1}k_\tau \in k\, K_x k^{-1},$$

that is, $\hat{k}_\tau^{-1}k_\tau$ lies in the intersection of all subgroups of K conjugate to K_x. In particular, $\hat{k}_\tau^{-1}k_\tau$ must lie in K_x itself.

The fiber bundle $H \rtimes S^1$ is a product manifold just in case the map $H \to k_\tau H$ is homotopic to the identity map. From this result we note that, if the group of homotopy classes of self-homeomorphisms of H is finite, then $H \rtimes S^1$ is finitely covered by the product $H \times S^1$. It is for this reason that we call any compact orbit a periodic orbit for the Lie dynamical system.

Remark 2 - Both the case of a periodic orbit and a line-manifold orbit are called non-stationary, in contradistinction to the case 1) of a stationary orbit. The stationary orbit $G(x)$ arises when the R-action curve $R(x)$ lies tangential to $K(x)$, that is, G_x projects onto R -- otherwise $G(x)$ is nonstationary. Also $G(x)$ is periodic if and only if G_x projects to a proper subgroup of R.

Let us next examine the behavior of the orbits of a Lie dynamical system $\{G,M,\Phi\}$, with $G = KxR$ as earlier, in the neighborhood of a given orbit $G(x)$ that is assumed stationary in the sense 1) of the Theorem 1. Especially, we consider the possibility of the "linearization of the dynamics" near the orbit $G(x) = H$.

Thus we assume that the orbit $G(x) = H = K(x)$ is stationary in the sense of Theorem 1. In this case the construction of an invariant tubular neighborhood of $K(x)$, with regard to the K-action only, provides a sort of linearization.

In the particular case when $K_x = K$, so $K(x)$ reduces to a single point, then the K-action in a neighborhood of $x \in M$ is equivariantly equivalent to a linear action of a subgroup of the orthogonal group about the origin of R^n (in accord with Bochner's theorem |3,6|). Hewever if $K_x \neq K$, so $K(x) = H$ is a manifold of dimension at least 1, then, if $R_x = R$ the stationary points of the R action are not isolated points but constitute all the entire manifold $K(x)$. In such a case the classical linearization theorems for R-flows are not applicable. We shall not pursue such questions of linearization further at this point.

The case of greatest interest in our theory of a Lie dynamical system $\{G,M,\Phi\}$, for a $G = KxR$ acting on the n-manifold M, is the situation where $\dim K(x) = n-2$ and $\dim G(x) = n-1$. In the simplest case dim $G = n-1$ and the group G acts locally freely at $x \in M$, so the orbit $G(x)$ has co-dimension 1 in the n-manifold M. For this case, where G acts locally freely at $x \in M$, $G(x)$ is a nonsingular orbit since G_x is discrete ; hence K_x is discrete and $H = K(x) = K/K_x$ is an orientable (n-2)-manifold in $G(x)$.

As a final question in this section we examine the problem of the existence of a Lie dynamical system $\{G,M,\Phi\}$, with $G = KxR$ acting nontrivally (locally effective) on the n-manifold M. We show that this problem can be reduced to the study of locally effective actions of the compact group K on M. For this latter problem there is a considerable literature regarding compact transformation groups |15,26,27,28,33|. For instance, it is known that the circle group SO(2,R) cannot act in a locally effective manner on the compact 3-manifold $M^3 = (KBxS^1) \# (P^2xS^1)$, here KB is the Klein Bottle surface, P^2 is the real projective plane, and the products $(KBxS^1)$ and (P^2xS^1) are combined as a connected sum upon the identification of the two spherical surfaces resulting from the excision of corresponding compact 3-balls.

Remark - Let $G = KxR$ act effectively (or locally effectively) on a differentiable n-manifold M, as a Lie dynamical system. Then the induced K-action on M is obviously also effective (locally effective).

On the other hand an effective (locally effective) K-action on M can always be extended to a corresponding action of $G = KxR$ as a Lie dynamical system on M. If K acts transitively on M, so that there is just one orbit $K(x) = M$. Then it is trivial to define the R-action to be only the identity (generated by the zero vector field on M) so as to obtain the desired action of $G = KxR$ on M. But in case the principal orbit $K(x) \neq M$, some further care is needed in defining the appropriate commuting R-action. This construction is clarified in the next THeorem 2, and the subsequent remark.

Theorem 2 - Let K be a connected compact Lie group with an effective (locally effective) action on a differentiable n-manifold M. Then there exists an effective (locally effective) action of $G = KxR$ as a Lie dynamical system on M, so the action Kx0 is the given K-action on M.

<u>Moreover, provided that a principal</u> K-<u>orbit</u> K(P) <u>has dimension less than</u> n, <u>we can require that the orbit</u> G(P) <u>is nonstationary with</u> dim G(P) = dim K(P)+1.

<u>Proof</u>

Let K act in a locally effective manner on the n-manifold M. As mentioned in the previous remark we can assume that a principal orbit $K(x) \neq M$, and we seek to construct an appropriate locally effective action of $G = K \times R$ on M.

Since K(x) is principal, every K-orbit suitable near K(x) has the same orbit-type, corresponding to the conjugacy class (K_x). Then there exists an K-invariant tubular neighborhood U of the principal orbit K(x) in M ; where U is the product of K(x) and a transverse slice N_x of the normal bundle - or rather a disk-neighborhood $U \approx K(x) \times B$ therein. Furthermore the K-action on U leaves fixed the coordinates of the Euclidean ball B and acts on K(x), and the other "horizontal K-orbits" in U, so as to preserve the product structure of $U \approx K(x) \times B$.

Next we define the action of R on M by means of the flow generated by a vector field v on M. Namely, take $v \equiv 0$ outside the tubular neighborhood U in M, but within the tube U we now specify v explicitly in terms of the product coordinates (y,r,θ). Here y is an arbitrary point in the orbit K(x), and r is a radial distance in B (as measured outwards from the center lying on K(x)), and θ is an appropriate angular multi-coordinate in the Euclidean ball B. We select coordinates in B so $0 \leq r \leq 1$, and then define the vector field v by the formulas

v) $\qquad \dot{r} = f(r), \quad \dot{\theta} = 0, \quad \dot{y} = 0 ,$

where $f(r) \geq 0$ is a real C^∞-function on $[0,1]$ which vanishes near the endpoints of the unit interval, but which is strictly positive at the midpoint r = 1/2. |Various modifications of the vector field v could be used - see remarks following the theorem|.

Because K acts on the tube U $\approx$ K(x)xB by "horizontal motions" only– that is, leaving the coordinates (r, θ) fixed, and preserving the class of "vertical slices" as coordinatized by y = constant, the actions of R along the flow generated by v are seen to commute with the actions of K. In this way we obtain an action of G = KxR on the entire n-manifold M. Moreover the orbit G(P), for K(P) a K-principal orbit near K(x), say at radial coordinate r = 1/2 in U, is nonstationary with dim G(P) = dim K(P)+1.

In order to show that this action of G = KxR is locally effective on M, we need only prove that each nonzero tangent vector u, at the identity element of the Lie group G, yields a nonzero vector field $\Lambda(u)$ on M ; where Λ is the infinitesimal generator of the transformation group $\{G,M,\Phi\}$, as earlier. Certainly if u is tangent to K, or if u is tangent to R in G = KxR, then $\Lambda(u)$ does not vanish identically on M (because both K and R act in a locally effective manner on M). Moreover $\Lambda(u)$ does not vanish identically within the tube U – because K(x) is a principal K-orbit (and each K-isotropy subgroup on M contains some K_y for $y \in K(x)$), and because $f(\frac{1}{2}) > 0$ in the respective two cases.

But suppose $u = u_K+u_R$, for u_K and u_R nonzero vectors tangent to K and R respectively in G = KxR. We examine $\Lambda(u)$ in the tube U where the "radial vector field" $\Lambda(u_R)$ does not vanish identically. Since $\Lambda(u_K)$ is a "horizontal vector field" in U (that is, tangent to the K-orbits), then the sum

$$\Lambda(u) = \Lambda(u_K) + \Lambda(u_R)$$

must have a nonzero radial component in the tube U where r = 1/2. Hence $\Lambda(u)$ does not vanish identically on M, and therefore G = KxR acts as a locally effective transformation group on M.

Finally assume that K acts effectively (and hence locally effectively) on M, and we shall note that the corresponding stronger property also holds for the constructed G-action. Consider the G = KxR action, as defined earlier, and take an element $g = (k,\tau) \in G$ that acts as the identity transformation on the manifold M. We must prove that $g = (k,\tau)$ is, in fact, the identity element g = (e,o) of KxR.

Since $g = (k, \tau)$ yields the identity transformation on that principal K - orbit K(x), where R acts trivially, we conclude that $k \in K$ leaves K(x) pointwise fixed. Thus $k \in (\bigcap K_y)$, where the intersection is over all isotropy subgroups conjugate in K to K_x. Since K(x) is a K-principal orbit (with minimal isotropy subgroup), k must leave each K-orbit on M pointwise-fixed. This means that $k = e$ is the identity element of K.

But now note that $g = (e, \tau)$ acts on the tube U so as to move some K-orbits on to other K-orbits, unless $\tau = 0$. Therefore, since g acts as the identity transformation on M, we conclude that $\tau = 0$ so $g = (e,0)$. Hence G acts effectively on M, as required. $\square$

Remark - As mentioned in the proof of Theorem 2 various choices for the vector field v on M can be of particular interest. We next redefine such a vector field $\hat{v}$, in the case where K(x) has codimension 2 in M, to illustrate the construction of a nonstationary periodic orbit $G(P_o)$ which is an example of a "limit cycle" for nearby G-orbits.

As in Theorem 2 assume that the connected compact Lie group K acts on the n-manifold M as a locally effective transformation group. Moreover, assume that the principal orbit K(x) has codimension 2 in M. Then, just as in Theorem 2, we seek to define a locally effective action of G = KxR as a Lie dynamical system on M ; but now with certain special types of G-orbits that are nonstationary with codimension 1 in the tube U around K(x) in M.

To modify the construction in the proof of Theorem 2 we consider a vector field $\hat{v}$ on M, vanishing outside the tube U, and given within U by the formulas

$$\hat{v}) \qquad \dot{r} = (\frac{1}{2} - r)\,\hat{f}(r), \quad \dot{\theta} = 1, \quad \dot{y} = 0.$$

Here (y,r,θ) are defined in U as earlier - so (r,θ) are plane polar coordinates in the 2-disk B, and where we select the real non-negative C^∞-function $\hat{f}(r)$ on $0 \leqslant r \leqslant 1$ to vanish near the endpoints and to be identically equal to 1 in a subinterval around the midpoint $r = 1/2$. As in Theorem 2 the R-action generated by $\hat{v}$ commutes with the K-action on U, and therefore on the entire manifold M.

Now note that the G-orbit of the point $P_0 = (y = 0,\ r = 1/2,\ \theta = 0)$ in U is diffeomorphic with the product $K(x) \times S^1$. Hence $G(P_0)$ is a periodic orbit in the sense of Theorem 1. Furthermore $G(P_0)$ serves as an example of a "limit cycle" such as arises in Theorem 3 and its subsequent corollaries.

4. Qualitative Theory of Lie Dynamical Systems of codimension 1 on Simply-Connected Manifolds, especially Spheres

In this section we commence our serious investigations of the actions of a Lie group $G = K \times R$ on a differentiable n-manifold, as a Lie dynamical system. Assume there exists an orbit $G(x)$ of co-dimension 1 in M, and then we have the immediate observations

a) If $G(x)$ is nonsingular, then the compact connected Lie group K has dimension $(n-2)$;
b) If $G(x)$ is nonstationary, then $\dim K(x) = n-2$ so we conclude only that $\dim K \geqslant (n-2)$.

However we shall not impose any such hypotheses on dim K unless they are explicitly specified.

Our first goal is to prove some lemmas leading to a generalization of the classical Poincaré-Bendixson theorem concerning flows on spheres, although the next two Lemmas are valid on general manifold and will play on important role in a later chapter.

Lemma 1 - Let $G = K \times R$ act on a differentiable n-manifold M as a Lie dynamical system, with a nonstationary orbit $G(x)$ of codimension 1. Then $K(x)$, and also each $K(y)$ for y sufficiently near to x in M, is a compact $(n-2)$-manifold.

Moreover, if M and $K(x)$ are orientable manifolds, then all such $K(y)$ for y sufficiently near x, are of the same K-orbit type as $K(x)$. In this case there exists a K-invariant tubular neighborhood U of $K(x)$ in M such that

i) U is K-equivariantly diffeomorphic with the product $K(x)$ and a 2-disk B ;

ii) K acts trivially on U as a product ; that is, the slice representation of K_x on the normal 2-disk at x is just the identity, and K acts on each horizontal section of U just as on $K(x)$, so as to preserve the vertical fibers B of the product.

Proof

The orbit $K(x)$ is in the nonstationary orbit $G(x)$ that has $\dim G(x) = n-1$; hence $\dim K(x) = n-2$. Therefore, for y sufficiently near to x in M, each K-orbit $K(y)$ is a compact manifold with $\dim K(y) \geqslant (n-2)$. Certainly $K(y) \neq M$, for otherwise we would have $K(x) = K(y) = M$, and thus $\dim K(y)$ must be either $(n-2)$ or $(n-1)$. We next show that $\dim K(y) = n-2$, for all y near to x in M.

Take a K-invariant tubular neighborhood U_1 of $K(x)$, with U_1 diffeomorphic to a fiber bundle over the base $K(x)$ with a fiber B that is an (open) 2-disk normal to $K(x)$ in M. (All goemetric properties like orthogonality can be interpreted in terms of a convenient auxiliary Riemann metric on M). Moreover the K-action on U_1 is entirely specified by the action of the stability subgroup K_x, acting as an orthogonal subgroup on the Euclidean 2-disk B. We shall show that this "slice representation" of K_x on B is, in fact, a finite subgroup of $O(2,R)$, which determines that each K-orbit $K(y)$ in U_1 is a finite covering space of $K(x)$.

Suppose K_x, as a subgroup of $O(2,R)$, is not finite. In such a case there would exist orbits $K(y)$ in U_1 whose intersection with the disk B are circles arbitrarily near to the center point $B \cap K(x)$. But then such an orbit $K(y)$ would be an $(n-1)$-manifold that constitutes the boundary of some narrow tubular neighborhood of $K(x)$ in U_1. However the action of $G = K \times R$ yields the orbit $G(x)$ that contains both the compact orbit $K(x)$, and its time-translates under the group R. Since $\dim G(x) > \dim K(x)$, we conclude that the curve $R(x)$ is not tangent to $K(x)$ but has a tangent at x with a nonzero component in the normal slice. This implies that $G(x)$ would meet the orbit $K(y)$, and accordingly that $K(x)$ would be diffeomorphic to $K(y)$. But this contradicts the condition that $\dim K(y) > \dim K(x)$, and hence we conclude that the slice representation of K_x on B is necessarily a finite group.

Since the slice representation of K_x on B is a finite group, each orbit K(y) near K(x) must meet the normal slice at x in only a finite number of points. Hence each orbit K(y) near K(x) must constitute a finite covering space of K(x), and thus
$\dim K(y) = \dim K(x) = n-2$.

Next assume that M and also K(x) are orientable manifolds. In this case we shall prove that K_x acts on the 2-disk B as the identity map. From this it follows that each orbit K(y) in U_1 covers K(x) exactly once, so the normal projection of K(y) on K(x) is a diffeomorphism, and also K(y) and K(x) are of the same orbit-type.

For this purpose we first construct a foliation of a K-invariant tubular neighborhood $U \subset U_1$ of K(x), with the leaves formed by segments of G-orbits. All the points selected hereafter in this argument will lie within U_1, or possibly some narrower tubular neighborhood $U \subset U_1$ about K(x) in M. Then consider a time-segment of the orbit G(x), namely for suitably small $\varepsilon > 0$ take

$$G_\varepsilon(x) = \{\Phi_{kt}(x) \mid k \in K, \ |t| < \varepsilon\}.$$

That is, the orbit-segment $G_\varepsilon(x)$ consists of all time-translates of K(x) for $|t| < \varepsilon$, and $\varepsilon > 0$ is fixed so small that $G_\varepsilon(x)$ lies in U_1. Since the infinitesimal generator of the G-action has the dimension (n-1) everywhere on $G_\varepsilon(x)$, it also has the dimension (n-1) everywhere in some neighborhood $U \subset U_1$ about $G_\varepsilon(x)$. Therefore the appropriate connected pieces of the G-orbits constitute the leaves of a foliation of codimension 1 in the open set U.

Now take a short open line-segment L in the 2-disk B normal to the manifold K(x) at the point x. Restrict the length of L so it lies in U, and then each leaf of the given foliation, lying suitably near $G_\varepsilon(x)$, meets the transversal L-perhaps in many points. In fact, nearby $G_\varepsilon(x)$ the foliation in U is just the saturation of L by pieces of G-orbits. Each such leaf of the foliation in U can be obtained as a segment of some G(z), for a point z on the transversal L (possibly further restricting ε, L, and U as necessary). We seek to show that the foliation near $G_\varepsilon(x)$ has a trivial product structure in U ; that is, we seek to show that the holonomy group h_x (defined as a group of germs of diffeomorphisms of L corresponding to loops in $G_\varepsilon(x)$ based at x) is just the identity map.

Since M and K(x) are orientable manifolds, so is G(x) orientable, and the neighborhood U is a topological product $U = G_\varepsilon(x) x L$. Choose one side of L, called L_+, so that $U_+ = G_\varepsilon(x) x L_+$ is an invariant set for the foliation, that is, U_+ is a union of leaves. Within the leaf $G_\varepsilon(x)$ take any loop ℓ_x (continuous image of $[0,1]$ in $G_\varepsilon(x)$ with $\ell_x(0) = \ell_x(1) = x$). Let ℓ_z, for each $z \in L_+$, be the corresponding lifted curve on the leaf of G(z) starting at the point z, and lying above ℓ_x. That is, use the normals to $G_\varepsilon(x)$ at each point on the loop ℓ_x to lift the loop ℓ_x to a continuous curve ℓ_z that lies on the covering space G(z) and initiates at $z \in L_+$ above x. Then the map from L into L described by

$$z \to \ell_z(1) = \text{endpoint of } \ell_z,$$

specifies a (germ of a) diffeomorphism of L_+ into itself, carrying the linear segment $[x,z]$ onto $[x,\ell_z(1)]$. Thus the loop ℓ_x specifies an element $h(\ell_x)$ of the holonomy group h_x of the foliation, with base point x. Of course, any loop $\hat{\ell}_x$ that is homotopic to ℓ_x in G (x), with fixed base point x, must yield the same holonomy element $h(\ell_x) = h(\hat{\ell}_x)$.

We next show that $h(\ell_x)$ is the identity element of h_x. Suppose ℓ_z is not a closed loop in G(z), but instead $\ell_z(1) < z$ on the transversal L_+ (otherwise treat the inverse loop to ℓ_x and the corresponding inverse holonomy map). Then $h(\ell_x)$ provides a diffeomorphism of the interval $[x,z]$ onto $[x,\ell_z(1)]$ for z sufficiently near to x along the transversal L_+. In this case the iterations of $h(\ell_x)$ generate an infinite cyclic subgroup of the holonomy group h_x.

Thus the prior supposition leads to the conclusion that the leaf on G(z) covers $G_\varepsilon(x)$ with infinitely many sheets, corresponding to the sequence of distinct points on L_+ that arise as the images of z under the successive iterations of $h(\ell_x)$. Note that each such leaf on G(z) can be homotopically retracted within itself onto the compact K-orbit K(z) ; and, in particular, the loop ℓ_x can be so deformed back to a loop $\hat{\ell}_x$ that lies in K(x).

We next construct a <u>collar</u> C around $G_\varepsilon(x)$ in M, so that the lift of $\hat{\ell}_x$ lies within C. Namely, let C be the saturation of the line segment L by K-orbits. Thus C is a K-invariant subset of M ; and moreover C is an (n-1)-manifold differentiably embedded (with inherited subspace topology) in the n-manifold M. This is clear because C has the geometry of an (n-1)-manifold locally near each point of L (by familiar Lie algebra integrability criteria), and its global manifold structure follows directly from the smoothness of the K-actions on M.

As an aside concerning the geometry of C, we can demand that L be orthogonal to the hypersurface $G_\varepsilon(x)$ and define an appropriate Riemann metric on M so that

i) C is everywhere normal to $G_\varepsilon(x)$ along the intersection manifold $K(x) = C \cap G_\varepsilon(s)$, and
ii) the Riemann metric in the tube U is invariant under R-actions, so that $\Phi_t(C) = C_t$ remains normal to $G_\varepsilon(x)$ along $\Phi_t(K(x))$.
In order to satisfy property i) we observe that the compact group K acts on the (n-1)-manifold C, so that we can use a K-invariant Riemann metric on C ; and then L and its K-translates (a field of lines still denoted by L) are all normal to K(x) within C. Since L is nowhere tangent to $G_\varepsilon(x)$ (because $U_+ = G_\varepsilon(x)xL_+$ is an invariant set), we can extend the domain of definition of the Riemann metric so that L is everywhere normal to the hypersurface $G_\varepsilon(x)$ along K(x). That is, pick a vector field N defined at points of K(x) in $G_\varepsilon(x)$, so that N is everywhere tangent to $G_\varepsilon(x)$ but nowhere tangent to K(x). Then define N to be a unit vector field that is orthogonal to K(x) and also to L.

The condition ii) can be met by propagating the Riemann metric from the hypersurface C throughout the region U so as to be R-invariant.

After this detour concerning the collar C over K(x) we now can use the field of lines L, normal to $G_\varepsilon(x)$, to lift the loop $\hat{\ell}_x \subset K(x)$ via the G-foliation of U, or equally well via the K-foliation of C. That is, we can study the K-orbit decomposition of the collar C as a foliation of codimension 1 in C, and we can consider the holonomy group h_x^C of this foliation, with base point $x \in K(x)$. Again h_x^C can be described as a group of (germs of) diffeomorphisms of the segment L (over x) into itself.

Now we return to the main argument of the proof of the lemma. As remarked earlier, the R-actions on U provide a homotopy deformation of U back onto the collar C, with each leaf G(z) near G(x) being deformed back to K(z). In this way there is induced an isomorphism between the fundamental groups

$$\pi_1(G_\varepsilon(x)) \leftrightarrow \pi_1(K(x)) \qquad |\ell_x| \leftrightarrow |\hat{\ell}_x|,$$

and accordingly an isomorphism of the holonomy group h_x onto h_x^C.

In this case the supposition that the lifted curves are not closed, namely $\hat{\ell}_z(1) = \ell_z(1) < z$, implies that h_x^C contains an infinite cyclic subgroup. But thus contradicts the fact that K(z) is compact and covers K(x) only finitely many times. Therefore the supposition $\ell_z(1) \neq z$ is false, and we conclude that ℓ_z is a closed loop for each z near x (both for $z \in L_+$ and similarly for $z \in L_-$). Thus the holonomy group h_x must consist of the identity map only.

Since the holonomy group h_x consists of only the identity map, the foliation of U, by the leaves formed by G-orbits, is trivial. Thus each closed loop ℓ_x based at $x \in G_\varepsilon(x)$ lifts to a closed loop $\hat{\ell}_z$ in each nearby K-orbit K(z) for $z \in L$, and K(z) is a simple covering space for K(x). But any K-orbit K(y) near K(x) in U is merely the time-translate of one such K(z), and hence K(y) is likewise a simple cover of K(x) and the normal projection of K(y) on K(x) is a diffeomorphism.

Since each K-orbit K(y), for y suitably near to x in M, covers K(x) exactly once, it follows that K(y) meets the normal 2-disk B (suitably restricted) in just a single point. Thus K_x acts on B, in the slice representation, by leaving fixed each point of B. That is, K_x acts as the identity transformation on B, and the K-invariant tubular neighborhood U (suitably restricted around K(x)) has the foliated structure of a product U = K(x)xB. That is, K acts trivially on the factor B as the identity and further the K-action on each "horizontal section" K(y) is such as to maintain the product structure of U. In particular we note that K(y) has the same orbit type as K(x). □

Remarks - Assume that M is an orientable n-manifold. Then the condition that $K(x)$ be orientable is fulfilled in lemma 1, provided that either

i) $K(x)$ is a principal K-orbit

or

ii) K has dimension $(n-2)$ so K_x is then finite.

Further, the nonstationary orbit $G(x)$ is orientable if and only if $K(x)$ is orientable.

In lemma 1 we obtain the product tube $U = K(x)xB$ under the asumption that the n-manifold M and the $(n-2)$-dimensional K-orbit $K(x)$ are both orientable. In the next lemma 2 we keep M orientable, but specify that $K(x)$ is nonorientable and obtain a corresponding tube $U = K(x) \tilde{\times} B$ which is a fiber bundle over the base $K(x)$, where the isotropy group K_x acts on the 2-disk B as the group of two elements, the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the reflection $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If M itself is nonorientable, then the orientable double cover $\hat{M}$ can be used for further analysis, but we do not investigate this case here ; we instead refer to the constructions in the next chapter.

Lemma 2 - *Let* $G = KxR$ *act on an orientable differentiable n-manifold* M *as a Lie dynamical system, with a nonstationary orbit* $G(x)$ *of codimension* 1.

If $K(x)$ *is nonorientable then it is an isolated K-orbit in the following sense. There exists a* K-*invariant tubular neighborhood* U *of* $K(x)$ *in* M *such that*

i) U *is diffeomorphic to a non-trivial fiber bundle* $K(x) \tilde{\times} B$ *with the compact orbit* $K(x)$ *as base, the 2-disk* B *as fiber* ;

ii) *The* K-*action on* U *is completely determined (up to* K-*equivariant diffeomorphism) by the group* K, *the isotropy subgroup* $K_x \subset K$, *and the slice (surjective) representation* $K_x \to \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\} \subset O(2,R)$ *of* K_x *acting on the disk* B ;

iii) *The K-orbits on* $G(x)$ *in* U *are all non-orientable compact* $(n-2)$ - *manifolds (each of orbit type of* $K(x)$*), but every other K-orbit in* U *is an orientable compact* $(n-2)$ - *manifold (each of the same orbit type) covering* $K(x)$ *twice*.

Proof

Take an open line segment L normal to $G(x)$ at the point $x \in K(x)$, and then the saturation of L by K-orbits is an $(n-1)$-manifold C differentiably embedded (with subspace topology) in the n-manifold M. The set C is the collar transverse to $G(x)$ along $K(x) = C \cap G(x)$, as in Lemma 1.

Since $G(x)$ is nonstationary, the time-trajectory $R(x)$ is nontangent to the orbit $K(x)$, and the same result holds true at each point on C near to $K(x)$. Thus the $(n-1)$-manifold C is an orientable submanifold of M. Because each K-orbit $K(y)$ in some neighborhood U of $K(x)$ in M must be a time-translate of a K-orbit $K(z)$ in C, with $z \in L$, we reduce the analysis of K-orbits to the K-invariant $(n-1)$-manifold C. Note that each such $K(z)$, and accordingly the time-translate $K(y)$, is a compact $(n-2)$-manifold.

Now we use the slice representation, based at $x \in K(x)$; first for a description of the K-action near $K(x)$, within the invariant $(n-1)$-manifold C. Since $K(x)$ has codimension 1 in C, the transversal segment L is an appropriate (undirected) normal slice to $K(x)$ in C and the stability subgroup K_x acts on L as either the identity $SO(1,R)$, or else as the group with two elements $O(1,R)$. In either case eaoh point $p \in K(z)$, where $K(z) \neq K(x)$ is an orbit near $K(x)$ on C, has a unique unit normal directed from $p \in K(z)$ towards $K(x)$. Hence $K(z)$ has a normal vector field within the orientable manifold C, wherein $K(z)$ has codimension 1, and so $K(z)$ must be orientable. Therefore we conclude that each $K(y)$ in U, but which is not on $G(x)$, must be orientable ; however, evidently, the K-orbits on $G(x)$ are time-translates of $K(x)$ and hence nonorientable.

Following the methods of lemma 1, we consider the holonomy group h_x^C of the K-foliation in C, as based at $x \in K(x)$. Each element of h_x^C is either the identity on L, or is a reflection that reverses the ends of L. But if h_x^C were just the identity, then the nearby orbits $K(z)$ in C would be diffeomorphic to $K(x)$ and so nonorientable, which contradicts earlier conclusions. Thus $h_x^C = O(1,R)$ contains two elements, and each $K(z) \neq K(x)$ must afford a 2-fold covering of $K(x)$. Therefore the isotropy subgroup K_x must also reverse the vector along L -otherwise the K-action restricted to C would have a trivial isotropy subgroup, and accordingly each $K(z)$ would be diffeomorphic with $K(x)$.

Now let the K-invariant tubular neighborhood U of K(x) in **M** have a suitably small (open) 2-disk transversal B normal to K(x) at the point x. Then the tube U is a fiber bundle over the base K(x) with a 2-disk fiber. But the product K(x)xB is a nonorientable n-manifold, and so cannot be an open set in **M**. Therefore $U = K(x) \rtimes B$ must be a nontrivial fiber bundle, with the isotopy group K_x acting on B according to the slice representation in O(2,R).

But note that K_x leaves fixed the curve R(x) in G(x), since the R-actions commute with the K-actions. Thus each element of K_x is either the identity $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, or the orthogonal matrix $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, so the slice representation is a surjective homomorphism $K_x \to \{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\}$. According to the general theory of compact transformation groups the K-action in U is completely determined by the group K, the subgroup K_x, and the slice representation of K_x in O(2,R).

The explicit geometric model of the K-action on the K-invariant tube U shows that each $K(z) \neq K(x)$ in C has the same orbit type, and provides a 2-fold cover of K(x). The conclusion iii) now follows from the observation that the R-orbit of the (n-1)-manifold C fills a neighborhood of K(x) in **M**. □

The next theorem is a generalization of the classical Poincaré-Bendixson Theorem to the actions of Lie dynamical systems on an n-sphere S^n, for any dimension $n \geqslant 2$.

Theorem 3 - Let G = KxR act on the sphere S^n as a Lie dynamical system, with a nonstationary orbit G(x) of codimension 1. If G(x) is future recurrent, then G(x) is a periodic orbit-that is, G(x) is a compact fiber bundle (possibly trivial)

$$G(x) = K(x) \rtimes S^1.$$

Proof

In Theorem 1 it was shown that the nonstationary orbit G(x) must be either periodic, that is, G(x) is a compact fiber bundle (possibly a product) of the fiber K(x) over the base circle S^1 ; or else G(x) is a line-manifold, that is, a differentiable injective immersion of the product K(x)xR in S^n. From the hypothesis that G(x) is future recurrent, we shall show that the second alternative cannot arise.

First assume that $K(x)$ is orientable. Then, according to lemma 1, $K(x)$ and all nearby K-orbits are compact (n-2)-manifolds. Furthermore, there then exists a K-invariant tubular neighborhood U of $K(x)$ in S^n, such that $U \approx K(x)xB$ and the isotropy group K_x acts as the identity on the 2-disk B. Thus in U we have a very explicit model of the K-action - namely K acts on the homogeneous space $K(x) = K/K_x$ as usual, and acts trivially on U so as to carry the vertical fibers (2-diks like B) onto vertical fibers, thereby preserving the product structure of U.

We next select an open subneighborhood $U_\varepsilon \subset U$ about the orbit-segment

$$G_\varepsilon(x) = \{\Phi_{kt}(x) | k \in K, |t| < \varepsilon\}, \text{ small } \varepsilon > 0.$$

To specify U_ε we take a short open line segment L normal to $G_\varepsilon(x)$ at the point x, and let U_ε be the union of the corresponding G-orbit-segments $G_\varepsilon(z)$ for all points $z \in L$. In Lemma 1 it is proved that the foliation of U_ε by these leaves of G-orbit-segments is trivial, that is U_ε is diffeomorphic to the product foliation of $G_\varepsilon(x)xL$.

As in lemma 1 we construct a collar C around $G_\varepsilon(x)$, by defining C as the K-saturation of the segment L. Then C is a K-invariant submanifold and is K-equivariantly diffeomorphic with the product $C \approx K(x)xL$. Accordingly we can coordinatize C by points (α,ℓ) with $\alpha \in K(x)$ and $\ell \in L$. Moreover the R-translates of C yield other segments $L(\tau)$ and $C(\tau)$ in U_ε, for times $|\tau| < \varepsilon$. Therefore we can coordinatize the neighborhood U_ε by triples (α,ℓ,τ) for $(\alpha,\ell) \in C$ and $|\tau| < \varepsilon$ (restricting L, ε, U_ε whenever necessary).

Now we return to the examination of the geometry of the nonstationary orbit $G(x)$. Take the suborbit $K(x)$ and consider its time-translates $\Phi_{et}(K(x))$ into the far future until there arises a time $T > \varepsilon$ when the point $x_T = \Phi_{eT}(x)$ returns to the neighborhood U_ε and lies suitably near to some point of the set $K(x)$. In this case $K(x_T)$ lies in U_ε and approximates the set $K(x)$. If $K(x_T) = K(x)$, then the orbit $G(x)$ is periodic - that is, $G(x)$ is a nonstationary compact orbit and we shall suppose that this is not the case.

Thus we shall suppose that $G(x)$ is a line-manifold orbit, and that no future time-translate $K(x_T)$ ever returns to meet $K(x)$, but that we can find appropriate $K(x_T)$ arbitrarily near to $K(x)$ in U_ε. In particular, we select the collar $C(\frac{\varepsilon}{2})$ and take the future $T > \varepsilon$ so $K(x_T) \subset C(\frac{\varepsilon}{2})$. Furthermore we can require that the time $t = T - \frac{\varepsilon}{2}$ is the first time after $t = \varepsilon$ that the R-orbit of $K(x)$ ever returns to meet the collar C in U_ε.

Next define a subset $S \subset S^n$ to be formed from the orbit-segment $G_\varepsilon(x)$ in U_ε between $\frac{\varepsilon}{4} \leqslant \tau \leqslant \frac{3\varepsilon}{4}$, and thereafter $G(x)$ with increasing t up to the time $T - \frac{\varepsilon}{4}$, together with a certain piece $\hat{C}$ of the collar $C(\frac{\varepsilon}{4})$. We shall show that this union

$$S = \hat{C} \cup \{\Phi_{kt}(x) \mid k \in K, \frac{\varepsilon}{4} \leqslant t \leqslant T - \frac{\varepsilon}{4}\}$$

is a closed hypersurface in the sphere S^n, and that S separates the two points x and x_T in distinct components of the complement $S^n - S$.

Note that the orbit-segment $\{\Phi_{kt}(x) \mid k \in K, \frac{\varepsilon}{4} \leqslant t \leqslant T - \frac{\varepsilon}{4}\}$ is a compact differentiable (n-1) - manifold with boundary in S^n. The boundary consists of two K-orbits in the collar $C(\frac{\varepsilon}{4})$, that meet the segment $L(\frac{\varepsilon}{4})$ at distinct points $\ell_1(\frac{\varepsilon}{4})$ and $\ell_2(\frac{\varepsilon}{4}) \in L(\frac{\varepsilon}{4})$. Define $\hat{C}$ to be the K-saturation of the closed interval of $L(\frac{\varepsilon}{4})$, whose endpoints are $\ell_1(\frac{\varepsilon}{4})$ and $\ell_2(\frac{\varepsilon}{4})$. Then $\hat{C}$ is also a compact differentiable (n-1) - manifold with the same two K-orbits in $C(\frac{\varepsilon}{4})$ serving as its boundary in S^n.

We seek to show that S is a piecewise smooth (n-1)-submanifold, without boundary in S^n. Clearly the "curved piece" of S, the G-orbit segment from $\varepsilon/4 < t < T - \varepsilon/4$, does not meet the "collar piece" $\hat{C} \subset C(\varepsilon/4)$, since $t = T - \varepsilon/2$ is the first return of $G(x)$ (after $t = \varepsilon$) back to C. Thus S consists of the union of two topologically embedded differentiable hypersurfaces, nonintersecting except on their two boundary K - orbits in $C(\frac{\varepsilon}{4})$. Hence S is a piecewise smooth compact (n-1) - submanifold, without boundary in S^n. In fact, since the orientable manifold $G_\varepsilon(x)$ has codimension 1 in U_ε, and also $K(x)$ is an orientable submanifold of codimension 1 in the orientable collar C, it is easy to use the explicit coordinatization of U_ε to "smooth-off" the junctions of $\hat{C}$ with $G(x)$ so as the produce a smooth hypersurface $\hat{S}$ that is compact and without boundary in S^n - and also $\hat{S}$ uniformly aproximates S. However we shall only need to refer to the piecewise smooth hypersurface S.

By the Alexander Duality Theorem the compact hypersurface S separates the sphere S^n into two components. Using the explicit geometry of the constructions in U_ε, we shall observe that the two points x_T and x lie in different components of $S^n - S$. In more detail, using the coordinatization of U_ε it is easy to construct a continuous curve $\mathcal{C}$ joining x_T to x in U_ε, such that $\mathcal{C}$ meets the hypersurface S in precisely one point, say $\mathcal{C} \cap S$ is a single point in $\hat{C}$.

Now we claim that the future trajectory $\Phi_{kt}(x)$ of x_T can never, for $t > T$, meet the hypersurface S. In more detail, $\Phi_{kt}(x)$ for $t > T$ cannot meet the curved piece of S, or in fact any point of $\Phi_{kt}(x)$ for $t < T$. For otherwise G(x) would be a periodic orbit, contrary to our supposition. Also the future trajectory of x_T cannot meet the collar piece $\hat{C}$ of S, since each R-trajectory meeting $\hat{C}$ is passing from the "past" to "future" components of S^n-S ; that is R(x) can cross $\hat{C}$ only when entering the component of S^n-S that contains the point x_T. But since x_T and its future trajectory (up to some hypothetical intersection with $\hat{C}$) lie within the component of S^n-S containing x_T, the trajectory can never meet the component of S^n-S containing x and hence the future trajectory of x_T cannot meet $\hat{C}$.

Therefore we are led to the conclusion that $\Phi_{kt}(x)$, for $t > T$, never meets the component of S^n-S that contains x. But this conclusion contradicts the hypothesis of the theorem that G(x) is future recurrent, or that x lies in the future limit set of the orbit G(x). Therefore the supposition that G(x) is a line-manifold orbit must be false, and hence G(x) is necessarily a periodic orbit as asserted in the theorem.

Finally we give the proof of the theorem in the case where K(x) is nonorientable in S^n. Then K(x) is isolated in the sense of Lemma 2. In this case the return of $\Phi_{et}(K(x))$, along G(x) as t increases, must produce a suborbit $K(x_T)$ very near to K(x)- say in the collar C constructed around G(x) at the point x as in Lemma 2. Now both K(x) and $K(x_T)$ are nonorientable manifolds, and also they both lie in C. Since K(x) is an isolated nonorientable K-orbit in C, we conclude that $K(x_T) = K(x)$. This means that G(x) must be a periodic orbit. Thus the theorem is proved in all cases. $\square$

The next two remarks give important details on the geometric structure of the periodic orbit $G(x) = K(x) \widetilde{\times} S^1$, and the structure of the other nearby G-orbits.

Remark 1 - Consider any closed hypersurface S, for instance the periodic orbit $G(x) = K(x) \widetilde{\times} S^1$ of codimension 1, in the n-sphere S^n. That is, S is a compact submanifold (without boundary) of dimension (n-1) in S^n. Then the Alexander Duality Theorem (mod 2) asserts that the complement S^n-S consists of precisely two components. Hence the closed hypersurface S has two sides in the ambient space S^n, and therefore S must be orientable.

Thus the periodic orbit $G(x) = K(x) \widetilde{\times} S^1$ of codimension 1 in S^n must be an orientable submanifold. From this it follows that K(x) itself is an orientable (n-2)-manifold in S^n -- for otherwise the fiber bundle $K(x) \widetilde{\times} S^1$ would constitute a closed hypersurface that would be nonorientable, which contradicts our previous topological conclusions. But the facts that G(x) and K(x) are orientable do not force the fiber bundle $G(x) = K(x) \widetilde{\times} S^1$ to be a trivial product bundle.

Remark 2 - The geometrical significance of Lemma 1 is that the K-orbit quotient space, in some neighborhood U about the orientable (n-2)-orbit K(x), is differentiably a 2-disk B. This geometric structure is exploited in the first part of the proof of Theorem 3 to show that the K-orbit space, in some neighborhood $\hat{U}$ of the periodic (n-1)-orbit $G(x) = K(x) \widetilde{\times} S^1$ is a 2-annulus or planar ring A. In fact, this neighborhood $\hat{U}$ can be regarded as a fiber bundle with base A and with the K-orbits (all of same orbit type as K(x)) as the fibers.

Then the R-flow acting on the K-orbits in $\hat{U}$ defines a classical flow on the 2-annulus A. The K-orbits lying in G(x) project to the points of a central periodic R-orbit in A. Hence if one nearby G-orbit spirals in towards the limit cycle G(x) as $t \to \infty$, then every G-orbit (on the selected side of G(x) in $\hat{U}$) must also make a similar spiral approach to the periodic G-orbit G(x). This geometric analysis explains the terminology "limit cycle" that often is used to describe such a periodic orbit G(x), as arises in Theorem 3.

We shall not rest our analyses of periodic G-orbits on the geometry of the R-flow on A, except for a few remarks in the next chapter, but shall usually refer directly to the geometry of the G-action on S^n, or whatever is the ambient n-manifold M.

The following two remarks deal with problems of generalizing Theorem 3 by relaxing the hypotheses on smoothness, or on the topology of the ambient n-manifold M.

<u>Remark 3</u> - Theorem 3 is phrased for a Lie C^∞-dynamical system, with the Lie group G = KxR acting C^∞-differentiably on S^n. However the constructions in the proof and the conclusions of the theorem are clearly valid under the hypotheses of only C^1-differentiability. In fact, it seems quite likely that by using more intricate methods and more complicated techniques |4,39|, the corresponding theorem could be proved for general topological actions of G = KxR on S^n.

<u>Remark 4</u> - Let S be a closed hypersurface (connected, compact submanifold - without boundary) with codimension 1 in a given compact n-manifold M. If M is an integral homology n-sphere then the Alexander Duality Theorem shows that S must be orientable, and the two components of M-S correspond to the two sides of S in M. The same conclusions hold if we assume only that M is a compact orientable n-manifold with integral homology $H_{n-1}(M) = 0$, see |11,40|. In particular, if M is a compact n-manifold that is <u>simply-connected</u>, so $\pi_1(M) = 0$, then M is necessarily orientable and moreover $H_1(M) = 0$, so $H^1(M) = 0$ and by Poincaré Duality we find $H_{n-1}(M) = 0$. Thus the conclusions of Theorem 3, and the subsequent remark on the orientability of the periodic orbit G(x), can be demonstrated for the case of a Lie dynamical system acting on any compact simply-connected n-manifold M.

For simplicity of exposition we shall usually assume that the simply-connected n-manifold M is the n-sphere S^n. However we shall return to the more general case of a simply-connected ambient n-manifold M, and emphasize the conclusions for such generalizations, in the discussion that follows Theorem 5.

<u>Corollary 1</u> - <u>Let</u> G = KxR <u>act on the sphere</u> S^n <u>as a Lie dynamical system, with a G-minimal set</u> Σ. <u>Assume that an orbit</u> $G(x) \subset \Sigma$ <u>has codimension</u> 1 <u>in</u> S^n. <u>Then</u> Σ <u>consists of just that one orbit</u> G(x) ; <u>in more detail, either</u>

i) Σ consists of a stationary orbit $G(x) = K(x)$, or

ii) Σ consists of a nonstationary periodic orbit $G(x) = K(x) \ltimes S^1$.

Proof

Since each G-oribt, for any initial point in Σ, is necessarily dense in the minimal set Σ, we conclude that each such G-orbit has dimension equal to $\dim G(x) = n-1$.

First assume that $G(x) = K(x)$ is a stationary G-orbit in Σ. Then, since $K(x)$ is a compact homogeneous space for the group K, we find that $G(x)$ is itself a G-minimal set. In this case $\Sigma = G(x) = K(x)$.

Next assume that $G(x)$ is nonstationary, and hence future recurrent--since $G(x)$ lies within the minimal set Σ. In this case Theorem 3 guarantees that $G(x)$ is a compact set, namely a periodic orbit, so that $\Sigma = G(x) = K(x) \ltimes S^1$. Hence the corollary is proved in both cases. □

Corollary 2 - Let $G = K \times R$ act on the sphere S^n as a Lie dynamical system. Assume that an orbit $G(x)$ has a future limit set $\omega(x)$ that consists entirely of nonstationary G-orbits of codimension 1 in S^n. Then $\omega(x)$ is just one periodic G-orbit towards which $G(x)$ spirals as $t \to \infty$, that is, $\omega(x)$ is a limit cycle (allowing the case where $G(x) = \omega(x)$ is itself a periodic orbit).

Proof

Let Σ be a G-minimal set contained in the compact G-invariant set $\omega(x)$. Then each G-orbit in Σ is nonstationary and of codimension 1. By the previous corollary 1 the minimal set Σ consists of a single compact periodic G-orbit, say $\Sigma = G(\bar{x}) = K(\bar{x}) \ltimes S^1$.

According to the remarks following Theorem 3, the compact (n-2)-manifold $K(\bar{x})$ must be orientable. By the earlier Lemma 1, each K-orbit near to $K(\bar{x})$ is also a compact orientable (n-2)-manifold of the same K-orbit type as $K(\bar{x})$. Moreover there exists a K-invariant tubular neighborhood $U \approx K(\bar{x}) \times B$ around $K(\bar{x})$ in S^n, and the group $K_{\bar{x}}$ acts trivially on the 2-disk B that is normal to $K(\bar{x})$ at the point $\bar{x} \in G(\bar{x})$.

Since $G(x)$ approaches arbitrarily near to $K(\bar{x})$, we conclude that the R-flow is nontangential to both $K(\bar{x})$ and also to the K-orbits in $G(x)$. Thus $G(x)$ is a nonstationary G-orbit of codimension 1 in S^n. We shall assume that $G(x)$ is a line-manifold orbit, since in the remaining case, where $G(x)$ is a periodic orbit, it is trivial that $G(x) = \omega(x)$.

As in the proof of Theorem 3 take a G-orbit piece, for some time duration $\varepsilon > 0$,

$$G_\varepsilon(\bar{x}) = \{\Phi_t(K(\bar{x})) \mid |t| < \varepsilon\} \quad ,$$

and then take a line segment L normal to $G_\varepsilon(\bar{x})$ at the point $\bar{x}$. Note that the neighborhood U is diffeomorphic to the product $G_\varepsilon(\bar{x})xL$ (for suitable restrictions of U, ε and L). Further construct a collar C around $G_\varepsilon(\bar{x})$, over the submanifold $K(\bar{x}) \subset G_\varepsilon(\bar{x})$, so that C is diffeomorphic to $K(\bar{x})xL$. More precisely, C is the K-saturation of the line segment L, and thus C forms a differentiable (n-1)-manifold transverse to $G_\varepsilon(\bar{x})$ and $C \cap G_\varepsilon(\bar{x}) = K(\bar{x})$, just as in the constructions of Theorem 3.

Following the procedures in the proof of Theorem 3 we take two successive times $t_1 < t_2$ for return passes of $K(x)$ to meet C very near to $K(\bar{x})$. Let ℓ_1 and ℓ_2 be the corresponding points of L where these successive time-translations of $K(x)$ meet C. Certainly $\ell_1 \neq \ell_2$, because we are assuming that $G(x)$ is not a periodic orbit.

Next construct a piecewise smooth hypersurface S, closed and without boundary in S^n, consisting of the piece of $G(x)$ that lies between the successive intersections with the collar C, and the appropriate piece $\hat{C}$ of C that is generated as the K-saturation of the subinterval of L lying between the endpoints ℓ_1 and ℓ_2. We wish to show that $0 < \ell_2 < \ell_1$, that is, ℓ_2 and ℓ_1 lie on the same end, say L_+, of L with ℓ_2 closer than ℓ_1 to the center point $\bar{x} \in L$. This will then demonstrate that $G(x)$ spirals towards the limit cycle $\Sigma = K(\bar{x}) \rtimes S^1$ as $t \to \infty$.

First examine the situation where ℓ_2 is supposed to lie on the opposite end of L from ℓ_1, that is, suppose $\bar{x}$ separates ℓ_2 and ℓ_1 on the segment L. In this case the orbit $G(\bar{x})$ crosses the collar C, as t increases, at $K(\bar{x})$ and thus some points of $G(\bar{x})$ lie inside the closed hypersurface S and some points lie outside S. But $G(x)$ always lies on just one side, say the inside, of S after its intersection with C at the instant t_2 ; but this contradicts the fact that all points of $G(\bar{x})$ belong to the future limit set $\omega(x)$. Therefore we conclude that ℓ_2 and ℓ_1 both lie on one end L_+ of the segment L.

Next suppose $0 < \ell_1 < \ell_2$ on L_+, so ℓ_2 lies farther than ℓ_1 from the central point $\bar{x} \in L$. This case is also impossible because then $G(x)$, after the instant t_2, would be separated away from some points of $G(\bar{x})$ by the closed hypersurface S. This is clear since it is easy to construct a continuous curve $\mathcal{C}$ from $\bar{x}$ to any point on $G(x)$, after time t_2 ; with the curve $\mathcal{C}$ meeting the hypersurface S at just a single point, say at $\ell_1 \in G(x) \cap C$. This analysis shows that $0 < \ell_2 < \ell_1$ along L_+.

From the recurrence inequality $0 < \ell_2 < \ell_1$ on L_+ we conclude that successive returns of the orbit $G(x)$ back to the collar C must yield points on L_+ that converge monotonically to $\bar{x} \in \omega(x)$. Now note that the time axis R acts with a specific smallest period $T > 0$ on the set $K(\bar{x})$ in the periodic orbit $G(\bar{x}) = K(\bar{x}) \ltimes S^1$. Therefore the time-translate of the K-orbit $K(x)$, namely $\Phi_t(K(x))$, on the duration $t_1 \leqslant t \leqslant t_1 + T + 1$ must certainly remain uniformly near to the set $G(\bar{x})$ -- provided $\ell_1 > 0$ is taken suitably small. Also, when $\ell_1 > 0$ is taken suitably small, the return duration $|t_2-t_1| < T+1$ and during this duration the points of $\Phi_t(K(x))$ lie uniformly near to the compact set $G(\bar{x})$ while $t_1 \leqslant t \leqslant t_2$.

Since the returns of $\Phi_t(K(x))$ back to the collar C are know to arise at points on L_+ that tend monotonically towards $\bar{x}$, the prior argument can be repeated for each successive duration between successive returns to C. Thus we conclude that the distance (in some convenient metric on S^n) between the compact sets $\Phi_t(K(x))$ and $G(\bar{x})$ tends to zero as $t \to \infty$. In this sense $G(x)$ spirals towards the limit cycle $G(\bar{x})$ as $t \to \infty$.

Because $\omega(x)$ is a compact set containing $G(\bar{x})$, and distance between $\{\Phi_t(K(x)),G(x))\} \to 0$ as $t \to \infty$, we conclude that no points of $\omega(x)$ can lie off the limit cycle $G(\bar{x})$. Therefore $\omega(x) = G(\bar{x})$ is a single periodic orbit, and $G(x)$ spirals towards this limit cycle $\omega(x)$. $\square$

Remark - If, in Corollary 2, we further assume that every G-orbit in the closure $\overline{G(x)}$ is nonstationary with codimension 1 in S^n, then we can conclude both $\alpha(x)$ and $\omega(x)$ are each periodic G-orbits that arise as limit cycles spirally approached by $G(x)$ as $t \to -\infty$ and $t \to +\infty$, respectively. In fact, $\alpha(x)$ and $\omega(x)$ are each closed hypersurfaces that constitute the boundary of an annular region in S^n, and moreover $G(x)$ spirals from one boundary towards the other.

Corollary 3 - Let $G = K\times R$ act on the sphere S^n as a Lie dynamical system. Then each orbit $G(x)$ of codimension 1 is topologically embedded in S^n.

Proof

An orbit $G(x)$ is topologically embedded in S^n provided either

i) $G(x)$ is compact--this being the case if $G(x)$ is stationary or periodic ;

or

ii) $G(x)$ is a line-manifold (a differentiable injection of $K(x)\times R$ into S^n) which is neither past nor future recurrent.

Thus $G(x)$ fails to be topologically embedded in S^n only under the circumstance that $G(x)$ is a line-manifold that meets its own limit set $\alpha(x) \cup \omega(x)$. But if a nonstationary orbit $G(x)$ of codimension 1 is future recurrent, then Theorem 3 asserts that $G(x)$ is necessarily a compact periodic orbit in S^n ; that is, $G(x)$ is not a line-manifold.

Therefore we conclude that the orbit $G(x)$ cannot fail to be topologically embedded in S^n. $\square$

Let $G = K\times R$ act on an n-manifold M^n as a Lie dynamical system. Then, in analogy to the concept of a critical point of a classical flow, we define a critical orbit to be a G-orbit that is either

i) stationary

or

ii) codimension $\geqslant 2$ in M^n.

A classical Theorem of Poincaré asserts that, inside each periodic orbit of a classical dynamical system on S^2, there necessarily exists a critical point. The next theorem proves the appropriate generalization for Lie dynamical systems on n-spheres, for $n \geqslant 2$.

Theorem 4 - Let $G = K \times R$ act on the sphere S^n as a Lie dynamical system. Also let $G(x)$ be a periodic orbit of codimension 1 in S^n. Then within each component of $S^n - G(x)$ there exists at least one critical G-orbit, that is, an orbit which is either

i) stationary

or

ii) codimension $\geqslant 2$ in S^n.

Proof

Fix one component of the complement S^n-$G(x)$, and call this the inside of the periodic orbit $G(x)$. The (nonstationary) periodic G-orbits of codimension 1 that lie inside $G(x)$ can be partially ordered under the relation of set-inclusion for their respective inside regions. Choose a maximal (inextendible) linearly ordered chain of periodic G-orbits of codimension 1, so each lies inside the preceding periodic orbits of the chain.

Choose a point z_γ on each such periodic orbit $G(z_\gamma)$ of the chain. Then there is a subset of $\{z_\gamma\}$ that converges to some point z lying inside (or lying on) each such $G(z_\gamma)$. Consider the G-orbit $G(z)$.

Since each K-orbit $K(z_\gamma)$ has dimension $(n-2)$, the K-orbits on $G(z)$ must have $\dim K(z) \leqslant (n-2)$. But if $\dim K(z) < (n-2)$, or if $\dim K(z) = n-2$ and $G(z)$ is stationary, then $G(z)$ is a critical orbit inside $G(x)$ and the theorem is proved. Hence we assume that $G(z)$ is a nonstationary orbit of codimension 1 in S^n.

Consider the future limit set $\omega(z)$ for the orbit $G(z)$. Then $\omega(z)$ contains a G-minimal set Σ which consists either of a stationary G-orbit, or else consists of nonstationary G-orbits each with codimension $\geqslant 2$ -- all critical orbits -- or else Σ consists of nonstationary G-orbits of codimension $\leqslant 1$. In this latter case such a G-orbit $G(P) \subset \Sigma$ must have $\dim G(P)$ equal to $(n-1)$ or n ; the value $\dim G(P) = n$ is impossible because then $\dim \Sigma = n$ so $\Sigma = S^n$, which contradicts the existence of the given periodic orbit $G(x)$. Therefore, the only case in which $G(P)$ fails to be a critical orbit is that case in which $G(P)$ is a nonstationary orbit of codimension 1 in S^n. But $G(P) \subset \Sigma$ is future recurrent, and hence $G(P)$ is necessarily a periodic orbit -- according to the earlier Theorem 3.

Thus we can proceed with the argument of the proof under the assumption that $G(P) = \omega(z)$ is a nonstationary periodic orbit of codimension 1 in S^n. Since the linearly ordered chain $\{G(z_\gamma)\}$ was maximally inextendible, we conclude further that $G(P)$ belongs to this chain and moreover that $G(P)$ lies strictly inside every other one of the periodic orbits $G(z_\gamma)$.

Next examine the inside region of the closed hypersurface $G(P)$. Take a point Q very near to P, but strictly inside the hypersurface $G(P)$. Then, since $\dim K(P) = n-1$, we find $\dim K(Q) = n-1$ -- according to the earlier Lemma 1. Then either $G(Q) = K(Q)$ is stationary, in which case $G(Q)$ is the required critical orbit inside $G(x)$, or else $G(Q)$ is a nonstationary orbit of codimension 1 in S^n. But in this second situation the orbit $G(Q)$ has a future minimal set $\omega(Q)$ that contains a G-minimal set Σ_1. However the argument presented earlier in the proof shows that Σ_1 contains either a critical G-orbit, or else a periodic G-orbit of codimension 1 -- which is impossible since the chain $\{G(z_\gamma)\}$ terminating in $G(P)$ is inextendible.

Hence $G(P)$ must contain a critical G-orbit within its inside region, and the theorem is proved in all cases. $\square$

An important conclusion of the Poincaré-Bendixson Theorem for classical flows on S^2 is that each vector field tangent to S^2 must possess a critical point. The next theorem asserts the validity of a generalization of this classical result for n-spheres S^n, for $n \geqslant 2$. It is of personal interest to the author that our Theorem 5 is historically related to the famous thesis on foliations by G. Reeb |34|.

Theorem 5 - Let $G = K \times R$ act on the sphere S^n as a Lie dynamical system. Then not every G-orbit can be nonstationary with codimension 1. Further, there must exist a critical G-orbit on S^n.

Proof

Suppose all G-orbits were nonstationary with codimension 1 in S^n. Then a G-minimal set Σ must be a periodic orbit, according to Corollary 1 of Theorem 3. But according to Theorem 4 there must exist a critical orbit inside the periodic orbit Σ, which contradits our supposition.

Hence our conclusion is that a Lie dynamical system $G = KxR$ acting on S^n must possess either a critical orbit, or else a nonstationary orbit of dimension n. Then let $G(y)$ be a nonstationary orbit of dimension n. We shall show that this case also forces the existence of a critical G-orbit on S^n. First of all note that we can discard the possibility that $G(y) = S^n$ because this would imply that S^n is a fiber bundle over the circle S^1 -- a known impossibility for $n \geqslant 2$ (see subsequent remarks). This holds since the nonstationary orbit $G(y) = S^n$ either has an isotropy subgroup G_y not entirely within K_y -- in which case $S^n = K(y) \times S^1$; or else $G_y = K_y$ -- in which case each K-orbit is K-principal and so the K-orbit space is a 1-dimension compact manifold, and S^n is a fibre bundle over $S^n/K \approx S^1$.

Thus we assume taht the n-dimensional nonstationary orbit $G(y)$ is a proper subset of S^n. The point-set boundary $\partial G(y)$ of the subset $G(y) \subset S^n$ is a compact G-invariant set, without interior in S^n. Let Σ be a G-minimal set in $\partial G(y)$. Then Σ either contains a critical G-orbit, or else Σ consists of a periodic G-orbit of codimension 1 in S^n. But in this second case Σ is a closed hypersurface and must contain a critical G-orbit within its inside -- according to Theorem 4.

Hence in every possible case, a Lie dynamical system acting on S^n must contain a critical orbit. □

Remarks - Theorems 3, 4, 5 and their Corollaries are proved for Lie dynamical systems consisting of actions of $G = KxR$ on a compact orientable differentiable n-manifold M that is specified to be the standard n-sphere S^n, for $n \geqslant 2$. However, note that the global topology of S^n enters the proofs of Theorems 3, 4 and 5 in the requirement that a closed hypersurface separate M into two components. As mentioned earlier in Remark 4 following Theorem 3, the hypothesis on the integral homology group $H_{n-1}(M) = 0$ is sufficient to insure this topological separation property.

In Theorem 5 a further requirement is that M cannot be a fiber bundle over the base circle S^1. But if there were such a bundle projection map

$$p \quad M \to S^1,$$

then the 1-cocycle $d\theta$ (where θ is the angular coordinate along the circle S^1) would lift to a nontrivial 1-cocycle in M. This would imply that for M the Betti numbers $b_1 > 0$ and $b_{n-1} > 0$ -- which is impossible for $M = S^n$, or for any compact orientable n-manifold M with integral homology $H_{n-1}(M) = 0$ (recall that the real and integral (n-1) homology groups are free and of the same rank for a compact orientable n-manifold M for $n \geq 2$).

Thus the assertions and conclusions of Theorems 3, 4, 5 and all their Corollaries ramain valid under the assumption that M *is any compact orientable differentiable* n*-manifold with integral homology* $H_{n-1}(M) = 0$. *In particular, these results all hold in case* M *is any compact simply-connected differentiable* n*-manifold, that is, when* $\pi_1(M) = 0$.

Furthermore, under these conditions we can conclude that each periodic orbit $G(x) = K(x) \rtimes S^1$ *of codimension* 1 *in* M *is an orientable closed hypersurface in* M. *Moreover both* K(x) *and also* G(x) *are orientable submanifolds of the compact orientable* n*-manifold* M. (For further details and discussions we refer back to the relevant Remarks following Theorem 3).

In the next set of theorems we state the corresponding results for a Lie dynamical system $G = K \times R$ acting on the vector space R^n for $n \geq 2$, that is, the ambient n-manifold is the noncompact n-dimensional real linear space. Here it is evident that a periodic orbit $G(x) = K(x) \rtimes S^1$ of codimension 1 in R^n must be an orientable closed hypersurface, and involve the compact orientable (n-2)-manifold K(x). But the other results, corresponding to Theorems 3, 4, 5 and Corollaries, require some minor modifications and are presented next as Theorems 3A, 4A, 5A, etc..., using the obvious parallel notation to indicate the close conceptual relations.

Theorem 3A - *Let* $G = K \times R$ *act on the linear space* R^n *as a Lie dynamical system, with a nonstationary orbit* G(x) *of codimension* 1. *If* G(x) *is future recurrent, then* G(x) *is a periodic orbit, that is,* G(x) *is a compact fiber bundle*

$$G(x) = K(x) \rtimes S^1$$

where $K(x)$ is a compact orientable $(n-2)$-manifold and $G(x)$ is a closed orientable hypersurface in R^n.

Proof

As in the prior Theorem 3 a closed hypersurface S is constructed from a "curved piece" of the future orbit of $G(x)$ and a "flat piece" of the collar C around $G(x)$ over the submanifold $K(x)$. Then S separates R^n into two components.

According to the argument in the proof of Theorem 3, the future recurrent orbit $G(x)$ must enter and remain within one component of $R^n - S$, which is a contradiction, unless $G(x)$ is a periodic orbit. Thus we conclude that $G(x) = K(x) \rtimes S^1$ is a periodic orbit.

But then the remarks following Theorem 3 apply to show that the closed hypersurface $G(x)$ is 2-sided in R^n ; and hence both $G(x)$ and also $K(x)$ must be orientable submanifolds of R^n. □

Corollary 1A - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system, with a compact G-minimal set Σ. Assume that an orbit $G(x) \subset \Sigma$ has codimension 1 in R^n. Then Σ consists of just that one orbit $G(x)$; in more detal, either

i) Σ consists of a stationary orbit $G(x) = K(x)$, or
ii) Σ consists of a nonstationary periodic orbit $G(x) = K(x) \rtimes S^1$.

Proof

As in the prior Corollary 1 to Theorem 3 the G-minimal set is compact, and each G-orbit $G(x) \subset \Sigma$ is future recurrent. Thus the same argument holds to prove that either

i) $G(x)$ is stationary, so $G(x) = K(x)$ is itself a compact K-homogeneous space (then $K(x)$ coincides with Σ), or
ii) $G(x)$ is nonstationary with codimension 1 in R^n, and future recurrent ; so $G(x) = K(x) \rtimes S^1$ is periodic according to Theorem 3A. Then $G(x)$ coincides with Σ.

Hence the corollary is proved. □

Corollary 2A - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system. Assume that an orbit $G(x)$ has a future limit set $\omega(x)$ that consists entirely of nonstationary G-orbits of codimension 1 in R^n. Assume further that $G(x)$ is future-bounded in R^n, that is, the future half-orbit of x lies within some compact subset of R^n.

Then $\omega(x)$ is just one periodic orbit towards which $G(x)$ spirals as $t \to \infty$, that is, $\omega(x)$ is a limit cycle (allowing the case where $G(x) = \omega(x)$ is itself a periodic orbit).

Proof

As in the proof of the prior Corollary 2 to Theorem 3 we find a compact G-minimal set Σ within the compact nonempty limit set $\omega(x)$ of $G(x)$. Then by Corollary 1A the minimal set Σ consists of a single periodic orbit, say $\Sigma = G(\bar{x}) = K(\bar{x}) \times S^1$, where $K(\bar{x})$ is a compact orientable (n-2)-manifold in R^n.

The argument given in Corollary 2 shows that $G(x)$ spirals towards the limit cycle $G(\bar{x})$ as $t \to \infty$. An examination or that proof reveals that all the geometric constructions occur in a compact neighborhood of $G(\bar{x})$, say generated by a compact tubular neighborhood $\bar{U}$ of $K(\bar{x})$ under time-translation for a specified bounded duration described in terms of the period T of $G(\bar{x})$. Hence the compact tube $\bar{U}$ and its time-translates for $0 \leqslant t \leqslant T + 1$, all lie within a compact subset of R^n that contains the future half-orbit of $G(x)$; and the argument of the earlier Corollary 2 applies to prove the conclusion of Corollary 2A. $\square$

Corollary 3A - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system. Then each orbit $G(x)$ of codimension 1 is topologically embedded in R^n.

Proof

As noted in Corollary 3 to Theorem 3 the only situation in which $G(x)$ fails to be topologically embedded in the ambient n-manifold, here R^n, is when $G(x)$ is a line-manifold that meets its own limit set $\alpha(x) \cap \omega(x)$.

Suppose that $G(x)$ were a line-manifold of codimension 1 in R^n, and that $G(x)$ meets $\omega(x)$. Then by Theorem 3A we find that $G(x)$ must instead be a compact periodic orbit. From this contradiction we conclude that $G(x)$ must be topologically embedded in R^n. □

Theorem 4A - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system. Also let $G(x)$ be a periodic orbit of codimension 1 in S^n. Then within the bounded component of $R^n - G(x)$ there exists at least one critical G-orbit, that is, an orbit which is either.

i) stationary

or

ii) codimension $\geqslant 2$ in R^n.

Proof

The periodic orbit $G(x)$ is an orientable closed hypersurface in R^n, and hence $G(x)$ separates R^n into two components, exactly one of which is bounded. Call the bounded component of $R^n - G(x)$ the inside region bounded by $G(x)$.

The proof of Theorem 4 now applies directly to the periodic orbit $G(x)$, and its inside region in R^n, to demonstrate the existence of the required critical orbit inside $G(x)$. □

The results corresponding to the assertions of Theorem 5, as referred to Lie dynamical systems in R^n, are rather complicated and are developed in our next two Theorems 5A and 6A.

Recall that Corollary 2A includes the case where the orbit closure $\overline{G(x)}$ consists entirely of nonstationary orbits of codimension 1 and in such circumstances concludes that - if $G(x)$ is future-bounded in R^n then $G(x)$ spirals towards a limit cycle $\omega(x)$ as $t \to \infty$. The next result investigates the case where $G(x)$ may not be future-bounded, but the prior condition holds for $\overline{G(x)}$. The surprising conclusion is that then $G(x)$ must spiral towards a limit cycle $\alpha(x)$ as $t \to -\infty$.

Lemma - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system. Also let $G(x)$ be an orbit whose closure $\overline{G(x)}$ consists entirely of nonstationary orbits of codimension 1 in R^n. In this case, if $\alpha(x) \cup \omega(x)$ is nonempty then either

i) $\omega(x)$ <u>is a periodic orbit towards which</u> $G(x)$ <u>spirals as</u> $t \to \infty$<u>(allowing the possibility</u> $G(x) = \omega(x)$).

<u>or</u>

ii) $\alpha(x)$ <u>is a periodic orbit towards which</u> $G(x)$ <u>spirals as</u> $t \to -\infty$ <u>(allowing the possibility</u> $G(x) = \alpha(x)$).

<u>Proof</u>

Assume that $\omega(x) \neq \emptyset$ for definiteness. If $G(x)$ were also future-bounded as $t \to \infty$, then the earlier Corollary 2A would yield the desired conclusion that $G(x)$ spirals towards the limit cycle $\omega(x)$. Therefore we need only deal with the case where $G(x)$ is not future-bounded, yet $\omega(x) \neq \emptyset$, and we henceforth make these assumptions.

Just as in the proof of Corollary 2A we take a point $\bar{x} \in \omega(x)$ and consider the orbit $G(\bar{x})$ that is nonstationary with codimension 1 in R^n. Next construct a collar C around $G(\bar{x})$ over the (n-2)-manifold $K(\bar{x})$, where C is the K-saturation of the line segment L normal to $G(\bar{x})$ at $\bar{x}$. Let $t_1 < t_2$ be successive times of return of the trajectory along $G(x)$ back to meet the collar C ; and let ℓ_1 and ℓ_2 be corresponding points of intersection of $G(x)$ with L. Note that $\ell_1 \neq \ell_2$, unless $G(x)$ is a periodic orbit -- in which case the Lemma is already proved.

Just as in the constructions of Corollary 2 and 2A let S be a closed hypersurface composed of a "curved piece of $G(x)$" between the intersection times t_1 and t_2, and a flat "piece of the collar C" obtained as the K-saturation of the segment of L with endpoints ℓ_1 and ℓ_2. First note that ℓ_1 and ℓ_2 lie on the same end L_+ of L, with ℓ_2 nearer than ℓ_1 to the point $\bar{x} \in L$. This assertion, that ℓ_2 separates $\bar{x}$ and ℓ_1 along L, is evident for otherwise the future half-trajectory along $G(x)$ is separated by S away from the point $\bar{x} \in \omega(x)$. The details of such geometric separation follows the arguments of Corollary 2.

For times $t > t_2$, $G(x)$ does not lie inside the closed hypersurface S, since $G(x)$ is not future-bounded in R^n. In such a case, $G(x)$ must lie inside S for all times $t < t_1$ and hence $G(x)$ is past-bounded in R^n. But then the result of Corollary 2A, as applied to times $t \to -\infty$, proves that $\alpha(x)$ is a periodic orbit. Furthermore $G(x)$ then spirals towards the limit cycle $\alpha(x)$ as $t \to -\infty$. □

It may seem difficult to picture the geometric situation arising in the previous Lemma. One configuration that might assist the geometric intuition is that of an action of $G = S^1 x R$ on R^3 where $\alpha(x)$ is a torus surface linking around an infinite cylindrical pipe $\omega(x)$, with

the S^1-orbits in $\alpha(x)$ around the longitude circles and the S^1-orbits in $\omega(x)$ around the meridian circumferences of the pipe. The G-orbit $G(x)$ then unwinds from around the torus $\alpha(x)$ and tends towards the pipe $\omega(x)$ in longer and longer time durations.

Corollary - Let $G = KxR$ act on the linear space R^n as a Lie dynamical system. Also let $G(x)$ be an orbit whose closure $\overline{G(x)}$ consists entirely of nonstationary orbits of codimension 1 in R^n. If neither $G(x)$ nor $\alpha(x)$ nor $\omega(x)$ is a periodic orbit, then $G(x)$ tends to infinity in R^n, with both ends of $G(x)$ as $|t| \to \infty$. (That is, $G(x)$ lies exterior to any prescribed compact subset of R^n for all suitably large times $|t|$).

Proof

Suppose $G(x)$ does not tend to infinity in R^n as $|t| \to \infty$. Then the limit set $\alpha(x) \cup \omega(x)$ is nonempty, say $\omega(x) \neq \emptyset$. But the previous Lemma then asserts that either $\alpha(x)$ or $\omega(x)$ is a periodic orbit (or $\alpha(x) = \omega(x) = G(x)$ is a periodic orbit).

This conclusion contradicts the hypothesis given, and hence we conclude that $G(x)$ must tend to infinity in R^n, with both ends of $G(x)$ as $|t| \to \infty$. $\square$

The prior Lemma and Corollary lead immediately to the assertion of the next Theorem 5A, and provide the details of the required proof.

Theorem 5A - Let $G = KxR$ act on the linear space R^n as a Lie dynamical system. If all G-orbits are nonstationary with codimension 1 in R^n, then each orbit $G(x)$ tends to infinity in R^n, with both ends of $G(x)$ as $|t| \to \infty$.

A more refined topological analysis will show that, under plausible hypotheses, the circumstances considered in Theorem 5A actually occur only in the classical case $n = 2$, where the dynamics reduce to flows in the plane.

Theorem 6A - Let $G = K \times R$ act on the linear space R^n as a Lie dynamical system. Assume that all G-orbits are nonstationary with codimension 1 in R^n, and also that every K-orbit is orientable.

Then $n = 2$, each K-orbit reduces to a single point, and each G-orbit is a curve in the plane R^2 -- each such curve $G(x)$ tends to infinity in R^2, with both ends of $G(x)$ as $|t| \to \infty$.

Proof

Since each K-orbit is a compact orientable (n-2)-manifold, each such $K(x)$ lies in a K-invariant tubular neighborhood in which all the K-orbits are diffeomorphic. Hence all the K-orbits in R^n are diffeomorphic and thus the K-orbit space R^n/K is a 2-manifold. Therefore R^n is a fiber bundle over the base manifold R^n/K, with the compact fiber type $K(x)$.

Suppose the dimension $n \geqslant 3$. Consider the exact sequence of homotopy groups for the fiber bundle $R^n \to R^n/K$, namely

$$\ldots \to \pi_2(R^n/K) \to \pi_1(K(x)) \to \pi_1(R^n) \to \pi_1(R^2/K) \to \pi_o(K(x)) \to \ldots .$$

Since $\pi_o(K(x)) = 1$, the map $\pi_1(R^n) \to \pi_1(R^n/K)$ is surjective, and thus $\pi_1(R^n/K) = 1$. Hence R^n/K is a simply-connected 2-manifold. Also R^n/K is noncompact since the time-translates of one $K(x)$ along $G(x)$ have no convergent subsequence, since $G(x)$ cannot be future-recurrent. Therefore R^n/K must be diffeomorphic with the plane R^2.

But then $R^n/K = R^2$ is contractible and so the fiber bundle $R^n \to R^2$ is diffeomorphic with a product bundle

$$R^n \approx K(x) \times R^2.$$

Clearly this is impossible since some homotopy group of the compact manifold is nontrivial, say $\pi_s(K(x))$ is nontrivial for some $1 \leqslant s \leqslant n-2$. (Because $H_{n-2}(K(x)) = Z$, and the conclusion follows from Hurewicz' Theorem on homotopy groups). This contradiction rules out the supposition that $n \geqslant 3$, and so we conclude that $n = 2$.

For $G = K \times R$ acting on the plane R^2 as specified in the hypotheses, it is necessary that each K-orbit is a single point, and each G-orbit is a curve in R^2 tending to infinity with both ends as $|t| \to \infty$. $\square$

We do not know whether the assumption that "every K-orbit is orientable" is necessary in Theorem 6A.

It is amusing to consider the related problem of constructing a fiber bundle of S^n over S^2. I have been assured by two distinguished topologists (who wish to remain un-named) that only S^3 can occur as a fiber bundle over S^2. {Hint - Consider appropriate loop spaces and the corresponding Pontryagin algebras, and then utilize the fundamental combinatorial formula $\binom{n-2}{k} \not\equiv 1$}.

5. Qualitative Theory of Lie Dynamical Systems of Codimension 1 on Multiply-Connected Manifolds, especially Tori

In the previous chapter we studied the action of a Lie dynamical system $\{G, M, \Phi\}$ on a simply-connected n-manifold M ; in particular the action of a Lie group G = KxR, with compact connected Lie group K, on the sphere S^n. The main Theorem 3, and its Corollaries, assert that a minimal set Σ that consists of orbits of codimension 1 must be a compact (n-1)-manifold ; in particular Σ cannot be the entire space S^n. In this chapter we shall show that a similar result holds on any multiply-connected n-manifold M, provided M is not the n torus T^n, or certain other n-manifolds that are fiber bundles over T^2.

We begin this chapter with a description of two basic constructions of general importance in the theory of Lie dynamical systems. The first construction gives the analogue of the Poincaré map around a classical periodic orbit, and the second gives the details of the lifting of a dynamical system from a nonorientable ambient space to an orientable double covering manifold.

Construction 1 - Poincaré map

Let G = KxR act on a differentiable n-manifold M as a Lie dynamical system, and let G(P) be a nonstationary orbit with codimension 1 in M. Assume that G(P) is future recurrent and we shall procede to define the Poincaré map (or first-return map)

$$\Psi \quad L \to L,$$

where L is a line-segment normal to the (n-1)-submanifold G(P) at the point P ϵ M. For simplicity we shall assume that both M and the (n-2)-manifold K(P) are orientable, so that the results of Lemma 1 in chapter 4 are available. In this case each K-orbit K(Q) near K(P) is a compact (n-2)-manifold that covers K(P) just once, and G(Q) is a nonstationary orbit of codimension 1.

At a selected point P on the orbit G(P) choose an open line-segment L normal to the (n-1)-submanifold G(P). (That is, L is a smooth nonsingular curve that is transverse to G(P), and so L appears as a normal line-segment in an appropriate local chart on M and for a suitable auxiliary Riemann metric on M). Using the constructions of the earlier Lemma 1 we place a collar C around G(P) over the base $K(P) = C \cap G(P)$; with C formed as the K-saturation of the line segment L. Of course, we restrict the segment L and the collar C to lie suitably near to K(P), as is convenient.

Since G(P) is future recurrent the time-translates of K(P) return to meet the collar C at infinitely many future instants. Each future recurrence of G(P) to meet C yields an intersection $C \cap G(P)$ that is a K-orbit which meets the segment L in precisely one point. We denote the first-return of P to be $\Psi(P) \in L$. Similarly, for each point $Q \in L$, sufficiently near to P, there is a first-return time t(Q), very near to t(P), with a corresponding first-return point $\psi(Q)$ near to $\Psi(P)$. In this way we define a first-return map (classically called) the Poincaré map).

$$\Psi \quad L \to L \qquad \mathrm{dom}\,(\Psi) \to L$$

from some open neighborhood dom(Ψ) of P in the segment L, with range in L. For definiteness take dom(Ψ) to consist of all points $Q \in L$ which have a first-return $\Psi(Q) \in L$.

In order to make evident the smoothness of the Poincaré map Ψ, and to investigate its dependence on the selection of P and L, we introduce polar coordinates within the collar C by means of the "radial coordinate" $\ell \in L$ and the "directional coordinates" $\alpha \in K(P)$, as in the previous description of the collar $C \approx K(P) \times L$ in Theorem 3.

Consider the first-return map

$$F \quad C \to C$$

where dom(F) is the K-saturation of dom(Ψ), so dom(F) is an open neighborhood of K(P) in the (n-1)-manifold C. Certainly F is a C^∞-map in a neighborhood of K(P), in particular for all small $|\ell|$, because this is a classical result of R-dynamical systems. We write the map F on dom(F) $\subset$ C in term of the polar coordinates (α, ℓ)

$$(\alpha,\ell) \to (A(\alpha,\ell),\ L(\alpha,\ell)) \ ,$$

where $A(\alpha,\ell)$ and $L(\alpha,\ell)$ are in class C^∞. But $L(\alpha,\ell) = L(\ell)$ since the family of K-orbits in C is preserved (as a family) under the map F. Thus the map F induces the map Ψ on the segment L,

$$\Psi \quad L \to L \qquad \ell \to L(\ell)$$

and hence Ψ can be considered as a real C^∞-function of the real variable ℓ that is the coordinate along L. It is clear that the properties of the Poincaré map Ψ are independent of the choice of the point P, the line-segment (smooth nonsingular curve) L, and the coordinate ℓ along L, as analyzed next.

Clearly the Poincaré map Ψ, in terms of the coordinate $\ell \in$ dom(Ψ) $\subset$ L satisfies the conditions

i) Ψ is a C^∞-map from an open set dom(Ψ) $\subset$ L into the open line-segment L ;
ii) Ψ dom(Ψ) $\to$ range(Ψ) is a bijection ;
iii) $d\Psi/d\ell \neq 0$ on dom(Ψ), since Ψ^{-1} is also a C^∞-map (where Ψ^{-1} has the domain specified as range (Ψ) or codom(Ψ)).

Further special properties of the Poincaré map Ψ hold in the important case where $\Sigma = \overline{G(P)}$ is a G-minimal set, especially the case when $\Sigma = K(P) \times S^1$ is a periodic orbit.

In the case when G(P) lies in a minimal set Σ, then $L \cap \Sigma$ belongs to the domain of the first-return map Ψ (as usual, we assume that Σ has empty interior in M and that the endpoints of L do not belong to Σ). Then we further conclude that

iv) Ψ maps $L \cap \Sigma$ onto itself, where $L \cap \Sigma$ is a compact, nowhere dense, linear subset of L ; hence Ψ induces a discrete dynamical system on $L \cap \Sigma$ which is then minimal.

and

v) the s^{th} iterate $\Psi^{(s)}$ is defined on some open neighborhood (depending on s = 0,1,2,3,...) of $L \cap \Sigma$; and furthermore

$$\Psi^{(s)}(\ell_0) = \ell_0 \text{ , for some iterate } s \geqslant 1 \text{ and } \ell_0 \in L \cap \Sigma,$$

holds just in case $G(\ell_0) = G(P)$ is a periodic orbit. Thus, unless $\Sigma = G(P)$, is a periodic orbit, the compact set $L \cap \Sigma$ is perfect.

It is known |13,38| that no C^2-function Ψ can satisfy the five properties i) ii) iii) iv) and v) (with no periodic points). We shall return to this conclusion in our investigation of minimal sets in Theorem 7.

In particular, the important case where G(P) is periodic, so $L \cap \Sigma = P$, is of special interest. In such circumstances, provided L is suitable short, the Poincaré map can be described by a germ of a C^∞-diffeomorphism of (R,0) – and this germ is specified up to conjugation in this group. Hence the characteristic multiplier $\mu = \left.\frac{d\Psi}{d\ell}\right|_{\ell=0}$ is a geometric invariant and the periodic orbit G(P) is locally asymptotically stable (as $t \to \infty$) whenever $|\mu| < 1$.

This discussion concludes our construction 1, and its analysis.

<u>Construction 2</u> – <u>Lifting to orientable double cover</u>

Let G = KxR act on a differentiable n-manifold M as a Lie dynamical system. We assume that M is nonorientable and we seek to lift the Lie dynamical system $\{G,M,\Phi\}$ to a covering system $\{\hat{G},\hat{M},\hat{\Phi}\}$ on the orientable double covering manifold $\hat{M}$. That is, we seek a compact connected Lie group $\hat{K}$, and an action of $\hat{G} = \hat{K}\text{x}R$ on $\hat{M}$, which projects equivariantly onto the action of G on M under the given projection covering map

$$\pi \quad \hat{M} \to M.$$

<u>For simplicity we assume that</u> G <u>acts locally effective on</u> M, <u>and furthermore that the compact group</u> K <u>acts effectively</u>. This is not a restrictive hypothesis, since we could otherwise factor K by its isotropy subgroup K_M so K/K_M acts effectively (hence we procede to assume that $K_M = e$ is the identity alone). More to the point, it is actually only the K-orbits, R-orbits, and G-orbits that are of significance in our geometric analyses – and these orbit structures are unchanged by the appropriated identifications corresponding to the collapse of the isotropy subgroups.

For the classical case of an R-flow on M it is easy to lift the infinitesimal generator of the flow to a vector field on $\hat{M}$, by means of the local diffeomorphism provided by the covering projection π.

In the case of a Lie dynamical system of $G = K \times R$ acting on M, it is also easy to lift the infinitesimal generator of the Lie dynamical system from M to a corresponding Lie algebra of vector fields on $\hat{M}$. But now we must further require that the corresponding group $\hat{K}$ is compact, and that $\hat{K}$ acts effectively on $\hat{M}$.

We begin with the covering space

$$\pi \quad \hat{M} \to M,$$

where $\hat{M}$ is an orientable double covering manifold of the given nonorientable n-manifold M. Let $\Lambda(G) = \Lambda(K) \times \Lambda(R)$ be the infinitesimal generator of the G-action on M, and using π, we lift this to a Lie algebra of vector fields $\hat{\Lambda}_K \times \hat{\Lambda}_R$ on $\hat{M}$. We take the obvious R-action generated by $\hat{\Lambda}_R$ on $\hat{M}$, and next find a compact group $\hat{K}$ with action generated by $\hat{\Lambda}_K$.

There exists a unique simply connected Lie group $\tilde{K}$ which acts (locally effectively) on M and $\hat{M}$ according to the infinitesimal generators $\Lambda(K)$ and $\hat{\Lambda}_K$, respectively. We denote the corresponding isotropy groups by N and $\hat{N}$, so $K = \tilde{K}/N$ and $\hat{K} = \tilde{K}/\hat{N}$. Here N is a discrete normal subgroup of $\tilde{K}$ and clearly $\hat{N} \subset N$.

Take an element $\nu \in N$ so that the action of ν on M yields the identity map, that is, $\nu(P) = P$ for each $P \in M$. Further ν acts on $\hat{M}$ and either yields the identity map of $\hat{M}$ (in which case $\nu \in \hat{N}$), or else ν interchanges two points P_1 and P_2 that lie on $\hat{M}$ above $P \in M$, that is, $\nu(P_1) = P_2$. But if ν interchanges two such points on $\hat{M}$, then ν leaves fixed no point $Q \in \hat{M}$ but $\nu(Q_1) = Q_2$, as earlier. This result is evident since, by continuity, the set of points of $\hat{M}$ left fixed by ν is open, and also the set of those interchanged is open in $\hat{M}$. Thus we conclude that either

i) $\nu \in \hat{N}$ acts as the identity on $\hat{M}$

or

ii) $\nu \in N-\hat{N}$ reverses the two sheets of $\hat{M}$ covering M.

In the second case ν^2 is the identity, and $\nu^{-1} = \nu$.

We conclude that $\hat{N}$ is a normal subgroup of N ; namely for $\mu \in \hat{N}$, $\nu^{-1}\mu\nu \in \hat{N}$ for each $\nu \in N$. Also for each pair ν_1 and ν_2 in $N-\hat{N}$ the product $\nu_1\nu_2^{-1} \in \hat{N}$ (since $\nu_1\nu_2^{-1}$ twice reverses the two sheets of $\hat{M}$ over M), and therefore ν_1 and ν_2 belong to the same coset of the quotient group $N/\hat{N}$. Hence $N/\hat{N}$ contains only one nontrivial coset, and so $N/\hat{N} = Z_2$, or else $\hat{N} = N$.

Now standard group theory specifies the isomorphisms

$$K \simeq \tilde{K}/N \simeq (\tilde{K}/\hat{N})/(N/\hat{N}) \simeq K/(N/\hat{N}).$$

Therefore we conclude that $\hat{K}$ is a two-fold cover of K, or else that $\hat{K}$ = K. In either case the connected Lie group $\hat{K}$ is compact, and acts effectively on $\hat{M}$ with the given infinitesimal generator $\hat{\Lambda}_K$.

Finally define the action of $\hat{G} = \hat{K}xR$ on $\hat{M}$, as generated by the Lie algebra $\hat{\Lambda}_K x \hat{\Lambda}_R$. Then $\hat{G}$ acts locally effectively on $\hat{M}$, since the isotropy subgroup is locally isomorphic to a subgroup of the isotropy group of G acting on M. Furthermore the $\hat{G}$-action on $\hat{M}$ projects, via π, equivariantly onto the given G-action on M – because π carries $\hat{\Lambda}_K x \hat{\Lambda}_R$ onto $\Lambda(G)$ and the required equivariance need only be verified locally near each point $\hat{P} \in \hat{M}$ above each P $\in$ M.

Our construction is completed for lifting the Lie dynamical system G = KxR on M to the Lie dynamical system $\hat{G} = \hat{K}xR$ that acts on the orientable double cover $\hat{M}$ over M. But we shall briefly analyze this lifting procedure as it applies to G-periodic orbits on M, and more generally to G-minimal sets on M.

Let Σ be a G-minimal set in the n-manifold M. Again assume that M is nonorientable with an orientable double covering space $\hat{M}$, with the given projection map

$$\pi \quad \hat{M} \to M,$$

and we seek to lift Σ to a $\hat{G}$-minimal set $\hat{\Sigma}$ in $\hat{M}$. The plausible choice $\Sigma_1 = \pi^{-1}(\Sigma)$ is compact but may not be connected, as so we must investigate further to find the required $\hat{G}$-minimal set $\hat{\Sigma}$ above Σ.

Take any point P $\in \Sigma$ and choose a point $\hat{P} \in \Sigma_1$ above P. Since Σ is minimal, the orbit closure $\overline{G(P)} = \Sigma$. Since the $\hat{G}$-action on $\hat{M}$ projects equivariantly to the G-action on M, the orbit $\hat{G}(\hat{P})$ projects onto G(P). Hence the closure $\overline{\hat{G}(\hat{P})}$ projects onto Σ. Thus $\overline{\hat{G}(\hat{P})}$ is a compact $\hat{G}$-invariant set, and this must contain a $\hat{G}$-minimal set which we call $\hat{\Sigma}$.

Certainly the $\hat{G}$-minimal set $\hat{\Sigma}$ is contained in Σ_1, and so $\pi(\hat{\Sigma}) \subset \Sigma$. But take a point $\hat{Q} \in \hat{\Sigma}$ with $Q = \pi(\hat{Q}) \in \Sigma$. Then $\overline{\hat{G}(\hat{Q})} = \hat{\Sigma}$ projects onto $\overline{G(Q)} = \Sigma$, that is, $\pi(\hat{\Sigma}) = \Sigma$. Therefore $\hat{\Sigma}$ is a $\hat{G}$-minimal set that covers the given G-minimal set Σ. Note particularly that if Σ has an interior in M (so Σ = M), then $\hat{\Sigma}$ must have an interior in $\hat{M}$ (so $\hat{\Sigma} = \hat{M}$).

In the important special case where Σ is the single periodic orbit $G(P) = K(P) \times S^1$, so G(P) is a compact submanifold of M, then $\hat{G}(\hat{P})$ is also a nonstationary compact orbit in $\hat{M}$. That is, $\hat{G}(\hat{P})$ is also a periodic $\hat{G}$-orbit. In this case $\hat{G}(\hat{P})$ must be a minimal set $\hat{\Sigma} = \hat{K}(\hat{P}) \times S^1$ that projects onto Σ. Because the $\hat{G}$-action projects equivariantly onto the G-action, we conclude that

$$\pi \quad \hat{\Sigma} \to \Sigma \quad ,$$

restricts to yield a covering (at most double) of the manifold $\hat{\Sigma}$ over Σ. From this geometric configuration we conclude that

$$\dim \Sigma = \dim \hat{\Sigma},$$

and

$$\dim \hat{K}(\hat{P}) = \dim K(P) = \dim\Sigma - 1.$$

Furthermore, the $\hat{G}$-minimal set $\hat{\Sigma} = \hat{G}(\hat{P}) = \hat{K}(\hat{P}) \times S^1$ must be a fiber bundle, with fiber type $\hat{K}(\hat{P})$, over a base circle S^1.

The two basic constructions, the "Poincaré map" and the "lifting to the orientable double cover", will be used in our later theorems concerning minimal manifolds. Before presenting our major theorems in these topics, we first give a few examples of minimal manifolds based on the classical construction of Kronecker that deals with an irrational slope flow of R on the torus surface T^2.

Example 1 - Let R act on the torus surface T^2 as a classical flow, say a "Kronecker flow of irrational slope", where each trajectory is almost periodic and dense in T^2. Hence T^2 is an R-minimal set.

Next take $G = KxR$, where K is the abelian toral group T^{n-2}, acting as a Lie dynamical system on the n-manifold $M = T^n = T^{n-2}xT^2$. That is, we let $K = T^{n-2}$ act on itself by the usual group translation, and then let R act on T^2, as described earlier. In this manner we define the required Lie dynamical system with the obvious product action of $G = T^{n-2}xR$ on $M = T^{n-2}xT^2$.

Each G-orbit is then nonstationary with codimension 1 in M. Moreover each G-orbit is a line-manifold that is dense in M. Hence M is the unique G-minimal set for this Lie dynamical system.

In the next example the ambient n-manifold M will be a nontrivial fiber bundle over the base torus surface T^2. Our later Theorem 8 will show that this example is typical of the general case of a minimal manifold, for a Lie dynamical system with nonstationary G-orbits of codimension 1 in M.

Example 2 - We shall define a Lie dynamical system with $G = KxR = SO(n-1,R)xR$ acting on a nontrivial fiber bundle $M = S^{n-2} \times T^2$, such that each G-orbit is nonstationary of codimension 1 and dense in M - so M is a G-minimal set.

First we define the ambient n-manifold $M = (S^{n-2} \times S^1)xS^1$. Here $(S^{n-1} \times S^1)$ is the fiber bundle (nontrivial when n is even) constructed over the base S^1, treated as $[0,1]$ (mod 1), using the antipodal map of the fiber S^{n-2} for the identification at the endpoints of the segment $[0,1]$. Next treat the meridian circle of the torus surface T^2 as the base circle of $(S^{n-2} \times S^1)$, and use the longitude circle of T^2 as the other factor S^1. Thus we obtain the n-manifold $M = (S^{n-2} \times S^1)xS^1$.

Equally well, we can regard $M = S^{n-2} \tilde{\times} T^2$, where the bundle over T^2 is nontrivial when n is even.

Next we define the required action of $G = K\times R$, where $K = SO(n-1,R)$, on the n-manifold M. Use the standard action of $K = SO(n-1,R)$ on the fiber S^{n-2} over each point of the base S^1 ; since the antipodal map of the sphere commutes with all rotations, K acts on the fiber bundle $S^{n-2} \tilde{\times} S^1$. Now use the given irrational flow of R on the base T^2, with the specified action of K on the fibers S^{n-2}, to define the required action of $G = SO(n-1,R)\times R$ on the n-manifold $M = S^{n-2} \tilde{\times} T^2$.

In this Lie dynamical system of $G = SO(n-1,R)\times R$ acting on $M = S^{n-2} \tilde{\times} T^2$ we note that each G-orbit is nonstationary with codimension 1, and each G-orbit is dense in M. Therefore M is a minimal set.

We observe further that M is nonorientable, when n is even, but that each G-orbit is an orientable (n-1)-manifold, in fact a line-manifold diffeomorphic to $S^{n-2}\times R$. It is also interesting to note that if we modify the R-action to be strictly periodic on the base T^2, then each G-orbit on $M = S^{n-2} \tilde{\times} T^2$ becomes a periodic orbit of codimension 1 in M - but in this case M would no longer be a minimal set for G.

In closing this discussion we remark that a slight modification of the construction in this example could replace the fiber S^{n-2} by a nonorientable projective space, and then each G-orbit would be nonorientable ; yet still dense in the ambient minimal n-manifold.

The final two theorems of this investigation show that the nature of a G-minimal set Σ, consisting of nonstationary and codimension 1 G-orbits, is either like those on the sphere - that is, Σ is a periodic orbit ; or else like those in the above examples - that is, $\Sigma = M$ is a fiber bundle over a torus surface.

Theorem 7- Let $G = K\times R$ act on the differentiable n-manifold M as a Lie dynamical system. Let Σ be a G-minimal set consisting of nonstationary orbits of codimension 1 in M. Then either

i) Σ consists of one periodic orbit, say $\Sigma = G(P) = K(P) \tilde{\times} S^1$,

or

ii) $\Sigma = M$.

Proof

First, consider the case where M is orientable ; the nonorientable case will be treated later using an orientable double cover M and an appropriate lifting of G to $\hat{G}$, as in our earlier construction.

Take a point $P \in \Sigma$ and consider the corresponding orbits K(P) and G(P) in Σ. By the Lemmas before Theorem 3 we note that K(P) is a compact (n-2)-manifold that lies within a tubular neighborhood U in which every K-orbit is of the same orbit type as K(P) - if K(P) is known to be orientable ; or else every K-orbit (other than R-translates of K(P) in U) is an orientable double covering of K(P). Unless $\Sigma = G(P) = K(P) \times S^1$ is a compact periodic orbit as in conclusion i), we can assume that G(P) is future recurrent back to any arbitrary neighborhood of K(P). Thus, excepting the case i) where $G(P) = \Sigma$ is a periodic orbit, we conclude that all K-orbits in U are of the same orbit type, and hence K(P) must itself be orientable. In summary, assuming M is orientable, we find that either

i) $\Sigma = G(P) = K(P) \times S^1$ is a periodic orbit,

or that each K-orbit in Σ is necessarily an orientable (n-2)-manifold- all of the same orbit type

We next recall the construction 1) for the Poincaré map

$$\Psi \quad L \to L \quad \mathrm{dom}(\Psi) \to L$$

for a selected line segment L normal to G(P) at the point P. As usual we suppose that Σ contains no interior in M (otherwise $\Sigma = M$ and the theorem is proved) and L is an open segment through P, with endpoints not in Σ.

The pertinent information for the Poincaré map Ψ, when G(P) lies in a minimal set Σ , which we suppose is not a periodic G-orbit, is expressed in conditions i) - v) of the earlier construction 1. In essence, $L \cap \Sigma$ is a perfect set, without periodic points and without interior, invariant under the map Ψ. But the famous analyses of A. Schwartz |10,38| (following the methods of A. Denjoy) show that these five conditions are impossible when $\Psi \in C^2$, in particular for a Lie dynamical C^∞-system. Thus the supposition of the existence of the G-minimal set Σ, that is neither a periodic G-orbit, nor fills all M, must be rejected. Therefore the theorem is proved in the case where M is orientable.

Now consider the remaining case where M is a nonorientable n-manifold. In this case let the nonorientable manifold M have the orientable double covering manifold $\hat{M}$, and lift the G-action on M to the $\hat{G}$-action on $\hat{M}$, as in the prior construction 2). Further let $\hat{\Sigma} \subset \hat{M}$ be a $\hat{G}$-minimal set that projects onto the G-minimal set $\Sigma \subset M$. Clearly $\hat{\Sigma}$ consists of nonstationary $\hat{G}$-orbits of codimension 1 in $\hat{M}$.

Suppose that the G-minimal set Σ contains no periodic G-orbit, nor does Σ contain any interior in M. We shall find a contradiction to these suppositions by considering the lifted $\hat{G}$-minimal set $\hat{\Sigma}$. As remarked in the earlier construction 2), $\hat{\Sigma}$ cannot then be a compact periodic $\hat{G}$-orbit ; and our supposition demands that $\hat{\Sigma}$ have empty interior in $\hat{M}$. But these conditions contradict the first part of our proof that deals with the $\hat{G}$-minimal set $\hat{\Sigma}$ in the orientable n-manifold $\hat{M}$. Thus we conclude that Σ must be either

i) $\Sigma = G(P) = K(P) \rtimes S^1$, a periodic orbit

or

ii) $\Sigma = M$, since Σ has interior in M.

Therefore the Theorem is proved in all cases. $\square$

Remark - As in the theorem let Σ be a G-minimal set, consisting of nonstationary orbits of codimension 1, in the n-manifold M. If M is assumed orientable, then we have noted that each K-orbit $K(x) \subset \Sigma$ is also orientable, and all such K-orbits are diffeomorphic.

But if M is nonorientable, then Lie dynamical systems, like those discussed in example 2), show that $M = K(x) \rtimes T^2$ can be a minimal manifold with K-orbits that are nonorientable.

<u>Corollary</u> - <u>Let</u> $G = K \times R$ <u>act on the differentiable</u> n-<u>manifold</u> M <u>as a Lie dynamical system. Let an orbit</u> $G(x)$ <u>have a nonempty compact future limit set</u> $\omega(x)$ <u>that consists entirely of nonstationary orbits of codimension</u> 1 <u>in</u> M. <u>Then either</u>

i) $\omega(x) = K(\bar{x}) \rtimes S^1$ <u>is a periodic orbit toward which</u> $G(x)$ <u>spirals as a limit cycle as</u> $t \to \infty$ (<u>for some</u> $\bar{x} \in \omega(x)$, <u>and allowing the possibility</u> $G(x) = \omega(x)$)

or

ii) $\omega(x) = M$, <u>so</u> M <u>is a</u> G-<u>minimal set</u>.

<u>Proof</u>

The compact G-invariant set $\omega(x)$ must contain a compact G-minimal set $\Sigma \subset M$, and Σ consists of nonstationary orbits of codimension 1. By Theorem 7 either

i) Σ is a periodic orbit $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ (some $\bar{x} \in \Sigma$)

or

ii) $\Sigma = M$.

In the second case ii) $\Sigma = M$ implies that M is a G-minimal manifold so $\omega(x) = M$ and the Corollary is proved.

Thus consider only the case i) where $\Sigma = G(\bar{x}) = K(\bar{x}) \rtimes S^1$ is a periodic orbit contained in $\omega(x)$. We must then show that $\omega(x) = \Sigma$ is a limit cycle towards which $G(x)$ spirals as $t \to \infty$ (allowing $G(x) = \omega(x)$).

We consider first the case where M is orientable, and deal later with nonorientable manifolds by the technique of orientable double coverings as in the prior construction 2). Hence assume that M is an orientable n-manifold, and begin with the hypothesis that $\Sigma = G(\bar{x}) = K(\bar{x}) \rtimes S^1$; and first assume that $K(\bar{x})$ is also orientable.

Since $K(\bar{x})$ is a compact orientable (n-2)-manifold, Theorem 3 and the subsequent remarks assert that there exists a neighborhood V about the periodic orbit $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ in which each K-orbit is diffeomorphic to $K(\bar{x})$. Moreover, the K-orbit space V/K can be classified, up to diffeomorphism, in terms of the line segment L normal to $G(\bar{x})$ at $\bar{x}$, and the self-diffeomorphism of L upon encircling the base circle S^1 of the periodic orbit $G(\bar{x})$. If $G(\bar{x})$ is itself orientable, then V/K is a planar 2-ring or annulus, but if $G(\bar{x})$ is nonorientable then V/K is a Möbius band. In either case V is a fiber bundle over the base 2-manifold A = V/K, with the fiber type $K(\bar{x})$.

Under the projection map of the fiber bundle V onto the base surface A, the central orbit $G(\bar{x})$ is carried to the central circle S^1 of A. and the G- action on V is projected equivariantly onto a classical R-flow on the surface A.

Since G(x) has the future limit set $\omega(x)$, which contains $G(\bar{x}) = K(\bar{x}) \rtimes S^1$, the projection of V onto A must carry the future half-orbit of G(x) onto a spiral trajectory in A approaching the central circle S^1. But the classical geometry concerning flows on surfaces shows that this spiral trajectory in A has a future limit set consisting entirely of S^1. Furthermore this spiral in A approaches its limit cycle S^1 with monotonically decreasing intercepts along the transversal L - from just one side of L if A is an orientable ring surface, otherwise alternately from opposite sides of L.

Thus in the case where M and $K(\bar{x})$ are assumed orientable, we reduce the study of the approach of G(x) towards its limit set $\omega(x)$ to a study of a curve in the surface A that spirals towards its limit cycle S^1. Hence we find that G(x) in V spirals towards the limit cycle $G(\bar{x}) = K(\bar{x}) \rtimes S^1$;and so $\omega(x)$ consists of $G(\bar{x})$ only. Furthermore , G(x) spirals towards its limit cycle $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ with monotonically decreasing intercepts along the transversal L - from just one side of L if $G(\bar{x})$ is an orientable hypersurface, otherwise alternately from opposite sides of L.

Next we assume that the n-manifold M is orientable, but that the K-orbit $K(\bar{x})$ is nonorientable. In this situation the Lemma 2 before Theorem 3 describes the geometry of the K-orbits in a tubular neighborhood V about the periodic orbit $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ in M. Namely, the K-orbit space V/K is classified, up to diffeomorphism, in terms of the half-closed line segment L_+ (coordinatized by $0 \leqslant \ell < 1$) in L normal to $G(\bar{x})$ at $\bar{x}$, and the self-diffeomorphism of L upon encircling the base circle S^1 of the periodic orbit $G(\bar{x})$.

Look first at the case when the diffeomorphism around S^1, attaching L onto itself, preserves the orientation of L. Then V/K is the product $L_+ xS^1$. Then the G-action in V projects equivariantly onto a classical R-flow in the half-closed annular ring $A_+ = L_+ xS^1$ – with the boundary circle S^1 of A_+ corresponding to the periodic orbit $G(\bar{x})$. Since G(x) has the future limit set $\omega(x)$ that contains $G(\bar{x})$, the projected trajectory in A_+ must spiral towards the boundary circle S^1, monotonically, with each encircling of the longitude around A_+. Hence G(x) must spiral towards to limit cycle $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ which thereby comprises all $\omega(x)$.

Next look at the case where the diffeomorphism around S^1, attaching L onto itself, reverses the orientation of L. In this case we can consider an appropriate double cover $\hat{V}$ of V, corresponding to a double encirclement of S^1, and then trivially lift the K-action and induced R-flow to $\hat{V}/K$. Then $\hat{V}/K$ is diffeomorphic to A_+, with the trajectory corresponding to $\Phi_t(K(x))$ reducing to a spiral approaching the boundary circle of A_+. From this geometric picture we again conclude that G(x) must spiral towards the limit cycle $\omega(x) = G(\bar{x}) = K(\bar{x}) \rtimes S^1$.

Our theorem has now been demonstrated in all the subcases in an orientable n-manifold M. Finally we now make the hypothesis that M is a nonorientable n-manifold. As in the earlier construction 2) we let $\hat{M}$, with the projection

$$\pi \quad \hat{M} \to M,$$

be an orientable double covering manifold of M ; and we lift the G-action on M to the $\hat{G}$-action on $\hat{M}$. That is, $\hat{G} = \hat{K}xR$ acts as a Lie dynamical system on the orientable n-manifold $\hat{M}$, and this $\hat{G}$-action projects equivariantly onto the specified action of G = KxR on M.

Consider the orbit G(x) in M with future limit set $\omega(x)$, and then consider the $\hat{G}$-orbit $\hat{G}(\hat{x})$ in $\hat{M}$; where $\hat{x}$ is a point of $\hat{M}$ that projects to $x \in M$. Let $\hat{\omega}(\hat{x})$ be the future limit set of $\hat{G}(\hat{x})$ in $\hat{M}$.

Since the G-system on M and the $\hat{G}$-system on $\hat{M}$ have infinitesimal generators that are locally isomorphic under the projection map π, it is evident that $\hat{G}(\hat{x})$ projects onto G(x), and by continuity arguments $\pi(\hat{\omega}(\hat{x})) \subset \omega(x)$. In fact, for each pair of points $\hat{Q}_1$ and $\hat{Q}_2$ above $Q \in \omega(x)$ at least one of them belongs to $\hat{\omega}(\hat{x})$; and hence $\pi(\hat{\omega}(\hat{x})) = \omega(x)$. Moreover, since $\hat{\omega}(\hat{x}) \subset \pi^{-1}(\omega(x))$ we find that $\hat{\omega}(\hat{x})$ must consist of nonstationary $\hat{G}$-orbits of codimension 1 in $\hat{M}$.

But $\hat{G} = \hat{K}xR$ acts on the orientable n-manifold $\hat{M}$ as a Lie dynamical system, and $\hat{G}(\hat{x})$ has the nonempty compact future limit set $\hat{\omega}(\hat{x}) \neq \hat{M}$, that consists entirely of nonstationary $\hat{G}$-orbits of codimension 1 in $\hat{M}$. By the first section of this proof, dealing with the case of an orientable ambient manifold $\hat{M}$, we conclude that $\hat{\omega}(\hat{x}) = \hat{G}(\hat{\bar{x}}) = \hat{K}(\hat{\bar{x}}) \rtimes S^1$ is a periodic $\hat{G}$-orbit, towards which $\hat{G}(\hat{x})$ spirals as a limit set when $t \to \infty$.

Since $\hat{\omega}(\hat{x}) = \hat{G}(\hat{\bar{x}})$ (for a point $\hat{\bar{x}} \in \hat{\omega}(\hat{x}) \subset \hat{M}$ above $\pi(\hat{\bar{x}}) = \bar{x} \in \omega(x) \subset M$), we conclude that $\pi(\hat{G}(\hat{\bar{x}})) = G(\bar{x})$ is a single periodic G-orbit. Therefore the future limit set $\omega(x)$ of $G(x)$ is the periodic orbit $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ in M. Hence the theorem has now been proved in the last remaining case where M is nonorientable. $\square$

<u>Remark</u> - The nature of the spiral approach of $G(x)$ towards its limit cycle $G(\bar{x}) = K(\bar{x}) \rtimes S^1$ is somewhat different in the various subcases of the given proof. At any rate we can assert that in every subcase the distance (in some convenient metric) from the time-translates $\Phi_t(K(x))$ to the compact limit cycle $G(\bar{x})$ decreases towards zero ;

$$\lim_{t\to\infty} \text{dist}\,\{\Phi_t(K(x)),\ G(\bar{x})\} = 0.$$

For the final theorem of this paper we shall prove an analogue of the famous result of Denjoy concerning <u>classical irrational (or ergodic) flows on the torus surface</u> T^2 - that is, classical flows on T^2 that are each topologically conjugate to a "linear flow with irrational slope" on the usual representation of the torus $T^2 = R^2/Z^2$.

<u>Theorem 8 - Let</u> $G = K \times R$ <u>act on a compact orientable</u> n-<u>manifold</u> M, <u>as a Lie dynamical system. Assume that each</u> G-<u>orbit is nonstationary, codimension</u> 1, <u>and also dense in</u> M - <u>so that</u> M <u>is a</u> G-<u>minimal space. Then the</u> K-<u>orbit space</u> M/K <u>is a torus</u> T^2, <u>and the quotient projection map</u>

$$M \to M/K = T^2$$

<u>defines</u> M <u>as a fiber bundle over</u> T^2 ; <u>that is</u>

$$M = K \rtimes T^2.$$

<u>Moreover, the</u> G-<u>action on</u> M <u>projects equivariantly onto an</u> R-<u>action on</u> T^2 <u>that is an ergodic minimal flow</u>.

<u>Proof</u>

Since each G-orbit $G(x)$ is future recurrent, but not periodic, $K(x)$ must be an orientable (n-2)-manifold in the orientable n-manifold M. Also, since M is a G-minimal manifold, all K-orbits are diffeomorphic. Thus M/K is a compact 2-manifold, and

$$M \to M/K$$

is a fiber bundle.

But each K-orbit K(x) has a trajectory under the time-flow $\Phi_t(K(x))$ that fills a dense subset of M. Accordingly, this R-flow projects equivariantly onto a classical flow on the compact surface M/K. Moreover, each trajectory of this R-flow is dense in M/K, that is, M/K is a minimal set. The classical theory of flows on surfaces guarantees that $M/K = T^2$, and that the induced R-flow on T^2 is topologically conjugate to a "linear irrational flow" or an ergodic minimal flow on T^2. □

If M is a nonorientable n-manifold, then M/K might not be a manifold (without boundary). In this case we can lift the minimal G-action on M to a minimal $\hat{G}$-action on the orientable double cover $\hat{M}$, as in the prior construction 2), and then apply the analysis of Theorem 8 to the orientable n-manifold $\hat{M}$. Therefore $\hat{M}$ must be a fiber bundle over a base torus surface.

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