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*Astérisque*, tome 107-108 (1983), p. 219-228
<http://www.numdam.org/item?id=AST_1983__107-108__219_0>

Société Mathématique de France
Astérisque 107-108 (1983) p.219-228

# COMMUTATIVE AND RELATED METRICS

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## 1. INTRODUCTION

The idea of a commutative metric of a Riemannian manifold first appeared in the paper by Roberts and Ursell [ RU ] in 1960. These authors were concerned with the problem of random walks on the sphere and with the possibility of generalising their problem to more general Riemannian manifolds. They restricted their attention to a special type of random walk in which the probability arising from two consecutive steps is independent of the order in which the steps are taken. Such Riemannian metrics were called commutative.

This paper was primarily concerned with the propagation of waves arising in seismology, and although, with hindsight, it is seen to contain ideas subsequently discovered independently and developed by Allamigeon, the non-invariant treatment makes it difficult to understand what is really going on. Nevertheless,they showed that a <u>necessary</u> condition for a commutative metric is that its curvature tensor should satisfy an infinite sequence of conditions. These conditions were re-obtained in a subsequent paper by H.S. Ruse [RE] in a more invariant manner.

These conditions had already appeared in work on harmonic spaces by Copson and Ruse [RU] in 1940, and later by Lichnerowicz [LZ1] in 1944, as a subset of the conditions for a Riemannian space to be harmonic. More precisely, the conditions for harmonicity appear naturally as an infinite sequence of conditions which must be satisfied by the curvature tensor and its covariant derivatives. The necessary conditions for a commutative space appeared as the second, fourth, sixth, ... conditions for a harmonic space. Now it can be proved, e.g. Vanhecke (1981) [VE] that in a harmonic space, the first, third, fifth, ... conditions imply the second, fourth, sixth, ..., so that these necessary conditions for a commutative space are automatically satisfied. Thus commutative spaces appear as a natural generalization of harmonic spaces, and include them as particular cases.

Round about the same time as the paper by Ruse on commutative spaces, there appeared in the Journal of Differential Geometry in 1969 a paper by D'Atri and Nickerson on 'Divergence-Preserving Geometric

Symmetry" [DN]. It turns out that the condition that they were imposing on Riemannian metrics was that the local geodesic symmetry about each point should be volume preserving except possibly for sign. When the metric is analytic a necessary and sufficient condition for this to hold is that the infinite set of necessary conditions for a commutative space should be satisfied. For convenience, we shall call a D'Atri space one which is characterized by this infinite set of conditions. Thus a commutative space is a D'Atri space, but whether a D'Atri space is a commutative space remains an open question.

## 2. THE SEQUENCE OF CURVATURE CONDITIONS

It is shown, for example, in the book by Ruse, Walker and Willmore [RWW], 1961, that a Riemannian metric is harmonic at m if and only if the volume function $\theta_m(p)$, when expressed in terms of normal coordinates centred at m, is a radial function. A Riemannian manifold is harmonic if its metric is harmonic at each point m. Using the method developed by A. Gray in 1973 [GR] we can obtain the expansion

$$(2.1)\quad \theta = 1 - \frac{1}{6}\rho_{ij}(m)x_ix_j - \frac{1}{12}(\nabla_i\rho_{jk})(m)x_ix_jx_k + \frac{1}{24}\left\{-\frac{3}{5}\nabla^2_{ij}\rho_{k\ell} + \frac{1}{3}\rho_{ij}\rho_{k\ell} - \frac{2}{15}R_{iajb}R_{ka\ell b}\right\}(m)x_ix_jx_kx_\ell + \frac{1}{120}\left\{-\frac{2}{3}\nabla^3_{ijk}\rho_{\ell h} + \frac{5}{3}(\nabla_i\rho_{jk})\rho_{\ell h} - \frac{2}{3}(\nabla_iR_{jakb})R_{\ell ahb}\right\}(m) \times x_ix_jx_kx_\ell x_h + a_{ijk\ell hg}(m)x_ix_jx_kx_\ell x_hx_g + \ldots$$

where

$$(2.2)\quad a_{ijk\ell hg} = \frac{1}{720}\left\{-\frac{5}{7}\nabla^4_{ijk\ell}\rho_{hg} + 3(\nabla^2_{ij}\rho_{k\ell})\rho_{hg} + \frac{5}{2}(\nabla_i\rho_{jk})(\nabla_\ell\rho_{hg}) - \frac{8}{7}(\nabla^2_{ij}R_{ka\ell b})R_{hagb} - \frac{5}{9}\rho_{ij}\rho_{k\ell}\rho_{hg} - \frac{15}{14}(\nabla_iR_{jakb})(\nabla_\ell R_{hagb}) - \frac{16}{63}R_{iajb}R_{kb\ell c}R_{hcga} + \frac{2}{3}\rho_{ij}R_{ka\ell b}R_{hagb}\right\},$$

and we have used the summation convention for repeated indices.

If we write

$$(2.3)\quad x^i = t^is,$$

where $t^i$ are components of the unit tangent vector at m to the unique geodesic joining m to p, and substitute in the expansion for $\theta$, by requiring $\theta$ to be a radial function we imply that $\theta$ must not involve $t^i$. Remembering the condition

$$g_{ij}t^i t^j = 1, \tag{2.4}$$

it follows on consideration of the quadratic term in the x's that

$$\rho_{ij} = k_1 g_{ij} , \tag{2.5}$$

for some constant $k_1$. This is the first condition for a harmonic space. However, when this condition is satisfied, the coefficient of the cubic term in the x's is identically zero. Thus the second condition arising from cubic terms is automatically satisfied. On consideration of the quartic term in the x's and use of the first condition we get

$$\mathfrak{S}_{i,j,k,\ell} R_{iajb} R_{ka\ell b} = k_2 \mathfrak{S}_{i,j,k,\ell} g_{ij} g_{k\ell} \tag{2.6}$$

for some constant $k_2$, where the notation denotes cyclic summation of indices $i,j,k,\ell$. This is the fourth condition. Again the fifth condition arising from the quintic terms in x is satisfied automatically because of the preceding conditions.

The sixth condition is

$$\mathfrak{S}_{i,j,k,\ell,h,g} a_{ijk\ell hg} = k_3 \mathfrak{S}_{i,j,k,\ell,h,g} g_{ij} g_{k\ell} g_{hg} \tag{2.7}$$

where $k_3$ is a constant.

Again the seventh condition is automatically satisfied because of the preceding conditions. That this is true in general, is the result obtained by Vanhecke [VE] referred to in the previous paragraph.

The problem of classifying harmonic metrics which satisfy the corresponding infinite set of conditions remains unsolved. The only known harmonic spaces apart from flat spaces are the locally symmetric spaces of rank one, and many of us believe the validity of the conjecture, originally made by Lichnerowicz, that these are the only ones.

Equation (2.5) shows that harmonic spaces of dimension 2 are necessarily of constant curvature. The same equation shows that harmonic spaces are Einstein spaces, and it is well-known that every 3-dimensional Einstein space is of constant curvature. In dimension 4 the matter is more complicated. It is known (see, for example, [RWW]) that every 4-dimensional harmonic space with positive definite metric is locally symmetric. Moreover, it was proved by Lichnerowicz (see [RWW]) that an n-dimensional harmonic space with metric of signature (1, n-1) is necessarily of constant curvature, so this applies in particular when $n = 4$. However examples of 4-dimensional harmonic metrics are known of signature (2,2) which are not locally symmetric. For $n \geq 5$, the conjecture that every harmonic space with positive definite metric remains completely open.

Another question which arises naturally is whether supporting a harmonic metric imposes any local conditions on the manifold. For $n \leq 4$ with positive definite metrics and for all n-dimensional metrics of signature (1, n - 1), harmonic spaces are locally symmetric and therefore the manifolds admit a real analytic structure. My guess is that this is the case for all harmonic spaces; but the problem remains unsolved to this day.

## 3. GEODESIC SYMMETRIES

Let m be a fixed point; and consider the geodesic symmetry $p \mapsto p'$ where p' lies on the unique geodesic joining p to m, equidistant from m but lying on the opposite side from p. This is sometimes called in the literature the "flip" map. It will be recalled that the space is locally symmetric in the sense of É. Cartan if the flip map is a local isometry for each point m. An equivalent condition is that the curvature tensor has zero covariant derivative. In a recent paper by Carpenter, Gray and Willmore [CGW] 1982, a complete classification was obtained of symmetric spaces which satisfied the first k non-trivial conditions for a harmonic space. An interesting result contained therein is that the exceptional Lie group $E_8$ satisfies what we have previously called conditions one to six inclusive but not condition seven.

Instead of requiring the flip map to be an isometry, we may weaken the requirement that the map is merely volume preserving. Since the flip map is represented in normal coordinates by the map $x_i \to -x_i$, it follows that the flip map is volume preserving if and only if the coefficients of odd powers of x's in the expansion of $\theta$ are all zero. This is precisely the infinite set of conditions for a D'Atri space. Clearly symmetric spaces are D'Atri spaces. That the converse is not true is shown by the following counterexample. It is well-known that the so-called Heisenberg group of real $3 \times 3$ matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

can be given the left-invariant metric.

$$ds^2 = dx^2 + dz^2 + (dy - zdx)^2 . \tag{3.1}$$

Direct calculation shows that this is a D'Atri metric but it is not that of a symmetric space.

## 4. HOMOGENEOUS METRICS AND NATURALLY REDUCTIVE SPACES

A metric on a manifold M is called <u>Riemannian homogeneous</u> if there exists a group G of isometries acting transitively on M. Then M is diffeomorphic to the coset space G/H where H is the isotropy group of some point in M. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H, and let $\mathfrak{g} = \mathfrak{h} + p$ be an adh invariant splitting, i.e. $[h,p] \subset p$. In particular if H is compact, then such a splitting always exists. The subset p can then be identified with the tangent space of G/H at the coset (H). A homogeneous space is called <u>naturally reductive</u> if

$[x,\cdot]_p : p \to p$ is skew-symmetric for all $x \in p$. The following theorem due to D'Atri and Ziller [DZ], 1979, states, in our terminology, that a naturally reductive Riemannian homogeneous metric is a D'Atri space. Until quite recently it was thought that the converse was probably true. Certainly the metric (3.1) of the Heisenberg group is naturally reductive. However, in a forthcoming paper to appear in the Bulletin of the London Mathematical Society [KN], Aroldo Kaplan has given a 6-dimensional example of a D'Atri space for which the metric is not naturally reductive. It appears that the restriction "naturally reductive" is algebraic rather than geometrical in nature.

In a pre-print sent to me by W. Ziller [ZR], he classifies the homogeneous Einstein metrics on compact symmetric spaces of rank 1. In particular he shows that $S^{15}$ with a suitable radius can be considered as a geodesic sphere at a point in the 16-dimensional Cayley plane. This sphere may be considered as the Riemannian homogeneous space Spin(9)/Spin(7). With the naturally induced metric such a space is not naturally reductive. It seems very likely that this will also be a D'Atri space. So the problem of classifying D'Atri spaces is still wide open.

## 5. MEAN VALUE THEOREMS

In an early paper, [WE], 1950, I prove that harmonic Riemannian spaces are completely characterized by the mean value property

$$M_m(r,f) = f(m) \tag{5.1}$$

which must hold for harmonic functions f. Here the left hand side is defined by

$$M_m(r,f) = \int_{\exp_m(S^{n-1}(r))} f * dr \Big/ V(\exp_m(S^{n-1}(r))), \tag{5.2}$$

where $\exp_m$ is the exponential map which maps a small neighbourhood of the origin in $M_m$ onto a neighbourhood of $m \in M$, and $*dr$ is the volume element of $\exp_m(S^{n-1}(r))$. Here $\exp_m(S^{n-1}(r))$ is the set of points of distance r from m, and $V(\exp_m(S^{n-1}(r)))$ is its volume.

In a joint paper by Alfred Gray and myself [*GW*] , *1982* , we obtained a formula for $M_m(r,f)$ for a smooth function f in an arbitrary Riemannian manifold. More specifically we obtained the relation

$$(5.3)\qquad M_m(r,f) = \frac{\sum_{k=0}^{\infty}\left(\frac{r}{2}\right)^{2k}\frac{1}{k!\,\Gamma(\frac{1}{2}n+k)}\tilde{\Delta}_m^k\,[f\theta]\,(m)}{\sum_{k=0}^{\infty}\left(\frac{r}{2}\right)^{2k}\frac{1}{k!\,\Gamma(\frac{1}{2}n+k)}\tilde{\Delta}_m^k\,[\theta]\,(m)},$$

where $(\tilde{\Delta}_m^k f)_m = (L^k f)_m$ and $L^k$ is a globally defined differential operator of degree 2k on M given by

$$(5.4)\qquad L^k f = \frac{1}{1.3\ldots(2k-1)}\sum_{i,\ldots,i_k=1}^{n}\{\nabla^{2k}_{i_1 i_1 \ldots i_k i_k} f + \ldots \ldots + \nabla^{2k}_{i_1 \ldots i_k i_k \ldots i_1} f\}.$$

## 6. COMMUTATIVE METRICS

An alternative way of defining a mean value over a geodesic is to use the formula

$$(6.1)\qquad \hat{M}_m(r,f) = \int_{\exp_m(S^{n-1}(r))} f * dr \Big/ V(S^{n-1}(r)).$$

Here we are averaging over what is effectively the sphere of directions in the tangent plane $M_m$. Suppose we are given two values of r, namely $r_1$ and $r_2$. From a given function f we can define a new function denoted by $f_{r_1}$ where

$$(6.2)\qquad f_{r_1}(m) = \hat{M}_m(r_1,f).$$

From $f_{r_1}$ we define a new function denoted by $(f_{r_1})_{r_2}$ where

$$(6.3)\qquad (f_{r_1})_{r_2}(m) = \hat{M}_m(r_2,f_{r_1}).$$

We seek the condition that

$$(6.4)\qquad (f_{r_1})_{r_2} = (f_{r_2})_{r_1}$$

holds for all smooth functions f and for all sufficiently small real numbers $r_1$, $r_2$. It turns out that the above condition is equivalent to

$$L^k(L^\ell f) = L^\ell(L^k f), \tag{6.5}$$

for all integers k and $\ell$. This is equivalent to the definition of a commutative space adopted by Roberts and Ursell in their original paper, and we take equation (6.5) as our definition of a commutative Riemannian metric.

We know that a commutative metric is a D'Atri metric, see, for example, [RU]. An interesting question is to discuss whether the converse is true. Since an isometry is a very special case of a volume preserving transformation (modulo sign), it is trivial that all symmetric spaces have D'Atri metrics. We now prove that all symmetric spaces have commutative metrics. This follows immediately from the following theorem first proved by Lichnerowicz [LZ2] in 1963.

Theorem 6.1 In a compact symmetric space the algebra of invariant differential operators is commutative.

The proof by Lichnerowicz is known but not well-known, so we sketch it below. Let $D_1$, $D_2$ be arbitrary invariant differential operators in a symmetric space. Let S be a symmetry associated with the space, and let $D_1^*$, $D_2^*$ be operators defined by

$$D_i^* = SD_iS^{-1}, \quad i = 1,2. \tag{6.6}$$

Lichnerowicz proves that $D_1^*$, $D_2^*$ are dual to the operators $D_1$, $D_2$.

Then we have

$$\begin{aligned} D_1^* D_2^* &= (SD_1S^{-1})(SD_2S^{-1}) \\ &= S(D_1D_2)S^{-1} \\ &= (D_1D_2)^* \\ &= D_2^* D_1^* , \end{aligned}$$

and this clearly implies that

$$D_1 D_2 = D_2 D_1 .$$

Since $L^k$, $L^\ell$ are invariant differential operators, it follows that all compact symmetric spaces have commutative metrics.

We also have

Theorem 6.2 Let $(M_1, g_1)$, $(M_2, g_2)$ be two Riemannian manifolds with commutative metrics. Then $(M_1 \times M_2, g_1 \times g_2)$ has a commutative metric.

The proof is immediate. However, the theorem is interesting because it is known [RWW] that the corresponding property is not enjoyed by manifolds with (positive definite) harmonic metrics, unless each manifold is flat.

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