MORIHIKO SAITO

Supplement to "Gauss-Manin system and mixed Hodge structure"

Astérisque, tome 101-102 (1983), p. 320-331
<http://www.numdam.org/item?id=AST_1983__101-102__320_0>

Supplement to "Gauss-Manin system and mixed Hodge structure"

by Morihiko Saito

This note is a supplement to "Gauss-Manin system and mixed Hodge structure"(cited as [Sa]), which is submitted for publication in Proceedings of the Japan Academy. In this supplement, we discuss the following questions, which we could not discuss in full detail in the paper:

1) the necessity of a unipotent base change in the formulation of the result of Scherk and Steenbrink (e.g., counter-examples to the first formulation of Scherk, Steenbrink and Pham, cf.[Ph]),

2) the diffference between the limit Hodge filtration of Schmid (which is obtained using a unipotent base change) and the limit of Hodge filtration which is obtained without a base change.

§1. The main point of the paper [Sa] is the following: in the formulation of the result of Scherk and Steenbrink, it is necessary to take a unipotent base change. We give two examples in which the first formulation of Scherk, Steenbrink and Pham as stated in the first version of [Ph] does not apply.

(1.1) First we review the notations in [Sa],[SS] and [Ph].

Let $f:\mathbb{C}^{n+1},o \to \mathbb{C},o$ be a holomorphic function with an isolated singularity, and let $f:X \to S$ be a Milnor fibration so that $H_X := R^n f_* \mathbb{C}_X|_{S^*}$ is a local system on $S^* = S - \{0\}$. There is a natural extension $\mathcal{A}$ of H_X to the origin as a locally free $\mathcal{O}_S$-Module with a regular singular connection ∇, such that the eigenvalues of $\operatorname{res}(t\nabla_{d/dt})$ are in $(-1,0]$. ($\mathcal{J}$ is denoted by $\mathcal{L}_X$ in [Sa,(1.3)].) There is another extention $\mathcal{H}_X^{(o)}$, which we call the Brieskorn lattice. $\mathcal{H}_X^{(o)}$ is a locally free $\mathcal{O}_S$-Module with a a regular singular connection such that $\mathcal{H}_{X,o}^{(o)} \cong \Omega_{X,o}^{n+1} / df\wedge d\Omega_{X,o}^{n-1}$. It is known that there is a natural inclusion $\mathcal{H}_{X,o}^{(o)} \hookrightarrow \mathcal{J}$ (by Malgrange), which is $\mathcal{O}_S$-linear, preserves the connection and induces an isomorphism on S^*. $\mathcal{H}_{X,o}^{(o)}$ is also a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ, where $\mathbb{C}\{\{\partial_t^{-1}\}\} = \{ \sum_{i\geq 0} a_i\partial_t^{-1}: \sum a_i r^i/i! < \infty \ \exists r>0 \}$ and $\partial_t = \nabla_{d/dt}$ (Malgrange, Pham).

The Gauss-Manin system $\int_f^{\circ}\mathcal{O}_X$ is defined as an integration of system (cf. [Ph],[Sa]). $\int_f^{\circ}\mathcal{O}_X$ contains $\mathcal{J}$ and $\mathcal{H}_{X,o}^{(o)}$ naturally, and it is a holonomic system on S such that $DR(\int_f^{\circ}\mathcal{O}_X) = R^n f_* \mathbb{C}_X$.($\int_f^{\circ}\mathcal{O}_X$ is denoted by $\mathcal{H}_X$ in [Sa].)

Let $X_\infty := X^* \times_{S^*} U$ be a base change of X^* by the universal covering $p:U \to S^*$. We set $H_\infty := H^n(X_\infty, C)(\simeq\Gamma(U,\rho^* H_X))$, i.e., H_∞ is the set of multivalued horizontal sections of H_X.

We have an isomorphism $H_\infty \xrightarrow{\sim} \mathcal{J}_o / t\mathcal{J}_o$, by $u \to \exp(-\log t \log M / 2\pi\sqrt{-1})\, u$, where M is the monodromy of H_X and the eigenvalues of $\log M$ are in $[o,1)$. Here we regard $\mathcal{J}$ as a subsheaf of $j_*(\mathcal{O}_{S^*} \otimes H_X)$, where $j: S^* \to S$ is an inclusion.

(1.2) The first formulation of Scherk, Steenbrink and Pham (cf.[Ph]) asserts the following.

Let $\{F_{St}^{\cdot}\}$ be the Hodge filtration of Steenbrink on H_∞, then we have

(1.2.1) $F_{St}^p = \partial_t^{n-p}\mathcal{H}^{(o)} \cap \mathcal{J}_o / \partial_t^{n-p}\mathcal{H}^{(o)} \cap t\mathcal{J}_o \ (\subset \mathcal{J}_o / t\mathcal{J}_o \simeq H_\infty)$

for any p, where we set $\mathcal{H}^{(o)} := \mathcal{H}_{X,o}^{(o)}$ and take intersections in $\int_f^{\circ}\mathcal{O}_X$.

By a result of Steenbrink, $\{F^{\cdot}_{St}\}$ is compatible with the monodromy decomposition $H_\infty = \oplus_\lambda H_{\infty,\lambda}$, where $H_{\infty\lambda} = \{u \in H_\infty : (M-\lambda)^{n+1} u = 0\}$. First we give an example for $n = p = 1$ that $\mathcal{H}^{(o)} / \mathcal{H}^{(o)} \wedge t\mathcal{A}_o$ is not compatible with the decomposition (hence (1.2.1) does not hold.)

(1.3) <u>Example</u> 1. $f = x^5/5 + y^5/5 + a x^3y^3/3$.

This is the first example in which b-function changes under a μ-constant deformation (i.e., $b(s) = (s+1)\ \Pi_{i=2}^{8}(s + i/5)$ for $a=0$, and $b(s) = (s+1)\ \Pi_{i=2}^{7}(s + i/5)$ for $a\neq 0$ (by T. Miwa).)

We assume now $a\neq 0$.

We have a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis $\{w_{ij} = x^{i-1}y^{j-1}dx\wedge dy\}_{i,j=1,\cdots,4}$ of $\mathcal{H}^{(o)}$. Let $\tilde{\mathcal{H}}^{(o)} = \sum_{i=0}^{1}(\partial_t t)^i\mathcal{H}^{(o)}$ be the saturation of $\mathcal{H}^{(o)}$. Then we have

$$\tilde{\mathcal{H}}^{(o)} = \sum_{(i,j)\neq(4,4)}\mathbb{C}\{\{\partial_t^{-1}\}\}w_{ij} + \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t w_{44}$$

Set

$$V^o := \sum_{j=1}^{3}\mathbb{C}\{\{\partial_t^{-1}\}\}w_{jj} + \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t w_{44}\ ,$$

$$V^k := \sum_{i-j\equiv k(\mathrm{mod}5)} \mathbb{C}\{\{\partial_t^{-1}\}\}w_{ij} \qquad \text{for } k=1,\cdots,4\ .$$

We can verify that for $k=o,\cdots,4$, V^k is an $\mathcal{E}(o)$-submodule of $\tilde{\mathcal{H}}^{(o)}$. ($\mathcal{E}(0) \doteq \mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$) For there is a decomposition

$$\mathbb{C}\{x,y\}dx\wedge dy = \oplus_{k=0}^{4}\{\ \sum_{i-j\equiv k(\mathrm{mod}5)} a_{ij}x^{i-1}y^{j-1}dx\wedge dy\}$$

of $\Omega^2_{X,o}$ which induces the one on $\mathcal{H}^{(o)}$ such that the action of t and ∂_t^{-1} are compatible with it.

Hence there is a decomposition $\mathcal{A} = \oplus_{i=0}^{4}\mathcal{A}^i$ (resp. $H_X = \oplus H_X^i$, resp. $H_\infty = \oplus H_\infty^i$) as locally free $\mathcal{O}_S$-Modules with connection (resp. as local systems, resp. as vector spaces with monodromy action) such that $V^i = \tilde{\mathcal{H}}^{(o)} \wedge \mathcal{A}^i$ (resp. $\mathcal{A}^i$ is an extension of H_X^i, resp. $H_\infty^i = \Gamma(U,\rho^* H_X^i)$).

The action of $\partial_t t$ on V^o / tV^o is given by the following matrix,

	w_{11}	$\partial_t w_{44}$	w_{22}	w_{33}
w_{11}	2/5	0	0	0
$\partial_t w_{44}$	-a/15	3/5	0	0
w_{22}	*	*	4/5	0
w_{33}	*	*	*	6/5

This implies

$$w_{33} \equiv 0 \qquad (\mathrm{mod}\ t\mathcal{L})$$
$$w_{22} \equiv t^{-1/5} \otimes u_4 \qquad (\mathrm{mod}\ t\mathcal{L})$$
$$\partial_t w_{44} \equiv t^{-2/5} \otimes u_3 \qquad (\mathrm{mod}\ t\mathcal{L} + \mathbb{C}w_{22})$$
$$w_{11} \equiv t^{-3/5} \otimes u_2 - (a/3)\, t^{-2/5} \otimes u_3 \qquad (\mathrm{mod}\ t\mathcal{L} + \mathbb{C}w_{22})$$

where $\{u_i\}_{i=1,\cdots,4}$ is a basis of H^o_∞ such that $M\, u_i = \exp(-2\pi\sqrt{-1}\ i/5)\ u_i$.

Thus we have $(\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{L}_o) \cap H^o_\infty = \mathbb{C}u_4 + \mathbb{C}(u_2-(a/3)u_3)$, hence $\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{L}_o$ is not compatible with the monodromy decomposition, because we have $\mathcal{H}^{(o)} = \oplus_i\ \mathcal{H}^{(o)} \cap \mathcal{L}^i$.

<u>Remark</u>. We have $F^1_{St} \cap H^o_\infty = \mathbb{C}u_2 + \mathbb{C}u_4$, because we have

$$\tilde{t}^3 \pi^* w_{11} \equiv 1 \otimes u_2 \ (\mathrm{mod}\ \tilde{t}\, \tilde{\mathcal{L}}),$$

where $\pi:\tilde{S} \rightarrow S$ is a 5-fold covering such that $\pi^* t = \tilde{t}^5$ and $\tilde{\mathcal{L}}$ (= $\tilde{\mathcal{L}}_X$ in [Sa]) is an extention of $\pi^* H_X$ as in (1.1) (cf.[Sa(3.2)]).

(1.4) <u>Example</u> 2.

Let $f:\mathbb{C}^2,o \rightarrow \mathbb{C},o$ be a holomorphic function such that $\{f=0\}$ is an irreducible and reduced curve. We show that $F^1_{St} \neq \mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{L}_o$ if f is not quasi-homogeneous.

Proof) By a result of Lê and A'Campo, the local monodromy is semi-simple and $H_{\infty,1} = \{0\}$. Suppose $F^1_{St} = \mathcal{H}^{(o)}/\mathcal{H}^{(o)} \cap t\mathcal{L}_o$ holds.

There is a basis $\{u_j\}_{j=1,\cdots,\mu}$ of H_∞ such that $F^1_{St} = \sum_{j=1}^{\mu/2} \mathbb{C}u_j$ and $M\, u_j = \exp(-2\pi\sqrt{-1}\ \alpha_j)\ u_j$, for F^1_{St} is compatible with the monodromy decomposition. We may assume that $-1 < \alpha_j < 0$ $(1 \leqq j \leqq \mu/2)$, $0 < \alpha_j < 1$ $(\mu/2 < j \leqq \mu)$ and $\alpha_j + \alpha_{\mu+1-j} = 0$ by the duality of exponents.

We set $v := t^{\alpha_j} \otimes u_j \in \mathscr{A}$ for $j=1,\cdots,\mu$ and $V := \sum_{j=1}^{\mu} \mathbb{C}\{t\}v_j \subset \mathscr{A}$. V is a free $\mathcal{O}_S$-Module containing $\mathscr{H}_X^{(o)}$, because of $F^1_{St} = \mathscr{H}^{(o)} / \mathscr{H}^{(o)} \cap t\mathscr{A}_o$.

Let $\{\gamma_i(t)\}_{i=1,\cdots,\mu}$ be a multivalued horizontal basis of $\bigsqcup_{t \in S^*} H_1(X_t, \mathbb{C})$ and $\{w_j\}_{j=1,\cdots,\mu}$ be a $\mathcal{O}_S$-basis of $\mathscr{H}_X^{(o)}$. Then $(\det(\int_{\gamma_i(t)} v_j))^2$ and $(\det(\int_{\gamma_i(t)} w_j))^2$ are both nowhere vanishing holomorphic functions on S, due to the duality of exponents and a lemma of Kyoji Saito.

Then we have $V = \mathscr{H}^{(o)}$, for there is a basis $\{e_i\}$ of V such that $\{t^{m_i} e_i\}$ is a basis of $\mathscr{H}^{(o)}$ $(m_i \geqq 0)$.

It is clear that $\mathscr{H}^{(o)} = V$ is saturated (i.e., $t\partial_t V \subset V$). Hence f is quasihomogeneous by a result of Kyoji Saito. Q.E.D.

Remark. In general, we can show the following.
Let $f : \mathbb{C}^2, o \rightarrow \mathbb{C}, o$ be a holomorphic function with an isolated singularity. We assume that the local monodromy of f is semi-simple. Then $\mathscr{H}^{(o)} / \mathscr{H}^{(o)} \cap t\mathscr{A}_o$ is compatible with the monodromy decomposition, if and only if f is quasihomogeneous.

Problem. For n=1, does the subspace $\mathscr{H}^{(o)} / \mathscr{H}^{(o)} \cap t\mathscr{A}_o$ of H_∞ determine the local moduli of f in the family of μ-constant deformation? In general, does $\mathscr{H}^{(o)} \subset \mathscr{A}$ determine the local moduli of f in the μ-constant family?

§2. The examples in §1 mean that the proof of the formulation of Scherk, Steenbrink and Pham such as stated in the first version of [Ph] is not complete. This contradiction comes from the following.

(2.1) Let $(H_{\mathbb{Z}}, \mathcal{F}^{\cdot})$ be a polarizable variation of Hodge structure of weight n on S^*: i.e., $H_{\mathbb{Z}}$ is a local system on S^*, $\mathcal{F}^{\cdot}$ are holomorphic subbundles of $\mathcal{O}_{S^*} \otimes H_{\mathbb{Z}}$ such that $\partial_t \mathcal{F}^p \subset \mathcal{F}^{p-1}$, and there is a bilinear form $H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$ such that they induce a polarized Hodge structure on $H_{\mathbb{C},t}$ for $\forall t \in S^*$. Here $H_{\mathbb{C}} = R^n \bar{f}_* \mathbb{C}_Y \mid_{S^*}$ and $\bar{f} : Y \to S$ is a compactification of a Milnor fibration $f : X \to S$, cf. [Sa,(1.1)].

Then $\mathcal{F}^{\cdot}$ can be extended to the origin as subbundles $\hat{\mathcal{F}}^{\cdot}$ of $\mathcal{T}$, where $\mathcal{T}$ is an extension of $H_{\mathbb{C}} = \mathbb{C} \otimes H_{\mathbb{Z}}$ as in (1.1). But the limit filtration $\hat{\mathcal{F}}^{\cdot}|_{t=0}$ of $H_{\mathbb{C},\infty} \simeq \mathcal{T}/ t\mathcal{T}$ is different from the filtration $F_{\infty}^{\cdot}$ of Schmid, which is obtained using a unipotent base change by Steenbrink. ($H_{\mathbb{C},\infty} := \Gamma(U, \rho^* H_{\mathbb{C}})$, cf.(1.1))

(2.2) First we show the existence of the extension $\hat{\mathcal{F}}^{\cdot}$.

We fix the coordinates t and z of S and U such that $S = \{|t|<1\}$, $U = \{\mathrm{Im}\, z>0\}$ and $\rho^* t = \exp(2\pi\sqrt{-1}\, z)$.

A natural isomorphism $H_{\mathbb{C},\infty} = \Gamma(U, \rho^* H_{\mathbb{C}}) \xrightarrow{\sim} (\rho^* H_{\mathbb{C}})_z$ induces a Hodge filtration $F_z^{\cdot}$ on $H_{\mathbb{C},\infty}$, which depends holomorphically on z. As we have $F_{z+1}^{\cdot} = M^{-1} F_z^{\cdot}$ for $\forall z \in U$, $\exp(z \log M)\, F_z^{\cdot}$ are filtrations on $H_{\mathbb{C},\infty}$ which depend only on $t = \exp(2\pi\sqrt{-1}\, z)$.

Let $M = M_s M_u$ be the Jordan decomposition of M and set $N := \log M_u$ (N is nilpotent). As M_s has a finite order e (cf. [Sc,(6.1)]), $\exp(z N)\, F_z^{\cdot}$ depends on $\tilde{t} := \exp(2\pi\sqrt{-1}\, z/e)$.

The Theorem of Schmid [Sc, 6.16] assures that there exists a limit $F_\infty^\cdot = \lim_{\mathrm{Im}\ z\to\infty} \exp(zN)\ F_z^\cdot$ in the flag manifold of $H_{\mathbb{C},\infty}$, such that the Hodge filtration $\{F_\infty^\cdot\}$ and the monodromy weight filtration $\{W.\}$ determine a mixed Hodge structure on $H_{\mathbb{Z},\infty}$.

Using this theorem we show the existence of $\hat{\mathcal{F}}^\cdot$.

If we choose an $\mathcal{O}_S$-basis of $\mathcal{J}$, the subbundles $\mathcal{F}^\cdot$ determine a holomorphic map $\Phi: S^* \to \mathrm{Flag}(\mathbb{C}^m)$, and the existence of $\hat{\mathcal{F}}^\cdot$ is equivalent to the extension of Φ on S.

Let $\{u_{ij}\}_{i=1,\cdots,\ell,\ j=0,\cdots,r_i-1}$ be a basis of $H_{\mathbb{C},\infty}$ such that $-N/(2\pi\sqrt{-1})\ u_{ij} = u_{i,j-1}$ $(u_{i,-1}:=0)$, $M_s u_{ij} = \exp(2\pi\sqrt{-1}\ a_i/e)\ u_{ij}$ $(a_i \in [0,e-1])$.

Then $\{v_{ij} = \exp(-\log t \log M/2\pi\sqrt{-1})\ u_{ij}\}_{ij}$ (resp. $\{\tilde{v}_{ij} = \exp(-\log \tilde{t}\ eN/2\pi\sqrt{-1})\ u_{ij}\}_{ij}$) is a $\mathcal{O}_S$-(resp. $\mathcal{O}_{\tilde{S}}$-) basis of $\mathcal{J}$ (resp. $\tilde{\mathcal{J}}$), where the eigenvalues of $\log M$ are in $[0,1)$. We remark that in general we have $\tilde{\mathcal{J}} \neq \pi^*\mathcal{J}$, i.e., there is a natural inclusion $\tilde{\mathcal{J}} \subset \pi^*\mathcal{J}$ such that $\pi^* v_{ij} = \tilde{t}^{-a_i}\ \tilde{v}_{ij}$.

Using these basis, $\mathcal{F}^\cdot$(resp. $\tilde{\mathcal{F}}^\cdot := \pi^*\mathcal{F}^\cdot$) can be identified with a holomorphic map $\Phi: S^* \to \mathrm{Flag}(H_{\mathbb{C},\infty})$ (resp. $\tilde{\Phi}: \tilde{S}^* \to \mathrm{Flag}(H_{\mathbb{C},\infty})$) such that $\Phi(t) = \exp(z \log M)\ F_z^\cdot$ (resp. $\tilde{\Phi}(\tilde{t}) = \exp(zN)\ F_z^\cdot$), for $t = \exp(2\pi\sqrt{-1}\ z)$ (resp. $\tilde{t} = \exp(2\pi\sqrt{-1}\ z/e)$).

Using Plücker coordinates, we can regard Φ(resp. $\tilde{\Phi}$) as $\Phi = (\phi_0(t):\cdots:\phi_k(t)): S^* \to \mathbf{P}^k$ (resp. $\tilde{\Phi} = (\tilde{\phi}_0(\tilde{t}):\cdots:\tilde{\phi}_k(\tilde{t})): \tilde{S}^* \to \mathbf{P}^k$), where ϕ_i (resp. $\tilde{\phi}_i$) are holomorphic functions on S^* (resp. $\tilde{S}^*$). Moreover, there are holomorphic functions g_i on $\tilde{S}$ such that $\phi_i(\pi(\tilde{t})) = g_i(\tilde{t})\ \tilde{\phi}_i(\tilde{t})$, because we have $\pi^* v_{ij} = \tilde{t}^{-a_i}\ \tilde{v}_{ij}$ and vector budles on S^* are trivial.

By the result of Schmid, $\tilde{\Phi}$ can be extended to the origin holomorphically. Hence there is a nowhere vanishing holomorphic function h on $\tilde{S}^*$ such that $h\cdot\tilde{\phi}_i$ and $h\cdot\pi^*\phi_i$ are holomorphic at the origin. Let $h(\tilde{t}) = \sum_{j=0}^{e-1} h_j(\pi(\tilde{t}))\cdot\tilde{t}^j$ be a decomposition of h such that h_j are holomorphic functions on S^*. Then $h_j\cdot\phi_i$ are extended to the origin, and also is Φ. Q.E.D.

(2.3) The reason why $\hat{\mathcal{F}}^{\cdot}|_{t=0} \neq F^{\cdot}_{\infty}$ is obvious from the proof. If $\tilde{\Phi} = \Phi\circ\pi$, they coincide, but this does not hold in general.

Example 3. Let $H_{\mathbb{Z}}$ be a local system on S^*, having a multi-valued basis $\{e_1,e_2\}$ such that $M\,e_1 = e_2$, $M\,e_2 = -e_1 - e_2$, where M is the monodromy of $H_{\mathbb{Z}}$.($M^3=1$) We define a skew symmetric bilinear form $<,>$ on $H_{\mathbb{Z}}$ by $<e_1,e_2> = 1$, and a Hodge subbundle $\mathcal{F}^1 := \mathcal{O}_{S^*}\,v \subset \mathcal{O}\otimes H_{\mathbb{Z}}$ by $v := g(t)\otimes e_1 + h(t)\otimes e_2$, where $g(t) := -a\,t^{-1/3} + \zeta\,t^{-2/3}$, $h(t) := a\,\zeta\,t^{-1/3} - t^{-2/3}$, $\zeta^3 = 1$, $\operatorname{Im}\zeta > 0$, $a \in \mathbb{C}$, $a \neq 0$ and $|a| \ll 1$.

It is easy to see that they form a polarized variation of Hodge structure of weight 1. (We set $\mathcal{F}^0 := \mathcal{O}\otimes H_{\mathbb{Z}}$, $\mathcal{F}^2 := \{0\}$.) For example, $\sqrt{-1}<v,\bar{v}> = -2\operatorname{Im} g\,\bar{h} > 0$ comes from $\operatorname{Im}\zeta > 0$ and $|a| \ll 1$.

We define another basis $\{u_1,u_2\}$ of $H_{\mathbb{C},\infty} = \Gamma(U,\rho^*H_{\mathbb{C}})$ by $u_1 := -e_1 + \zeta\,e_2$, $u_2 := \zeta\,e_1 - e_2$ such that $M\,u_1 = \zeta\,u_1$, $M\,u_2 = \zeta^{-1}\,u_2$ and $v = a\,t^{-1/3}\otimes u_1 + t^{-2/3}\otimes u_2$.

Then we have

$$\Phi(t) = \mathbb{C}\,(a\,u_1 + u_2)\quad (\subset H_{\mathbb{C},\infty})\qquad \text{for } \forall t \in S^*,$$

$$\tilde{\Phi}(\tilde{t}) = \mathbb{C}\,(a\,\tilde{t}\,u_1 + u_2)\quad (\subset H_{\mathbb{C},\infty})\qquad \text{for } \forall\tilde{t} \in \tilde{S}^*.$$

Hence $\Phi(0) = \mathbb{C}\,(a\,u_1 + u_2) \neq \tilde{\Phi}(0) = \mathbb{C}\,u_2$ ($\because a \neq 0$).

§3. Some remarks.

(3.1) The use of the Gauss-Manin system $\int^{\bullet}\mathcal{O}_X$ in the formulation of the result of Scherk-Steenbrink was first claimed by F. Pham (cf. [Ph]). One might think that $\int^{\bullet}\mathcal{O}_X$ and $\int^{o}\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}] = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}]$ would produce the same filtration, because we are considering the limit of the filtration on $S^* = S-\{0\}$. But this is not true, because the fundamental short exact sequence

$$0 \longrightarrow \oplus\mathcal{O}_S \longrightarrow \int^{o}\mathcal{O}_Y \longrightarrow \int^{o}\mathcal{O}_X \longrightarrow 0$$

does not split as $\mathcal{D}_S$-Modules in general, and we have an inclusion $\int^{o}\mathcal{O}_Y \hookrightarrow \int^{o}\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}]$ (cf. [Sa (2.5),(3.5)]). (The above exact sequence was found independetly by F. Pham (cf.[Ph 4.1]).)

(3.2) The rest of the proof of Theorem (3.2) in [Sa] is almost the same as Lemma 2 in [Va]. It is possible to prove the theorem without using it. For we can show the following. Let $\bar{Y} \to Y$ be a modification which is isomorphic on S^*.($\bar{Y}$ is smooth) Then $\int\mathcal{O}_Y$ is a direct factor of $\int\mathcal{O}_{\bar{Y}}$ as a filtered complex (cf.[Sa]').

(3.3) Let R be the residue of $t\partial_t : \tilde{\mathcal{H}}^{(o)} \to \tilde{\mathcal{H}}^{(o)}$. Then $\exp(-2\pi\sqrt{-1}\,R)$ and the monodromy M are conjugate to each other as matrices for $n = 1$ (i.e., $\{f=0\}$ is a plane curve). Combined with the result of Malgrange (Springer Lect. Note, 459, p. 115, Theorem (5.4)), we have the following. Let $b(s) = (s+1)\,\Pi_i(s+\alpha_i)^{m_i}$ be the b-function of f, and let $a(s) = \Pi_j(s-\lambda_j)^{r_j}$ be the minimal polynomial of the monodromy. Then we have $r_j = \max\{ m_i : \exp(-2\pi\sqrt{-1}\,\alpha_i) = \lambda_j\}$ for $n = 1$.

In fact, let $\{u_{ij}\}_{ij}$ be a basis of H_∞ such that $\{u_{ij}\}_{j=1,\dots,\ell_i}$ is a basis of $Gr_i^W H_\infty$ for $i = 0,1,2$, where W is the weight filtration [St]. Since F and W are compatible with the monodromy decomposition, we may assume that $u_{ij} \in F^1H_\infty$ for $i=1$, $j>\ell_1/2$ or $i=2$, and $M_s u_{ij} = \exp(-2\pi\sqrt{-1}\,\alpha_{ij})\,u_{ij}$ with $\alpha_{ij} \in (-1,0]$, where $M = M_s M_u$ is the Jordan decomposition. Since $N = \log M_u$ acts on H_∞ as the morphism of type $(-1,-1)$, we have $N u_{ij} = 0$ for $i\le 1$, and we may assume that $-N/(2\pi\sqrt{-1})\,u_{2j} = u_{0j}$ for $j\le\ell_0$ and $N u_{2j} = 0$ (hence $\alpha_{2j} = 0$) for $j>\ell_0$.

We set

$$v_{ij} = \exp(-\log t \log M/2\pi\sqrt{-1})\,u_{ij} = \begin{cases} t^{\alpha_{ij}}\,u_{ij} & \text{for } i\le 1 \text{ or } i=2,\ j>\ell_0 \\ t^{\alpha_{2j}}\,u_{2j} + t^{\alpha_{2j}}(\log t)u_{0j} & \text{for } i=2,\ j\le\ell_0\ , \end{cases}$$

so that $\{v_{ij}\}$ is a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis of $\mathcal{A}$.

By [Sa,(3.2)], there is an element $w_{ij} \in \mathcal{H}^{(o)}$, such that

$$\tilde{t}^{-e\alpha_{ij}}\,\pi^*(v_{ij} - w_{ij}) \in \tilde{t}\tilde{\mathcal{A}} \qquad \text{for } u_{ij} \in F^1H_\infty$$

and

$$\tilde{t}^{-e\alpha_{ij}}\,\pi^*(v_{ij} - \partial_t w_{ij}) \in \tilde{t}\tilde{\mathcal{A}} \qquad \text{for } u_{ij} \notin F^1H_\infty,$$

where $\pi: \tilde{S} \ni \tilde{t} \mapsto t=\tilde{t}^e \in S$ is a unipotent base change and $\tilde{\mathcal{A}}$ is the canonical extension for π^*H_X. Hence

$\{\partial_t w_{ij}\}_{i=0 \text{ or } i=1, j\le\ell_1/2} \cup \{w_{ij}\}_{i=1,j>\ell_1/2 \text{ or } i=2}$ is a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis of $\mathcal{A}$, and we have $\partial_t^{-1}\mathcal{A} \subset \mathcal{H}^{(o)}$. Then by the induction on the eigenvalue α_{ij}, we can show that v_{0j}, v_{1j} $(j>\ell_1/2)$ and v_{2j} are contained in $\tilde{\mathcal{H}}^{(o)}$.

For example, let $v = \sum v_i$ be an element of $\tilde{\mathcal{H}}^{(o)}$, such that $(t\partial_t - \alpha_i)^2 v_i = 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. We may assume that $(t\partial_t - \alpha_i) v_i = 0$ for $i \geq 2$ by the induction hypothesis, where $v = \sum v_i$ is the expansion of w_{2j} $(j \leq \ell_o)$ modulo $\partial_t^{-1}\mathcal{A} + \sum_{\alpha_{2j'} > \alpha_{2j}} \mathbb{C}\, v_{2j'}$. Then v_i and $(t\partial_t - \alpha_i) v_i$ are contained in $\tilde{\mathcal{H}}^{(o)}$, because we have the following identity:

$$\det \begin{pmatrix} 1, & 0, & 1, & \dots, & 1 \\ \alpha_1, & 1, & \alpha_2, & \dots, & \alpha_k \\ \alpha_1^2, & 2\alpha_1, & \alpha_2^2, & \dots, & \alpha_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^k, & k\alpha_1^{k-1}, & \alpha_2^k, & \dots, & \alpha_k^k \end{pmatrix}$$

$$= \pm \Pi_{i<j} (\alpha_i - \alpha_j)\, \Pi_{i>1} (\alpha_i - \alpha_1).$$

Thus $\tilde{\mathcal{H}}^{(o)}$ has a basis $\{v_{0j}\} \cup \{v_{1j}\}_{j>m} \cup \{v_{2j}\} \cup \{\partial_t^{-1} v_{1j}\}_{j \leq m}$ with $m \leq \ell_1/2$ (by changing $\{u_{1j}\}$ if necessary), which gives the desired result.

In general, we have that $\partial_t^{-n}\mathcal{A} \subset \mathcal{H}^{(o)}$, which implies that $f^{n+1} \in \sum \mathcal{O}\, \partial f/\partial x_i$. But I do not know whether $\exp(-2\pi\sqrt{-1}\, R)$ is conjugate to M for $n \geq 2$.

REFERENCES

[Ph] F. Pham, Structures de Hodge mixtes associées à un germe de fonction à point critique isolé (preprint).

[Sa] M. Saito, Gauss-Manin system and mixed Hodge structure. (to appear in Proc. Japan Acad., Vol.58, Ser.A, No.1 pp.29-32, 1982).

[SS] J. Scherk and J. Steenbrink, On the mixed Hodge structure on the cohomology of the Milnor fiber (preprint).

[Sc] W. Schmid, Variation of Hodge structure: the singularities of the period mapping. Inv. math., 22, 211-319 (1973).

[St] J. Steenbrink, Mixed Hodge structure on the vanishing cohomology. Proc. Nordic Summer School, Oslo (1976).

[Va] A. Varchenko, The asymptotics of holomorphic forms determine a mixed Hodge structure. Soviet Math. Dokl., 22(3) 772-775 (1980).

[Sa]' M. Saito, Hodge filtrations on Gauss-Manin systems (preprint).

Morihiko Saito

Research Institute for
Mathematical Sciences
Kyoto University
Kyoto 606 JAPAN

(Home Institute)
Dept. of Math.
Faculty of Sciences
University of Tokyo
Tokyo 113 JAPAN

Jan. 21, 1982