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Singularities, character formulas, and a q -analog of weight multiplicities

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1. The purpose of this paper is to discuss examples in which the intersection cohomology theory of Deligne-Goresky-MacPherson [4] enters in an essential way in the character formula for some irreducible representation of a semisimple group or Lie algebra. Thus, sections 3-5 are an exposition of the connection between singularities of Schubert varieties and multiplicities in Verma modules. In sections 6-11 we give an interpretation in terms of intersection cohomology for the multiplicities of weights in a finite dimensional representation of a simple Lie algebra. I wish to thank J. Bernstein for allowing me to use his unpublished results on the center of a Hecke algebra. (I learned about his results from D. Kazhdan.) These are used in the proof of Theorem 6.1 ; the original proof of that Theorem was based on [10] and on Macdonald's formulas for spherical functions.

2. Notations. For an irreducible complex algebraic variety X , we denote by $H^i(X)$ the i -th cohomology sheaf of the intersection cohomology complex of X .

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra, $\mathfrak{h}^*$ its dual space. Let $W \subset \text{Aut}(\mathfrak{h}^*)$ be the Weyl group, and let $S \subset W$ be the set of simple reflections (with respect to $\mathfrak{b}$). $Q \subset \mathfrak{h}^*$ is the subgroup generated by the roots.

$P \subset \mathfrak{h}^*$ is the subgroup consisting of those elements of $\mathfrak{h}^*$ which take integral values on any coroot. Then Q has finite index in P .

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$\tilde{W}_a \subset \{\text{affine transformations of } \underline{h}^*\}$ is the semidirect of W and of P (acting by translations). We shall regard $\tilde{W}_a$ as acting on the right on $\underline{h}^*$. The transform of $\lambda \in \underline{h}^*$ under $w \in \tilde{W}_a$ will be denoted $(\lambda)w$.

W_a is the subgroup of $\tilde{W}_a$ generated by W and Q . This is the affine Weyl group. It is a Coxeter group whose set S_a of simple reflections is S together with the reflection in W_a whose fixed point set is $\{x \in \underline{h}^* \mid \langle x, \alpha_0^\vee \rangle = 1\}$; here $\alpha_0^\vee \in \underline{h}$ is the highest coroot. Let Ω be the normalizer of S_a in $\tilde{W}_a$. Then $\tilde{W}_a$ is a semi-direct product $\Omega \cdot W_a$.

For $\lambda \in P$, we denote by p_λ the same element, regarded in $\tilde{W}_a$. Since the group law in $\tilde{W}_a$ is written multiplicatively, we have $p_{\lambda+\lambda'} = p_\lambda \cdot p_{\lambda'}$ for $\lambda, \lambda' \in P$. ℓ is the length function on the Coxeter group W_a . We extend it to $\tilde{W}_a$ by $\ell(\gamma w) = \ell(w\gamma) = \ell(w)$, $w \in W_a$, $\gamma \in \Omega$. For $s \in S$, let $\alpha_s \in Q$ be the corresponding simple root and let $\alpha_s^\vee \in \underline{h}$ be the corresponding simple coroot.

Let $P^{++} = \{p \in P \mid \langle p, \alpha_s^\vee \rangle \geq 0, \forall s \in S\}$. Then P^{++} parametrizes the double cosets $W \tilde{W}_a / W : \lambda \mapsto W p_\lambda W$. For $\lambda \in P^{++}$, W_λ denotes the stabilizer of λ in W , m_λ is the element of minimal length of $W p_\lambda W$, n_λ is the element of maximal length of $W p_\lambda W$, v_λ is the number of reflections in W_λ , $P_\lambda = \sum_{w \in W_\lambda} q^{\ell(w)}$ (q is an indeterminate). For $\lambda = 0$, we set $v_0 = v$, $P_0 = P$; $\rho \in P$ denotes half the sum of all positive roots; $\check{\rho} \in \underline{h}$ denotes half the sum of all positive coroots.

The fundamental alcove A_0 is the open simplex in $P \otimes \mathbb{R}$ (embedded in $\underline{h}^*$) bounded by the fixed hyperplanes of the various reflections in S_a . An alcove is an open simplex in $P \otimes \mathbb{R}$ of the form $(A_0)w$, $w \in W_a$ (which is unique). Define a new (left) action of W_a on the set of alcoves (denotes $A \rightarrow yA$) by the rule $y((A_0)w) = (A_0)yw$. For each $\lambda \in P$, we denote $A_\lambda^+ = (A_0)p_\lambda$, $A_\lambda^- = (-A_0)p_\lambda$. Let $\leq$ be the standard partial order on the Coxeter group W_a . It is generated by the relations $s_1 s_2 \dots \hat{s}_i \dots s_n \leq s_1 s_2 \dots s_n$ for any reduced expression $s_1 \dots s_n$ ($s_i \in S_a$), $1 \leq i \leq n$. We extend it to a partial order $\leq$ on $\tilde{W}_a$ by $\gamma w \leq \gamma' w' \iff \gamma = \gamma'$ and $w \leq w'$ ($\gamma, \gamma' \in \Omega$, $w, w' \in W_a$). Let $\leq$ be the partial order on P defined by

$\lambda \leq \lambda' \Leftrightarrow \lambda' - \lambda$ is a linear combination of positive roots, with ≥ 0 integral coefficients. If $\lambda, \lambda' \in P^{++}$, we have $\lambda \leq \lambda'$ if and only if $n_\lambda \leq n_{\lambda'}$ (in $\tilde{W}_a$). For $\lambda \in \underline{h}^*$, M_λ denotes the Verma module for $\underline{g}$ with highest weight λ (with respect to $\underline{b}$) and L_λ denotes the unique irreducible quotient $\underline{g}$ -module of M_λ .

3. We will restrict our attention to the Verma modules $M_{-\rho w - \rho}$ ($w \in W$). In the Grothendieck group of $\underline{g}$ -modules, $L_{-\rho w - \rho}$ is a linear combination with integral coefficients of the $\underline{g}$ -modules $M_{-\rho y - \rho}$ ($y \leq w$). The $\underline{g}$ -module $M_{-\rho w - \rho}$ appears with coefficient 1, but the other coefficients were rather mysterious. A study of representations of Hecke algebras has led Kazhdan and the author [7] to give a (conjectural) algorithm for these coefficients and to interpret them in terms of singularities of Schubert varieties. Let us define the Schubert varieties. Consider the adjoint group G of $\underline{g}$, and let B be the Borel subgroup corresponding to $\underline{b}$, G_w the B - B double coset of G containing a representative of $w \in W$, $O_w = G_w/B \subset G/B$. The Zariski closure $\overline{O}_w$ of O_w in G/B is said to be a Schubert variety. It is the union of the various O_y for $y \leq w$.

The following result was conjectured by D. Kazhdan and the author [7],[8] and was proved by J.L. Brylinski and M. Kashiwara [3] and independently by A.A. Beilinson and J.N. Bernstein [1], using the theory of holonomic systems.

Theorem 3.1. In the Grothendieck group of $\underline{g}$ -modules, we have, for any $w \in W$:

$$(3.2) \quad L_{-\rho w - \rho} = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} \left(\sum_i (-1)^i \dim H^i_{O_y}(\overline{O}_w) \right) M_{-\rho y - \rho}$$

where $\dim H^i_{O_y}(\overline{O}_w)$ is the dimension of the stalk of $H^i(\overline{O}_w)$ at a point in O_y .

4. We shall now describe the integers $\dim H^i_{O_y}(\overline{O}_w)$ following [7],[8]. Let us recall the definition of the Hecke algebra H associated to (W, S) . It consists of all formal linear combinations $\sum_{w \in W} a_w T_w$ with $a_w \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with multiplication defined by the rules $T_w T_{w'} = T_{ww'}$, if $\ell(ww') = \ell(w) + \ell(w')$ and $(T_s + 1)(T_s - q) = 0$ if $s \in S$; here $q^{1/2}$ is an indeterminate. There is a unique ring involution $h \rightarrow \bar{h}$ of H which takes $q^{1/2}$ to $q^{-1/2}$ and T_w to T_w^{-1} ($w \in W$).

It is semilinear with respect to the ring involution $q^{1/2} \rightarrow q^{-1/2}$ of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

According to [7, 1.1], for each $w \in W$, there is a unique element $C'_w \in H$ of the form $C'_w = q^{-\ell(w)/2} \sum_{y < w} P_{y,w} T_y$, where $P_{y,w}$ are polynomials in q satisfying $P_{w,w} = 1$ and $\deg P_{y,w} \leq 1/2(\ell(w) - \ell(y) - 1)$ for $y < w$, and such that $\overline{C}'_w = C'_w$. The uniqueness of C'_w holds also if $P_{y,w}$ for $y < w$ is only assumed to be a polynomial in q and q^{-1} in which only powers q^i with $i \leq 1/2(\ell(w) - \ell(y) - 1)$ are allowed to occur. It follows automatically that the $P_{y,w}$ are polynomials in q . The proof in [7] applies without change. (The discussion so far in this section, applies to an arbitrary Coxeter group and in particular to (W_a, S_a) . It also applies word by word to $(\widetilde{W}_a, S_a)$ which although is not a Coxeter group, possesses the length function and the partial order $\leq$ which give a sense to the previous definitions and results.)

We can now state

Theorem 4.1. Let $y \leq w$ be two elements in the Weyl group W . Then

$$(4.2) \quad \dim H^i_{O_y}(\overline{O}_w) = 0 \quad \text{if } i \text{ is odd}$$

$$(4.3) \quad \sum_i \dim H^{2i}_{O_y}(\overline{O}_w) q^i = P_{y,w}.$$

Besides the original proof in [8], there is another proof in [12] which has the advantage that it also applies in the case where $\overline{O}_w$ is replaced by the closure of a K -orbit on G/B , where K is the centralizer of an involution in G . (This plays a role in a character formula for real semisimple Lie groups.) Both proofs make use of reduction to characteristic > 1 and of a form of Weil's conjectures. Combining Theorems 3.1, 4.1, we can rewrite (3.2) in the form

$$(4.4) \quad L_{-\rho w - \rho} = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} P_{y,w}(1) M_{-\rho y - \rho}$$

where $P_{y,w}(1)$ is the value of $P_{y,w}$ at $q = 1$. Using the inversion formula [7, 3.1] for the matrix $(P_{y,w})$, this can be also written as

$$(4.5) \quad M_{\rho w - \rho} = \sum_{w \leq y} P_{w,y}(1) L_{\rho y - \rho}.$$

5. Remarks. (a) In the case where $y, w \in W_a$, the polynomials $P_{y,w}$ have been interpreted in [7] in terms analogous to (4.3), as intersection cohomology of certain generalized Schubert varieties. (In particular, they have ≥ 0 coefficients).

(b) There is a (conjectural) formula analogous to (3.2) for the characters of irreducible rational representations of a semisimple group over an algebraically closed field of characteristic > 1 . It involves the polynomials $P_{y,w}$ for y, w in an affine Weyl group. (See [9] for a precise statement).

6. If $\lambda \in P^{++}$, the $\mathfrak{g}$ -module L_λ is finite dimensional. With respect to the action of $\mathfrak{h}$, it decomposes into direct sum of weight spaces parametrized by elements $\mu \in P$. For $\mu \in P^{++}$, we denote $d_\mu(L_\lambda)$ the dimension of the μ -weight space in L_λ . It is well known that $d_\mu(L_\lambda) = 0$ unless $\mu \leq \lambda$. The remainder of this paper is mainly concerned with the proof of the following result.

Theorem 6.1. If $\mu, \lambda \in P^{++}$, $\mu \leq \lambda$, then $d_\mu(L_\lambda) = P_{n_\mu, n_\lambda}(1)$.

Here, P_{n_μ, n_λ} is defined in terms of the Hecke algebra of $\tilde{W}_a$, see section 4.

(This Hecke algebra will be denoted $\tilde{H}$; from now on, we shall reserve the letter H to denote the Hecke algebra of W_a . It is a subalgebra of $\tilde{H}$.) Note that

$P_{\gamma y, \gamma w} = P_{y, w}$ ($\gamma \in \Omega$, $y, w \in W_a$) so that the polynomials $P_{y', w'}$, for $y', w' \in \tilde{W}_a$, have ≥ 0 coefficients. For type A, Theorem 6.1 follows from the results of [11], where P_{n_μ, n_λ} are interpreted as Green-Foulkes polynomials. In general, 6.1 would be a consequence of the conjecture 5(b) together with the Steinberg tensor product theorem. The integers $d_\mu(L_\lambda)$ are given by Weyl's character formula. To state the formula, we consider the elements

$$(6.2) \quad k_\lambda = \frac{1}{|W|} \sum_{w \in W} w, (\lambda \in P^{++}), j_\lambda = \left(\sum_{w \in W} (-1)^{\ell(w)} w^{-1} \right) p_\lambda \left(\sum_{w \in W} w \right), (\lambda \in P^{++} + \rho)$$

of the group algebra $\mathbb{Q}[\tilde{W}_a]$. Then k_λ ($\lambda \in P^{++}$) form a $\mathbb{Z}$ -basis for the subgroup

$K^1 = \{x \in \frac{1}{|W|} \mathbb{Z}[\tilde{W}_a] : \left(\sum_{w \in W} w \right) x = x \left(\sum_{w \in W} w \right) = |W| \cdot x \subset \mathbb{Q}[\tilde{W}_a] \text{ and } j_\lambda (\lambda \in P^{++} + \rho) \text{ form a } \mathbb{Z}\text{-basis for the subgroup}$

$$J^1 = \{y \in \mathbb{Z}[\tilde{W}_a] : \left(\sum_{w \in W} (-1)^{\ell(w)} w^{-1} \right) y = y \left(\sum_{w \in W} w \right) = |W| \cdot y \}.$$

It follows that K^1 is a subring of $\mathbb{Q}[\tilde{W}_a]$ with unit element $\frac{1}{|\tilde{W}|} \sum_{w \in W} w$ and that, with respect to the product in $\mathbb{Q}[\tilde{W}_a]$, we have $J^1 \cdot K^1 \subset J^1$, i.e. J^1 is a right K^1 -module. Moreover, the map $K^1 \longrightarrow J^1$ given by $k \rightarrow j_\rho \cdot k$ is an isomorphism of right K^1 -modules. (This is a reformulation of [2, Ch. VI, 3.3, Prop. 2(iii)] . We can now state Weyl's character formula as follows

(6.3) For $\lambda \in P^{++}$, let $C_\lambda^1 = \sum_{\mu \in P^{++}} d_\mu(L_\lambda) k_\mu \in K^1$. Then C_λ^1 is the unique element in K^1 such that $j_\rho C_\lambda^1 = j_{\lambda+\rho}$.

(This is equivalent to the usual formulation in which the character of L_λ appears as a quotient of two alternating expressions.)

We wish to consider a q -analog of the multiplicity $d_\mu(L_\lambda)$. The q -analogs of the elements (6.2) are the following elements of the Hecke algebra $\tilde{H}$:

$$(6.4) \quad K_\lambda = \frac{1}{\bar{p}} \sum_{w \in W_{p_\lambda} W} T_w = \frac{q^{-v+\nu_\lambda}}{\bar{p} \cdot p_\lambda} \left(\sum_{w \in W} T_w \right) T_{p_\lambda} \left(\sum_{w \in W} T_w \right), \quad (\lambda \in P^{++})$$

$$(6.5) \quad J_\lambda = \left(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1} \right) q^{-\ell(m_\lambda)/2} T_{m_\lambda} \left(\sum_{w \in W} T_w \right)$$

and therefore

$$J_\lambda = q^{-v/2} \left(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1} \right) q^{-\ell(p_\lambda)/2} T_{p_\lambda} \left(\sum_{w \in W} T_w \right) \quad \text{for } \lambda \in P^{++} + \rho.$$

Then K_λ ($\lambda \in P^{++}$) form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -basis for

$$K = \{x \in \frac{1}{\bar{p}} \tilde{H} : \left(\sum_{w \in W} T_w \right) x = x \left(\sum_{w \in W} T_w \right) = p \cdot x\} \subset \tilde{H} \otimes \mathbb{Q}(q^{1/2})$$

and J_λ ($\lambda \in P^{++} + \rho$) form a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -basis for

$$J = \{y \in \tilde{H} : \left(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1} \right) y = y \left(\sum_{w \in W} T_w \right) = p \cdot y\}.$$

Note that K is a subring of $\tilde{H} \otimes \mathbb{Q}(q^{1/2})$ with unit element $\frac{1}{\bar{p}} \sum_{w \in W} T_w$ and that, with respect to the product in $\tilde{H} \otimes \mathbb{Q}(q^{1/2})$, we have $J \cdot K \subset J$, i.e. J is a right K -module.

In the statement of the following theorem, we shall give a meaning to $J_\lambda \in J$ for arbitrary $\lambda \in P$: if $(\lambda)w \neq \lambda$ for all $w \in W$, $w \neq e$, we set

$$J_\lambda = (-1)^{\ell(w)} J_{(\lambda)w} \quad \text{where } w \text{ is the unique element of } W \text{ such that } (\lambda)w \in P^{++} + \rho.$$

For the remaining $\lambda \in P$, we set $J_\lambda = 0$.

Theorem 6.6. For any $\lambda \in P^{++}$, we have

$$(6.7) \quad J_\lambda \cdot (q^{-\ell(p_\lambda)/2} K_\lambda) = \frac{1}{\overline{p}_\lambda} \sum_I (-q)^{-|I|} J_{\lambda+\rho-\alpha_I}$$

(sum over all subsets I of the set of positive roots); here α_I denotes the sum of the roots in I .

The proof will be given in Section 7.

If I is as in the previous sum and if $w \in W$ is such that $\lambda+\rho-\alpha_I = (\lambda'+\rho)w$, $\lambda' \in P^{++}$, then $\lambda-\lambda' = \lambda-(\lambda)w^{-1}-(\rho)w^{-1}+(\alpha_I)w^{-1} = \lambda-(\lambda)w^{-1}+\alpha_J$ where J is the set of positive roots β such that $(\beta)w \in I$ or such that $-(\beta)w$ is positive, $\notin I$. Since $\lambda \geq (\lambda)w^{-1}$ ($\lambda \in P^{++}$) and $\alpha_J \geq 0$, it follows that $\lambda \geq \lambda'$. Thus, the right hand side of (6.7) is a linear combination of elements $J_{\lambda'+\rho}$ ($\lambda' \leq \lambda$) with formal power series in q^{-1} without terms of form q^i ($i > 0$) as coefficients; moreover for $\lambda' < \lambda$, the coefficient doesn't have a constant term. On the other hand, since the left hand side of (6.7) is in J , these coefficients must be polynomials in $q^{1/2}$, $q^{-1/2}$. It follows that they are polynomials in q^{-1} (without constant term if $\lambda' < \lambda$). The coefficient of $J_{\lambda+\rho}$ is equal to 1; this follows from the identity $\frac{1}{\overline{p}_\lambda} \sum_I (-q)^{-|I|} = 1$.

$$\alpha_I = \rho - (\rho)w$$

Since a triangular matrix with 1's on diagonal has an inverse of the same form, we see that for any $\lambda \in P^{++}$, the element $J_{\lambda+\rho}$ is a linear combination of elements $J_\rho(q^{-\ell(p_{\lambda'})/2} K_{\lambda'})$, $\lambda' \leq \lambda$, with coefficients polynomials in q^{-1} (without constant term, if $\lambda' < \lambda$ and $\equiv 1$, if $\lambda' = \lambda$). Hence we have

Corollary 6.8. For any $\lambda \in P^{++}$, there is a unique element $C'_\lambda \in K$ such that

$$(6.9) \quad J_\rho \cdot C'_\lambda = J_{\lambda+\rho}.$$

It is of the form

$$(6.10) \quad c'_\lambda = q^{-\ell(p_\lambda)/2} \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_\mu(L_\lambda; q) K_\mu$$

where $d_\mu(L_\lambda; q)$ are polynomials in q and q^{-1} with integer coefficients; moreover, the powers q^i appearing in $d_\mu(L_\lambda; q)$ satisfy $i < \frac{1}{2}(\ell(p_\lambda) - \ell(p_\mu))$ if $\mu < \lambda$ and $d_\lambda(L_\lambda; q) \equiv 1$. In particular, the map $h \rightarrow J_\rho h$ defines an isomorphism of right K -modules of K onto J .

Note that, if $\mu \leq \lambda$, then $\frac{1}{2}(\ell(p_\lambda) - \ell(p_\mu))$ is an integer. Indeed, it is known [5] that, for $\lambda \in P^{++}$,

$$(6.11) \quad \ell(p_\lambda) = \langle \lambda, 2\rho^\vee \rangle.$$

Hence $\frac{1}{2}(\ell(p_\lambda) - \ell(p_\mu)) = \frac{1}{2}(\langle \lambda, 2\rho^\vee \rangle - \langle \mu, 2\rho^\vee \rangle) = \langle \lambda - \mu, \rho^\vee \rangle$ and this is an integer since $\lambda - \mu \in Q$.

We shall now show that $d_\mu(L_\lambda; q)$ are actually polynomials in q with ≥ 0 coefficients.

We have

Theorem 6.12. $c'_\lambda = q^{v/2} p^{-1} c'_{n_\lambda}$ ($\lambda \in P^{++}$). In particular, for $\mu \leq \lambda$ in P^{++} , we have

$$(6.13) \quad d_\mu(L_\lambda; q) = p_{n_\mu, n_\lambda}$$

hence $d_\mu(L_\lambda; q)$ is a polynomial in q with ≥ 0 coefficients.

For the proof of 6.12, we need the following result.

Lemma 6.14. If $\lambda \in P^{++}$, then $\bar{J}_{\lambda+\rho} = J_{\lambda+\rho}$.

In the case where $\lambda \in Q \cap P^{++}$, this is just Lemma 11.7 of [10]. The general case is proved in the same way.

The definition of K shows that K is stable under $h \rightarrow \bar{h}$ (which is extended to a ring involution of $\tilde{H} \otimes \mathbb{Q}(q^{1/2})$. (Note that $\overline{p^{-1} \sum_{w \in W} T_w} = p^{-1} \sum_{w \in W} T_w$.) From (6.9) it then follows that $J_\rho \bar{c}'_\lambda = J_{\rho+\lambda}$. Thus $J_\rho (c'_\lambda - \bar{c}'_\lambda) = 0$ and, since $c'_\lambda - \bar{c}'_\lambda \in K$, we have $c'_\lambda = \bar{c}'_\lambda$, by the last sentence in Corollary 6.8.

The element $q^{-v/2} p c'_\lambda$ is also fixed by $h \rightarrow \bar{h}$, since $\overline{q^{-v/2} p} = q^{-v/2} p$. This ele-

ment is equal to

$$q^{-\ell(n_\lambda)/2} \sum_{y \leq n_\lambda} d_\mu(y) (L_\lambda; q) T_y$$

where $\mu(y) \in P^{++}$ is defined by $y \in W_{p_\mu(y)}^W$.

We now use the bounds on the powers of q appearing in $d_\mu(L_\lambda; q)$ given in Corollary 6.8. It follows that $q^{-\ell/2} p_{C'_\lambda}$ satisfies the defining property of C'_{n_λ} , hence is equal to it. Thus Theorem 6.12 follows from Theorem 6.6. On the other hand, it implies Theorem 6.1. Indeed, under the specialization $\mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathbb{Z}$, given by $q^{1/2} \rightarrow 1$, $\tilde{H}$ becomes the group ring $\mathbb{Z}[\tilde{W}_a]$, K_λ becomes $k_\lambda (\lambda \in P^{++})$, J_λ becomes $j_\lambda (\lambda \in P^{++} + \rho)$ and (6.9) becomes (6.3). It follows that for $\mu, \lambda \in P^{++}$, $\mu \leq \lambda$, $d_\mu(L_\lambda)$ is the value of $d_\mu(L_\lambda; q)$ at $q=1$ and theorem 6.1 follows.

7. For the proof of Theorem 6.6 we shall need several preliminary steps. We shall begin with a definition (due to J. Bernstein) of a large commutative subalgebra of $\tilde{H}$, which is a q -analogue of the subring $\mathbb{Z}[P]$ of $\mathbb{Z}[\tilde{W}_a]$. To each $\lambda \in P$, Bernstein associates an element $\tilde{T}_{p_\lambda} \in \tilde{H}$ defined by $\tilde{T}_{p_\lambda} = (q^{-\ell(p_{\lambda_1})/2} T_{p_{\lambda_1}})^{-1} (q^{-\ell(p_{\lambda_2})/2} T_{p_{\lambda_2}})^{-1}$ where λ_1, λ_2 are elements of P^{++} such that $\lambda = \lambda_1 - \lambda_2$. This is independent of the choice of λ_1, λ_2 , since for $\lambda', \lambda'' \in P^{++}$ we have the identity $T_{p_{\lambda'}} T_{p_{\lambda''}} = T_{p_{\lambda''}} T_{p_{\lambda'}} = T_{p_{\lambda'} + p_{\lambda''}} = T_{p_{\lambda' + \lambda''}}$. (Indeed, we have $\ell(p_{\lambda'}) + \ell(p_{\lambda''}) = \ell(p_{\lambda'} + p_{\lambda''})$, by (6.11).) It follows also that if $\lambda', \lambda'' \in P$, we have $\tilde{T}_{p_{\lambda'}} \tilde{T}_{p_{\lambda''}} = \tilde{T}_{p_{\lambda'} + p_{\lambda''}}$ and $\tilde{T}_{p_{\lambda'}}^{-1} = \tilde{T}_{p_{-\lambda'}}$. We shall prove the following

Lemma 7.1. (J. Bernstein) Let $\lambda \in P$ and let $s \in S$. We have

$$T_s (\tilde{T}_{p_\lambda} + \tilde{T}_{p_{(\lambda)s}}) = (\tilde{T}_{p_\lambda} + \tilde{T}_{p_{(\lambda)s}}) T_s.$$

Proof : We may clearly assume that $\langle \lambda, \check{\alpha}_s \rangle \geq 0$. Assume first that $\langle \lambda, \check{\alpha}_s \rangle = 0$. We can write $\lambda = \lambda_1 - \lambda_2$ with $\lambda_1, \lambda_2 \in P^{++}$, $\langle \lambda_1, \alpha_s \rangle = \langle \lambda_2, \check{\alpha}_s \rangle = 0$. To prove the identity $T_s \tilde{T}_{p_\lambda} = \tilde{T}_{p_\lambda} T_s$, we are thus reduced to the case where $\lambda \in P^{++}$, $\langle \lambda, \check{\alpha}_s \rangle = 0$. But then $\ell(sp_\lambda) = \ell(p_\lambda s) = \ell(p_\lambda) + 1$ hence $T_s T_{p_\lambda} = T_{sp_\lambda} = T_{p_\lambda s} = T_{p_\lambda} T_s$ as required.

Next, we consider the case where $\langle \lambda, \alpha_s^\vee \rangle = 1$, i.e. $(\lambda)_s = \lambda - \alpha_s$. In this case, the result follows from Lemma 4.4.(b) in (G. Lusztig, Some examples of square integrable representations of semisimple p -adic groups, preprint IHES, 1982).

Next, we assume that $\langle \lambda, \alpha_s^\vee \rangle = d \geq 2$ and that the result is already known when d is replaced by d' , $0 \leq d' < d$. We can write $\lambda = \lambda_1 + \lambda_2$ where $\langle \lambda_1, \alpha_s^\vee \rangle = d-1$, $\langle \lambda_2, \alpha_s^\vee \rangle = 1$. Then $\langle \lambda_1 + (\lambda_2)_s, \alpha_s^\vee \rangle = d-2$. The induction hypothesis is applicable to λ_1, λ_2 and to $\lambda_1 + (\lambda_2)_s$. Hence T_s commutes with $A = \tilde{T}_{P\lambda_1} + \tilde{T}_{P(\lambda_1)_s}$, $B = \tilde{T}_{P\lambda_2} + \tilde{T}_{P(\lambda_2)_s}$, $C = \tilde{T}_{P\lambda_1 + (\lambda_2)_s} + \tilde{T}_{P(\lambda_1)_s + \lambda_2}$. But $\tilde{T}_{P\lambda_1 + \lambda_2} + \tilde{T}_{P(\lambda_1 + \lambda_2)_s} = A \cdot B \cdot C^{-1}$ hence T_s commutes with $\tilde{T}_P + \tilde{T}_{P(\lambda)_s}$. The lemma is proved.

We now define, for any $\lambda \in P$ an element $\tilde{J}_\lambda \in J$ by the formula

$$\tilde{J}_\lambda = q^{-\nu/2_\theta} \cdot \tilde{T}_{P\lambda} \theta$$

where $\theta = \sum_{w \in W} T_w$, $\theta' = \sum_{w \in W} (-q)^{\ell(w)} T_w^{-1}$. When $\lambda \in P^{++}_{+p}$, we have clearly $\tilde{J}_\lambda = J_\lambda$. In general, we have

Lemma 7.3. $\tilde{J}_{(\lambda)_w} = (-1)^{\ell(w)} \tilde{J}_\lambda$ for any $\lambda \in P$, $w \in W$; hence, $\tilde{J}_\lambda = J_\lambda$ for all $\lambda \in P$.

Proof : We may assume that $w = s \in S$. Note that $T_s \theta = q\theta$, $\theta' T_s^{-1} = -\theta'$, hence

$$\begin{aligned} \tilde{J}_\lambda + \tilde{J}_{(\lambda)_s} &= q^{-\nu/2_\theta} \cdot (\tilde{T}_\lambda + \tilde{T}_{(\lambda)_s}) \theta \\ &= q^{-\nu/2_\theta} \cdot T_s^{-1} (\tilde{T}_\lambda + \tilde{T}_{(\lambda)_s}) T_s \theta \\ &= -q \cdot q^{-\nu/2_\theta} \cdot (\tilde{T}_\lambda + \tilde{T}_{(\lambda)_s}) \theta \quad \text{by lemma (7.1)} \\ &= -q (\tilde{J}_\lambda + \tilde{J}_{(\lambda)_s}) \end{aligned}$$

Thus, $\tilde{J}_\lambda + \tilde{J}_{(\lambda)_s} = 0$, as required.

Lemma 7.4. There is a unique function $f : Q + p \rightarrow \mathbb{Z}[q, q^{-1}]$ with finite support satisfying properties (i), (ii), (iii) below :

(i) $f(\varphi) = q^\nu$

(ii) $f(\lambda) \neq 0 \Rightarrow \lambda \leq \rho$

(iii) Let $X \subset Q+\rho$ be an α_s -string : $X = \{x+n\alpha_s, n \in \mathbb{Z}\}$, where x is any fixed
element of $Q+\rho$ and α_s is any fixed simple root. Let $a > 0$ be an integer such
that $\langle \lambda, \check{\alpha}_s \rangle \equiv a \pmod{2}$ for all $\lambda \in X$. Then

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \geq a}} f(\lambda) = -q^{-(a-1)} \sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha}_s \rangle \leq -a}} f(\lambda) .$$

This function is given by the formula

$$(7.5) \quad f(\lambda) = (-1)^v \sum_{\substack{I \\ \alpha_I = \lambda + \rho}} (-q)^{|I|} q^{-\langle \lambda - \rho, \check{\rho} \rangle} ,$$

where I runs through the subsets of the set of positive roots, and α_I is defined
as in 6.6.

Proof : The function f defined by (7.5) clearly satisfies (i) and (ii). We now
 verify that it satisfies (iii). We shall set $\alpha_s = \alpha, \check{\alpha}_s = \check{\alpha}$. We have, with the no-
 tations of (iii) :

$$\begin{aligned} \sum_{\substack{\lambda \in X \\ \langle \lambda, \check{\alpha} \rangle \geq 0}} f(\lambda) &= (-1)^v \sum_{\substack{\lambda \in X \\ I \\ \alpha_I = \lambda + \rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda - \rho, \check{\rho} \rangle} \\ &= (-1)^v \sum_{\substack{\lambda \in X \\ I \not\ni \alpha \\ \alpha_I = \lambda + \rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda - \rho, \check{\rho} \rangle} + \Sigma' \end{aligned}$$

where

$$\begin{aligned} \Sigma' &= (-1)^v \sum_{\substack{\lambda \in X \\ I \ni \alpha \\ \alpha_I = \lambda + \rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|} q^{-\langle \lambda - \rho, \check{\rho} \rangle} = (-1)^v \sum_{\substack{\lambda \in X \\ I' \not\ni \alpha \\ \alpha_{I'} = \lambda - \alpha + \rho \\ \langle \lambda, \check{\alpha} \rangle \geq a}} (-q)^{|I|+1} q^{-\langle \lambda - \rho, \check{\rho} \rangle} = \\ &= (-1)^v \sum_{\substack{\lambda' \in X \\ I' \not\ni \alpha \\ \alpha_{I'} = \lambda' + \rho \\ \langle \lambda' + \alpha, \check{\alpha} \rangle \geq a}} (-q)^{|I'|+1} q^{-\langle \lambda' + \alpha - \rho, \check{\rho} \rangle} = (-1)^v \sum_{\substack{\lambda \in X \\ I \not\ni \alpha \\ \alpha_I = \lambda + \rho \\ \langle \lambda, \check{\alpha} \rangle \geq a-2}} (-q)^{|I|} q^{-\langle \lambda - \rho, \check{\rho} \rangle} \end{aligned}$$

Hence

$$(7.6) \quad \sum_{\substack{\lambda \in X \\ \langle \lambda, \alpha \rangle \geq a}} f(\lambda) = -(-1)^v \sum_{\substack{\lambda \in X \\ I \nmid \alpha \\ \alpha_I = \lambda + \rho \\ \langle \lambda, \alpha \rangle = a-2}} (-q) |I|_q^{-\langle \lambda - \rho, \rho \rangle^v}$$

A similar computation shows that

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \alpha \rangle \leq -a}} f(\lambda) = (-1)^v \sum_{\substack{\lambda \in X \\ I \nmid \alpha \\ \alpha_{I^v} = \lambda + \rho \\ \langle \lambda, \alpha \rangle = -a}} (-q) |I|_q^{-\langle \lambda - \rho, \rho \rangle^v}.$$

Now the simple reflection s maps the set of positive roots $\neq \alpha$ onto itself. Hence the last sum is equal to

$$\begin{aligned} (-1)^v \sum_{\substack{\lambda \in X \\ I \nmid \alpha \\ \alpha_{(I)s} = (\lambda + \rho)s \\ \langle \lambda, \alpha \rangle = -a}} (-q) |I|_q^{-\langle \lambda - \rho, \rho \rangle^v} &= (-1)^v \sum_{\substack{\lambda' \in X \\ I' \nmid \alpha \\ \alpha_{I'} = \lambda' + (a-1)\alpha + \rho \\ \langle \lambda', \alpha \rangle = -a}} (-q) |I'|_q^{-\langle \lambda' - \rho, \rho \rangle^v} \\ &= (-1)^v \sum_{\substack{\lambda' \in X \\ I' \nmid \alpha \\ \alpha_{I'} = \lambda' + \rho \\ \langle \lambda' - (a-1)\alpha, \alpha \rangle = -a}} (-q) |I'|_q^{-\langle \lambda' - (a-1)\alpha - \rho, \rho \rangle^v} \\ &= (-1)^v q^{a-1} \sum_{\substack{\lambda' \in X \\ I' \nmid \alpha \\ \alpha_{I'} = \lambda' + \rho \\ \langle \lambda', \alpha \rangle = a-2}} (-q) |I'|_q^{-\langle \lambda' - \rho, \rho \rangle^v} \end{aligned}$$

Comparing with the right hand side of (7.6), we conclude that f satisfies (iii).

To prove the converse it is enough to show that if a function $g: Q + \rho \rightarrow \mathbb{Z}[q, q^{-1}]$ with finite support satisfies $g(\rho) = 0$, $g(\lambda) \neq 0 \Rightarrow \lambda \leq \rho$ and the identity (iii) with f replaced by g , then $g \equiv 0$. Assume that $g \not\equiv 0$, and let $x \in Q + \rho$ be an element of maximal possible length (with respect to some positive definite, W -invariant scalar product on $P \otimes \mathbb{R}$) such that $g(x) \neq 0$. Let X be the string through x corresponding to the simple root α_s . Then $x' = (x)s$ is also in X . Let a be the absolute value of $\langle x, \alpha_s^v \rangle = \langle x', \alpha_s^v \rangle$. If $y \in X$ satisfies

$|\langle y, \alpha_s^\vee \rangle| > a$ then clearly the length of y is strictly bigger than that of x hence $g(y) = 0$. Hence the identity (iii) for g , and X , a , as above, reduces to $g(x) = -q^{a-1}g(x')$. It follows that $g(x') \neq 0$. Note also that x, x' have the same length. Iterating this, we see that $g((x)w) \neq 0$ for all $w \in W$; moreover, $(x)w$ has the same length as x . For suitable $w \in W$, we have $\langle (x)w, \alpha_s^\vee \rangle \geq 0$ for all simple roots α_s . Replacing x by $(x)w$, we may thus assume that $\langle x, \alpha_s^\vee \rangle \geq 0$ for all simple roots α_s . If we had $\langle x - \rho, \alpha_s^\vee \rangle \geq 0$ for all simple roots α_s then it would follow that $\langle x - \rho, \alpha_s^\vee \rangle \geq 0$; since $g(x) \neq 0$, we would have $\rho - x \geq 0$, hence $\rho - x = \sum n_s \alpha_s$ (α_s simple, $n_s \geq 0$ integers), hence $\langle -\sum n_s \alpha_s, \alpha_s^\vee \rangle \geq 0$. Thus $-\sum n_s = 0$, hence $n_s = 0$ for all simple roots α_s , hence $x = \rho$. But $g(\rho) = 0$ and this is a contradiction with $g(x) \neq 0$. Thus, there exists a simple root α_s such that $\langle x - \rho, \alpha_s^\vee \rangle < 0$; since $\langle x, \alpha_s^\vee \rangle \geq 0$, it follows that $\langle x, \alpha_s^\vee \rangle = 0$. Consider the string X through x corresponding to the simple root α_s . The equality $\langle x, \alpha_s^\vee \rangle = 0$ shows that among the elements of X , the element x has minimal length. It follows that $g(y) = 0$ for all $y \in X$, $y \neq x$. Let us now write the identity (iii) for g , this X , and $a = 0$. We get $g(x) = -q^{-1}g(x)$ hence $g(x) = 0$. This contradiction shows that $g = 0$ and the Lemma is proved.

We shall now introduce as in [10] an H -module M as follows. M is the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ module with basis (A) where A are the various alcoves in $P \otimes \mathbb{R}$. For each $s \in S_a$, we define an endomorphism T_s of this $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -module by

$$T_s(A) = \begin{cases} sA, & \text{if } \exists \text{ positive coroot } \alpha^\vee \text{ with } \langle x, \alpha^\vee \rangle = n \text{ for} \\ & x \in sA, \langle x, \alpha^\vee \rangle < n \text{ for } x \in A \\ q \cdot sA + (q-1)A, & \text{otherwise.} \end{cases}$$

These endomorphisms make M into an H -module.

Let W' be the subgroup of W_a generated by those $s \in S_a$ for which $s(A_\rho^+)$ contains ρ in its closure. (This is a parabolic subgroup of W_a conjugate to W under an element in Ω .)

Lemma 7.7. Let $y \in W_a$. We define a function $f : Q+p \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$ as follows:
 $f(\lambda)$ is the coefficient with which A_λ^- appears in

$$\left(\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1} \right) T_y \left(\sum_{w \in W} T_w \right) A_0^- \in M.$$

Then

(i) If $y(A_0^+) = A_\lambda^-$, $\lambda \in P^{++}$, $\lambda \in Q+p$, then $f(\lambda) = q^v$; moreover
 $\lambda' \in Q+p$, $f(\lambda') \neq 0$ implies $\lambda' \leq \lambda$.

(ii) In general, let $X \subset Q+p$ be an α_s -string (α_s a simple root) and let $a \geq 0$ be an integer such that $\langle \lambda, \alpha_s^\vee \rangle \equiv a \pmod{2}$ for all $\lambda \in X$. Then

$$\sum_{\substack{\lambda \in X \\ \langle \lambda, \alpha_s^\vee \rangle \geq a}} f(\lambda) = -q^{-(a-1)} \sum_{\substack{\lambda \in X \\ \langle \lambda, \alpha_s^\vee \rangle \leq -a}} f(\lambda).$$

Proof : (i) Follows from [10, 4.2 (a)] and (ii) is a consequence of [10, 9.2] applied to the element $T_y \left(\sum_{w \in W} T_w \right) A_0^-$.

Corollary 7.8. If y in the previous lemma is such that $y(A_0^+) = A_\rho^-$, then

$$(7.9) \quad \left(\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1} \right) T_y \left(\sum_{w \in W} T_w \right) A_0^- = q^{-v} \left(\sum_{w \in W'} (-q)^{\ell(w)} T_w^{-1} \right) \sum_{\lambda \in Q+p} f(\lambda) h_\lambda A_\lambda^- ,$$

where, for $\lambda \in Q+p$, $f(\lambda)$ is given by (7.5), and h_λ is an element of H such that $h_\lambda A_\lambda^- = A_\lambda^-$.

Proof : In our case, the function f of Lemma 7.7 satisfies the conditions (i), (ii), (iii) of Lemma 7.4, hence is given by (7.5). It follows that for any $\lambda \in Q+p$, A_λ^- appears with the same coefficient in the two sides of (7.9) and the corollary follows.

Since the H -module M is faithful, we can erase A_0^- from the two sides of (7.9) and we obtain an identity in H . We can rewrite this identity as follows. Let $\gamma \in \Omega$ be such that $\gamma W' \gamma^{-1} = W$. We multiply both sides of our identity on the left by T_γ . Note that $T_\gamma T_y = T_{\gamma y} = T_{m_\rho}$. Moreover $T_\gamma h_\lambda = q^{\ell(p\lambda)/2} T_{p_\lambda}$. Thus, we have

$$\begin{aligned}\theta' T_{\mathfrak{m}_\rho} \theta &= \left(\sum_{w \in W} (-q)^{\ell(w)} T_w^{-1} \right) T_{\mathfrak{m}_\rho} \sum_{w \in W} T_w = \\ &= q^{-\gamma} \sum_{w \in W} (-q)^{\ell(w)} T_w^{-1} \sum_{\lambda \in Q+\rho} f(\lambda) q^{\ell(p_\lambda)/2} \tilde{T}_{p_\lambda} \theta.\end{aligned}$$

We can now compute for $\lambda \in P^{++}$:

$$\begin{aligned}J_\rho(q^{-\ell(p_\lambda)/2} K_\lambda) &= q^{-\ell(m_\lambda)/2} \theta' T_{\mathfrak{m}_\rho} \theta \cdot \frac{1}{p \cdot p_\lambda} \theta \cdot q^{-\ell(p_\lambda)/2} q^{-v+\nu\lambda} T_{p_\lambda} \theta \\ &= \frac{1}{p_\lambda} q^{-\ell(m_\rho)/2} \cdot q^{-2v+\nu\lambda} \cdot \sum_{\mu \in Q+\rho} f(\mu) q^{<\mu, \rho>} \theta' \tilde{T}_{p_\lambda} \cdot \tilde{T}_{p_\mu} \theta \\ &= \frac{1}{p_\lambda} q^{-<\rho, \rho>+v/2} \cdot q^{-2v} (-1)^v \sum_I (-q)^{|I|} \cdot q^{<\rho, \rho>} q^{v/2} J_{\lambda+\alpha_I} \\ &= \frac{1}{p_\lambda} \sum_I (-q)^{|I|} q^{-v} J_{\lambda+\alpha_I-\rho}.\end{aligned}$$

Here I runs through the subsets of the set of positive roots. We make a change of variable $I \longrightarrow I' = \text{complement of } I$. Then $\alpha_I + \alpha_{I'} = 2\rho$, $|I| + |I'| = v$ hence

$$\frac{1}{p_\lambda} \sum_I (-q)^{|I|} q^{-v} J_{\lambda+\alpha_I-\rho} = \frac{1}{p_\lambda} \sum_{I'} (-q)^{-|I'|} J_{\lambda+\rho-\alpha_{I'}},$$

and Theorem 6.6 is proved.

8. The following result describes the centre Z of $\tilde{H}$.

Theorem 8.1. (J. Bernstein). Let $\lambda \in P^{++}$ and let $(\lambda)W$ be its W -orbit in P . Then $z_\lambda = \sum_{\lambda' \in (\lambda)W} \tilde{T}_{p_{\lambda'}}$ is in Z . Moreover, Z is the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -module with basis z_λ ($\lambda \in P^{++}$).

Proof : Let $s \in S$. Then $T_s z_\lambda = z_\lambda T_s$ by 7.1. It follows that $T_w z_\lambda = z_\lambda T_w$ for all $w \in W$. It is obvious that, for any $\mu \in P^{++}$, T_{p_μ} commutes with z_λ . But the elements T_w ($w \in W$) and T_{p_μ} ($\mu \in P^{++}$) generate H as an algebra. Hence $z_\lambda \in Z$.

Let z_λ^1 be the specializations of z_λ under the homomorphism $H \rightarrow \mathbb{Z}[\tilde{W}_a]$ given by $q^{1/2} \rightarrow 1$. Then clearly z_λ^1 form a set of $\mathbb{Z}$ -generators for the centre of $\mathbb{Z}[\tilde{W}_a]$: the elements of P are the only elements of $\tilde{W}_a$ whose conjugacy class is finite. Using a version of Nakayama's lemma it follows that any element z of Z

is a linear combination of the elements z_λ with coefficients being allowed to be in the localization of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ at the ideal generated by $q^{1/2}-1$. Since $z \in H$, these coefficients must automatically be in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. The fact that the elements z_λ are linearly independent is obvious. The Theorem is proved.

Let us now define, for $\lambda \in P^{++}$, an element

$$(8.2) \quad S_\lambda = \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_\mu(L_\lambda) z_\mu \in Z.$$

It is clear that for $\lambda, \lambda' \in P^{++}$, we have

$$(8.3) \quad S_\lambda S_{\lambda'} = \sum_{\lambda'' \in P^{++}} m(\lambda, \lambda'; \lambda'') S_{\lambda''}$$

where the ≥ 0 integers $m(\lambda, \lambda'; \lambda'')$ are the multiplicities in the tensor product of $\mathfrak{g}$ -modules :

$$(8.4) \quad L_\lambda \otimes L_{\lambda'} = \sum_{\lambda'' \in P^{++}} m(\lambda, \lambda'; \lambda'') L_{\lambda''}.$$

By Weyl's character formula (6.3) we have

$$\left(\sum_{w \in W} (-1)^{\ell(w)} \tilde{T}_{(\rho)w} \right) S_\lambda = \sum_{w \in W} (-1)^{\ell(w)} \tilde{T}_{(\lambda+\rho)w}.$$

It follows that

$$\begin{aligned} J_\rho S_\lambda &= |W|^{-1} \sum_{w \in W} (-1)^{\ell(w)} J_{(\rho)w} S_\lambda \\ &= |W|^{-1} \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \theta \cdot \tilde{T}_{(\rho)w}^\theta S_\lambda \quad \text{by lemma (7.3)} \\ &= |W|^{-1} \theta \cdot \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \tilde{T}_{(\rho)w}^\theta S_\lambda \\ &= |W|^{-1} \theta \cdot \sum_{w \in W} q^{-\nu/2} (-1)^{\ell(w)} \tilde{T}_{(\lambda+\rho)w}^\theta \\ &= |W|^{-1} \sum_{w \in W} (-1)^{\ell(w)} J_{(\lambda+\rho)w} \\ &= J_{\lambda+\rho}. \end{aligned}$$

The identity

$$(8.5) \quad J_\rho \cdot S_\lambda = J_{\lambda+\rho} \quad (\lambda \in P^{++})$$

shows that the map $Z \longrightarrow J$ given by $x \longrightarrow J_\rho z$ is an isomorphism of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -modules. From this we shall deduce

Proposition 8.6 : The map $Z \longrightarrow K$ given by $z \longrightarrow (\frac{1}{p} \sum_{w \in W} T_w)z = p^{-1}\theta z$ is an isomorphism of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ -algebras preserving the unit element. Under this isomorphism $S_\lambda \in Z$ correspond to $C'_\lambda \in K$, i.e. $C'_\lambda = p^{-1}\theta S_\lambda$.

Indeed, we have a commutative diagram

$$\begin{array}{ccc} & & F \\ & \nearrow^{J_\rho} & \uparrow \\ Z & & \\ & \searrow_{p^{-1}\theta} & \\ & & K \end{array} \quad J_\rho$$

(since $p^{-1}J_\rho\theta = J_\rho$) and the maps $Z \longrightarrow J$, $K \longrightarrow J$ given by multiplication by J_ρ are known to be isomorphisms (see 6.8). Our map $Z \longrightarrow K$ preserves multiplication : $p^{-1}\theta z \cdot p^{-1}\theta z' = p^{-2}\theta^2 zz' = p^{-1}\theta zz'$. Finally $S_\lambda \in Z$ corresponds to $C'_\lambda \in K$, since both correspond to $J_{\lambda+\rho} \in J$ (see (6.9), (8.5)). The isomorphism $Z \longrightarrow K$ is a version of the Satake isomorphism. It shows in particular that K is a commutative algebra.

Corollary 8.7. If $\lambda, \lambda' \in P^{++}$, we have

$$C'_\lambda \cdot C'_{\lambda'} = \sum_{\lambda'' \in P^{++}} m(\lambda, \lambda'; \lambda'') C'_{\lambda''}$$

where $m(\lambda, \lambda'; \lambda'')$ are defined by (8.4).

(The remarkable fact in (8.7) is that the coefficients with which $C'_{\lambda''}$ appears in the decomposition of $C'_\lambda \cdot C'_{\lambda'}$ are independent of q .)

Corollary 8.8. For any $\lambda \in P^{++}$, we have $\bar{z}_\lambda = z_\lambda$.

Indeed, the isomorphism given in 8.6 is compatible with $h \longrightarrow \bar{h}$ (since $\overline{p^{-1}\theta} = p^{-1}\theta$). Since $\overline{C'_\lambda} = C'_\lambda$, it follows that $\bar{S}_\lambda = S_\lambda$. But z_λ is a $\mathbb{Z}$ -linear combination of element $S_{\lambda'}$, ($\lambda' \leq \lambda$) hence $\bar{z}_\lambda = z_\lambda$.

Corollary 8.9. If $\lambda \in P^{++}$, we have

$$(8.10) \quad \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} \frac{P}{P_{\mu}} q^{<\mu, 2\rho^{\vee}> - \nu + \nu_{\mu}} d_{\mu}(L_{\lambda}; q) = \frac{\prod_{\alpha > 0} (q^{<\lambda + \rho, \alpha^{\vee}> - 1})}{\prod_{\alpha > 0} (q^{<\rho, \alpha^{\vee}> - 1})}$$

(product over all positive roots α)

Proof : The left hand side of (8.10) is $\chi(q^{\ell(p_{\lambda})/2} C'_{\lambda})$ (see 6.10) where

$\chi : \tilde{H} \longrightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$ is the algebra homomorphism defined by $\chi(T_w) = q^{\ell(w)}$, $\forall w \in \tilde{W}_a$. Note that $\chi(\tilde{T}_{p_{\mu}}) = q^{<\mu, \rho^{\vee}>}$ for any $\mu \in P^{++}$, (see (6.11)). We have

$$\begin{aligned} \chi(q^{\ell(p_{\lambda})/2} C'_{\lambda}) &= \chi(q^{\ell(p_{\lambda})/2} p^{-1} \theta S_{\lambda}) \\ &= q^{\ell(p_{\lambda})/2} \chi(S_{\lambda}) \\ &= q^{<\lambda, \rho^{\vee}>} \sum_{\substack{\mu \in P^{++} \\ \mu \leq \lambda}} d_{\mu}(L_{\lambda}) \sum_{\mu' \in (\mu)W} q^{<\mu', \rho^{\vee}>} \end{aligned}$$

and this is known to be equal to the right hand side of (8.10). (See the proof of Weyl's character formula in [6]).

9. Let $\mu \leq \lambda$ be two elements of P . According to [10] if $\tau \in P$ is such that $<\tau, \alpha_s^{\vee}> \gg 0$ for all $s \in S$ (so that, in particular, $\mu + \tau \in P^{++}$, $\lambda + \tau \in P^{++}$), the polynomial $P_{n_{\mu+\tau}, n_{\lambda+\tau}}$ is independent of the choice of τ . In particular, it only depends on the difference $\lambda - \mu$. Using now (6.13), we see that there exists a well defined function

$$\hat{P} : \{\kappa \in Q \mid \kappa \geq 0\} \longrightarrow \mathbb{Z}[q^{-1}]$$

such that for any $\mu \leq \lambda$ in P , with $\lambda - \mu = \kappa$, we have

$$(9.1) \quad q^{-<\kappa, \rho>} d_{\mu+\tau}(L_{\lambda+\tau}; q) = \hat{P}(\kappa)$$

for any $\tau \in P$ such that $<\tau, \alpha_s^{\vee}> \gg 0$, for all $s \in S$.

Proposition 9.2.

$$(9.3) \quad \hat{P}(\kappa) = \sum_{\substack{n_1, \dots, n_{\vee} \geq 0 \\ n_1 \alpha_1 + \dots + n_{\vee} \alpha_{\vee} = \kappa}} q^{-(n_1 + \dots + n_{\vee})}.$$

Here $\alpha_1, \dots, \alpha_\nu$ is the list of all positive roots and $n_1, \dots, n_\nu$ are required to be integers. In particular for $q = 1$, $P(\kappa)$ reduces to the Kostant partition function.

Proof : The formulas (6.7), (6.9), (6.10) show that $\hat{P}(\kappa)$ satisfies the recurrence relation

$$\sum_I (-q)^{-|I|} \hat{P}(\kappa - \alpha_I) = \begin{cases} 1 & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa > 0 \end{cases}$$

(sum over all subsets I of the set of positive roots), with the convention that $\hat{P}(\kappa) = 0$ if $\kappa \not\leq 0$. From this, the required formula for $\hat{P}(\kappa)$ follows immediately.

It may be conjectured that, for any $\mu \leq \lambda$ in P^{++} , we have

$$(9.4) \quad q^{-\langle \lambda - \mu, \rho \rangle} d_\mu^\vee(L_\lambda; q) = \sum_{w \in W} (-1)^{\ell(w)} \hat{P}((\lambda + \rho)w - (\mu + \rho))$$

For $q = 1$ this reduces to a well known formula of Kostant.

(Note added May 1982 : Conjecture (9.4) has been recently proved by S. Kato, to appear in *Inventiones Math.*)

For type A, formula (9.4) follows from a statement in [13, p. 131]; indeed, in that case, the left hand side of (9.4) is a Green-Foulkes polynomial (cf. [11]).

The right hand side of (9.4), in the special case $\mu = 0$, appears also in the work of D. Peterson, in connection with the $\underline{g}$ -module structure of the (graded) coordinate ring of the nilpotent variety of $\underline{g}$.

10. If λ is the highest root, we have $d_\mu(L_\lambda; q) = 1$ for any $\mu \in P^{++}$, $0 < \mu \leq \lambda$. Indeed, the multiplicity $d_\mu(L_\lambda)$ is 1 in this case (it is a dimension of a root space in the adjoint representation of $\mathfrak{g}$). Since $d_\mu(L_\lambda; q)$ has ≥ 0 coefficients and constant term 1, it must be identically 1. If we write the formula (8.10) for λ , the only unknown term is, therefore, $d_0(L_\lambda; q)$. We can compute it from (8.10) and we find $d_0(L_\lambda; q) = \sum q^{e_i - 1}$ where e_i ($i = 1, \dots, \text{rk}(\underline{g})$) are the exponents of $\underline{g}$.

11. We shall now describe the (generalized) Schubert varieties $\overline{O}_\lambda$ ($\lambda \in P^{++}$) with the following properties :

- a) $\overline{O}_\lambda$ is an irreducible, projective complex variety of dimension $\langle \lambda, 2\rho^\vee \rangle$.
- b) If $\mu, \lambda \in P^{++}$, are such that $\mu \leq \lambda$ then $\overline{O}_\mu \subset \overline{O}_\lambda$.
- c) Let $x \in \overline{O}_\lambda$ be such that $x \in \overline{O}_\mu$ ($\mu \leq \lambda$) but $x \notin \overline{O}_{\mu'}$, for any $\mu' < \mu$. Then the stalks $H_x^i(\overline{O}_\lambda)$ are zero if i is odd and $\sum_i \dim H_x^{2i}(\overline{O}_\lambda) q^i = d_\mu(L_\lambda; q) = P_{n_\mu, n_\lambda}$.

Let $\underline{g}'$ be a simple complex Lie algebra which is dual to $\underline{g}$ in the following sense. There is a Cartan subalgebra $\underline{h}' \subset \underline{g}'$ with a given isomorphism onto $\underline{h}^*$ which carries the set of coroots of $\underline{g}'$ with respect to $\underline{h}'$ onto the set of roots of $\underline{g}'$ with respect to $\underline{h}$. Let $\hat{\underline{g}}' = \underline{g}' \otimes \mathbb{C}((t))$. For each coroot $\check{\alpha} \in \underline{h}$ of $\underline{g}$ we denote by X_α a non-zero vector in the corresponding root space of $\underline{g}'$. For each $\lambda \in P^{++}$, we denote by L_λ the $\mathbb{C}[[t]]$ -submodule of $\hat{\underline{g}}'$ generated by the vectors $t^{\langle \lambda, \check{\alpha} \rangle} X_\alpha$ and by $\underline{h} \otimes \mathbb{C}[[t]]$. This is a lattice in $\hat{\underline{g}}'$ (i.e. a $\mathbb{C}[[t]]$ -submodule of maximal rank.) It is moreover an order in $\hat{\underline{g}}'$ (i.e. a lattice closed under the Lie bracket). Let $(,)$ be the Killing form on $\underline{g}'$; we extend it to a symmetric bilinear form on $\hat{\underline{g}}'$ with values in $\mathbb{C}((t))$. Then $L_\lambda = L_\lambda^\#$ where for any lattice L we denote by $L^\#$ the dual lattice $\{x \in \hat{\underline{g}}' \mid (x, y) \in \mathbb{C}[[t]] \text{ for all } y \in L\}$. It is easy to check that if L is any order in $\hat{\underline{g}}'$, then $L \subset L^\#$. It follows that any self dual order is a maximal order, hence, by a theorem of Bruhat-Tits, it is a "maximal parahoric" order. It moreover, must correspond to a special vertex of the extended diagram of $\underline{g}'$. Indeed, if L is a maximal parahoric order corresponding to a non-special vertex v , then $\dim(L^\# / L)$ is equal to the number of roots of $\underline{g}'$ minus the number of roots in a proper semisimple subalgebra of $\underline{g}'$ (whose Coxeter diagram is obtained by removing v from the extended diagram of $\underline{g}'$); hence L is not self-dual. It follows that the group G' of automorphisms of the Lie algebra $\hat{\underline{g}}'$ inducing identity on the Weyl group, acts transitively on the set X of all self dual orders in $\hat{\underline{g}}'$. Let G'_0 be the stabilizer of L_0 in G' . It is known that the sets O_λ ($O_\lambda = G'_0$ -orbit of L_λ in X) ($\lambda \in P^{++}$) are disjoint and cover the whole of X . For any integer $n \geq 0$, we consider the subset

$X_n \subset X$ defined by $X_n = \{L \in X \mid t^n L_0 \subset L \subset t^{-n} L_0\}$. Then $X_0 \subset X_1 \subset X_2 \subset \dots$ and their union is X : indeed for any lattice L we can find $n \geq 0$ such that $t^n L_0 \subset L$ and we then have by duality $L \subset t^{-n} L_0$.

We will show that X_n is in a natural way a projective algebraic variety. To give a self-dual lattice L , $t^n L_0 \subset L \subset t^{-n} L_0$, is the same as to give a subspace $\bar{L}$ of $t^{-n} L_0 / t^n L_0$ which is t -stable and is maximal isotropic for the symmetric $\mathbb{C}$ -bilinear form on $t^{-n} L_0 / t^n L_0$ defined by $\text{Res}(x, y)$. Moreover, L gives rise to a subspace $\tilde{L} \subset t^{-n} L_0 / t^{2n} L_0$ of codimension $= \dim L_0 / t^n L_0$. Now $t^{-n} L_0 / t^{2n} L_0$ carries a canonical alternating 3-form with values in $\mathbb{C}$, defined by $\text{Res}([x, y], z)$. The condition that L is an order (if we assume that L is already known to be a self-dual lattice) is that this 3-form is identically zero on $\tilde{L}$.

Thus, we have a 1-1 correspondence $L \leftrightarrow \bar{L}$ between X_n and the set of maximal isotropic subspaces of $t^{-n} L_0 / t^n L_0$, stable under the nilpotent endomorphism t , and whose inverse image in $t^{-n} L_0 / t^{2n} L_0$ is such that the canonical alternating 3-form vanishes identically on it.

This is a subset of a Grassmannian, defined by algebraic equations, hence is a projective algebraic variety. Thus X can be regarded as an increasing union of projective varieties. If $\lambda \in P^{++}$ satisfies $\langle \lambda, \alpha^\vee \rangle \leq n$ for all roots then $O_\lambda \subset X_n$. It is then a locally closed subset of X_n , since it can be regarded as an orbit of the algebraic group $G'_0 / \{g' \in G'_0 \mid g' \equiv 1 \text{ on } L_0 / t^n L_0\}$ acting on X_n .

We then define $\bar{O}_\lambda$ to be the Zariski closure of O_λ in X_n . One could define similarly the varieties $\bar{O}_\lambda$ over a finite field F_{p^S} (instead of over $\mathbb{C}$). The number of rational points (over F_{p^S}) of $\bar{O}_\lambda$ (in the sense of intersection cohomology) i.e., with each rational point x counted with a multiplicity equal to the trace of the Frobenius map on $\Sigma(-1)^{i_H^i}(\bar{O}_\lambda)$ is the left hand side of (8.10), hence it is given by the right hand side of (8.10), with q replaced by p^S .

In particular, the Euler characteristic of $\bar{O}_\lambda$ (in the sense of intersection cohomology) is equal to $\dim(L_\lambda)$.

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