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Sieve identities and gaps between primes

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SIEVE IDENTITIES AND GAPS BETWEEN PRIMES

by

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The purpose of this lecture is to describe two identities for estimating sums over primes. They are related to the well known Vaughan identity, but are distinctly stronger, and should be more illuminating.

In the first identity we write

$$M(s) = \sum_{n \leq X} \mu(n)\, n^{-s} \ .$$

We then have :

LEMMA 1. - Let $k \in \mathbb{N}$. Then

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \zeta(s)^{j-1} \zeta'(s)\, M(s)^{j} + \frac{\zeta'(s)}{\zeta(s)} (1 - \zeta(s)\, M(s))^{k} \ . \qquad (1)$$

If one is interested in the sum

$$S = \sum_{x/2 < n \leq x} \Lambda(n)\; f(n)$$

one may apply Lemma 1 with a fixed positive integer k , and $X^k \geq x$. Then, on picking out the coefficients of n^{-s} on either side, one sees that the final term on the right of (1) makes no contribution, since

$$\zeta(s)\, M(s) = 1 + \sum_{n > X} c_n\, n^{-s} \ .$$

Thus S may be written as a linear combination of $O_k(1)$ sums

$$\Sigma = \sum_{n_1,\ldots n_{2j}} (\log n_1)\, \mu(n_{j+1}) \ldots \mu(n_{2j})\, f(n_1 n_2 \ldots n_{2j}) ,$$

subject to the conditions $x/2 < n_1 n_2 \ldots n_{2j} \leq x$ and $n_i \leq X$ $(j < i \leq 2j)$. We may subdivide the ranges of summation into intervals $N_i < n_i \leq 2N_i$, where $2^{-k-1} x \leq \Pi N_i \leq x$ and $N_i \leq X$ for $j < i \leq 2j$; there are then $O_k((\log x)^{2k})$ sums Σ. Thus the "unknown" coefficients $\mu(.)$ are summed over ranges which can be made as short as desired, by taking k to be large, and X to be a small power of x .

In contrast Vaughan's identity yields at most a triple sum, and there is always an "unknown" coefficient corresponding to a range $\gg x^{\frac{1}{2}}$. This may be rectified by iterating Vaughan's identity, and Lemma 1 may be thought of as a simplified k-fold iteration.

We illustrate the use of Lemma 1 by applying it to the sum

$$\sum_{x-y<n\leq x} \Lambda(n) = \psi(x) - \psi(x-y) .$$

Here we take $y = x^{\theta}$ and $\frac{1}{2} + \delta \leq \theta \leq 1 - \delta$, δ being a fixed positive constant. We choose $k = 6$ and $X = x^{1/6}$. If some N_i (necessarily with $i \leq j$) satisfies $N_i \geq x^{\frac{1}{2}}$, then Σ may be evaluated elementarily. Otherwise we apply Perron's formula, and obtain

$$\Sigma = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT_o}^{\frac{1}{2}+iT_o} \frac{x^s - (x-y)^s}{s} F_1(s) \ldots F_{2j}(s)\, ds + O(y(\log x)^{-A}) , \tag{2}$$

in which $T_o = x^{1-\theta+\varepsilon}$ (with a fixed $\varepsilon > 0$), A is any positive constant and

$$F_1(s) = \sum_{N_1<n\leq 2N_1} (\log n)\, n^{-s} , \qquad F_\ell(s) = \sum_{N_\ell<n\leq 2N_\ell} n^{-s} , \; (2\leq \ell \leq j), \tag{3}$$

$$F_\ell(s) = \sum_{N_\ell<n\leq 2N_\ell} \mu(n) n^{-s}, \; (j < \ell \leq 2j) . \tag{4}$$

For ease of reference we shall call sums of type (3) "ζ-factors" and sums of the type (4) "μ-factors". Moreover, by the "length" of $F_\ell(s)$ we shall mean N_ℓ .

In estimating the integral on the right of (2) one uses the following lemma.

LEMMA 2. - Suppose each ζ-factor F_ℓ has length $\le x^{\frac{1}{2}}$, and each μ-factor F_ℓ has length $\le x^{2\theta-1-\delta}$. Then there exists $\varepsilon = \varepsilon(\delta)$ such that, if $j \le 4$,

$$\int_T^{2T} |F_1 \dots F_{2j}(\tfrac{1}{2}+it)|\, dt \ll_{A,\delta} x^{\frac{1}{2}} (\log x)^{-A}, \quad (\text{any } A>0)$$

uniformly for

$$\exp((\log x)^{\frac{1}{3}}) \le T \le x^{1-\theta+\varepsilon} .$$

If $\theta \ge \frac{11}{20}+\delta$ one may take $j=5$, and if $\theta \ge \frac{7}{12}+\delta$ one may take $j=6$.

The proof of Lemma 2 is long. It uses the Halász lemma, and also requires Vinogradov's zero-free region, since one needs a good bound for $F_\ell(\frac{1}{2}+it)$ when $j<\ell\le 2j$.

There is no room here to show how Lemma 2 is used to obtain an asymptotic formula for Σ. However we may at least observe that if $\theta = \frac{7}{12}+\delta$, then we can take $k=6$ and

$$X = x^{2\theta-1-\delta} = x^{(1/6)+\delta} .$$

Then Lemma 2 may be applied to each term Σ, and one obtains an asymptotic formula for $\psi(x)-\psi(x-y)$. We thus recover the well known result of Huxley.

We now turn to our second identity. Here we write

$$\pi(s) = \prod_{p<z} (1-p^{-s}) .$$

LEMMA 3. - Let $k \in \mathbb{N}$. Then

$$\log(\zeta(s)\,\pi(s)) = \sum_{e=1}^{\infty} \sum_{p\ge z} \frac{1}{e p^{es}}$$
$$= \sum_{j=1}^{\infty} (-1)^{j-1} j^{-1} (\zeta(s)\,\pi(s)-1)^j .$$

A result of this shape has been used by Linnik, but the factor $\pi(s)$ plays only a subsidiary rôle in his work.

We may apply Lemma 3 to the estimation of

$$S = \sum_{x/2<p\le x} f(p) .$$

We take k to be a fixed positive integer with $z^k > x$. Then, on picking out the relevent coefficients of n^{-s} on each side, one sees that the terms with $j \ge k$ make no contribution to S, since

$$\zeta(s)\,\pi(s) = 1 + \sum_{n\geq z} a_n n^{-s} \ . \qquad (5)$$

If we are interested again in $\psi(x) - \psi(x-y)$ we may proceed as before. There is however a technical difficulty in that $\pi(s)$ is not a μ-factor. This may be circumvented in fact, and a result similar in principle to Lemma 2 will apply. Thus the contribution from $(\zeta(s)\,\pi(s) - 1)^j$ will have j ζ-factors, and every μ-factor will have length $\leq z$. We shall choose $z = x^{2\theta-1-\delta}$.

For $\theta = \frac{7}{12} + \delta$ the condition $z^k > x$ will be satisfied if $k = 6$. Since the range for j is now $j < k$, Lemma 2 is more than sufficient. Again we may recover Huxley's theorem.

In case $\frac{4}{7} + \delta \leq \theta \leq \frac{7}{12} + \delta$ we need $k = 7$. Here Lemma 2 covers $j \leq 5$, but not $j = 6$. We have therefore to allow seperately for the term Σ_6 (say) given by

$$\Sigma_6 = \sum_{x-y<rstuvw\leq x} a_r a_s a_t a_u a_v a_w \ .$$

(Here a_n is defined by (5).) For our range of θ this becomes

$$\Sigma_6 = \sum_{\substack{x-y<p_1\,\ldots\,p_6\leq x \\ p_i\geq z}} 1 \ .$$

We may thus obtain an asymptotic formula of the shape

$$\pi(x) - \pi(x-y) + \frac{1}{6}\,\Sigma_6 \sim C(\theta)\,\frac{y}{\operatorname{Log} x} \ ,$$

for $y = x^{\theta}$, $z = x^{2\theta-1-\delta}$, $\frac{4}{7} + \delta \leq \theta \leq \frac{7}{12} + \delta$. In contrast to our application of Lemma 1 we can now see precisely what has gone wrong at $\theta = \frac{7}{12}$; we can no longer evaluate Σ_6 accurately.

We may think of Σ_6 as being small. For example, if $\theta = \frac{7}{12} - \delta$ then each of the primes p_i in (6) is restricted to the range $x^{1/6-3\delta} \leq p_i \leq x^{1/6+15\delta}$. By using a crude Selberg upper bound sieve one finds that

$$\frac{1}{6}\,\Sigma_6 \leq \frac{1}{5}\,\frac{y}{\operatorname{Log} x}$$

for $\theta \geq \frac{7}{12} - \frac{1}{6\,000}$. Consequently

$$\pi(x) - \pi(x-y) \gg \frac{y}{\operatorname{Log} x}$$

in this range. Moreover by working uniformly in θ one may show that $\Sigma_6 = o(y(\log x)^{-1})$ as θ tends to $\frac{7}{12}$ from below. By making this precise one can prove :

THEOREM 1. - Let $\varepsilon(x) \to 0$ as $x \to \infty$. Then

$$\pi(x) - \pi(x-y) \sim \frac{y}{\operatorname{Log} x}$$

as $x \to \infty$, uniformly for

$$x^{(7/12)-\varepsilon(x)} \leq y \leq x \quad .$$

THEOREM 2. - We have

$$\pi(x) - \pi(x-y) = \frac{y}{\operatorname{Log} x} + O(y(\log x)^{-(45/44)})$$

uniformly for $x^{7/12} \leq y \leq x$.

Finally we describe an interesting sieve property of Lemma 3. Let $a_z(n) = e^{-1}$ if $n = p^e$ and $p \geq z$, and put $a_z(n) = 0$ otherwise. Write $a_z(n, j)$ for the coefficient of n^{-s} in $(\zeta(s)\,\pi(s) - 1)^j$. Then Lemma 3 yields

$$a_z(n) = \sum_{j=1}^{k-1} (-1)^{j-1} j^{-1} a_z(n, j)$$

for $z^k > n$. However we also have :

LEMMA 4. -

$$a_z(n) \begin{Bmatrix} \leq & (J \text{ odd}) \\ \geq & (J \text{ even}) \end{Bmatrix} \sum_{j=1}^{J} (-1)^{j-1} j^{-1} a_z(n, j) \quad .$$

For example the case $J = 2$ says that if n $(\neq 1)$ has no prime factor below z, then $a_z(n) \geq 2 - \frac{1}{2} d(n)$. Unfortuately the only proof of Lemma 4 so far obtained is not at all illuminating.

By taking $J = 4$ we may derive a lower bound for $\pi(x) - \pi(x-y)$. The corresponding Dirichlet polynomials will have at most 4 ζ-factors, and so Lemma 2 applies throughout the range $\frac{1}{2} < \theta < 1$. Thus one obtains

$$\pi(x) - \pi(x-y) \geq c(\theta) \frac{y}{\operatorname{Log} x} \qquad (\tfrac{1}{2} < \theta < 1) \ .$$

The value of θ_o for which $c(\theta_o) = 0$ has not been calculated, although it has been shown that $\frac{5}{9} < \theta_o < \frac{4}{7}$.

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