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Astérisque, tome 94 (1982), p. 175-185
<http://www.numdam.org/item?id=AST_1982__94__175_0>

A PROGRESS REPORT ON DISCREPANCY

by

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1. - Definitions

Let X be a set of normalized measure $\mu(X) = 1$. Let I be a subset of $\mathbb{N}$ and J an arbitrary set. Let $\{X_j\}_{j\in J}$ be a set of μ-measurable subsets of X. Furthermore, let $\omega = \{\xi_i\}_{i\in I}$ be a set of elements of X, repetitions allowed. The counting function $A(X_j, N)$, for $j \in J$, $N \in \mathbb{N}$, is defined as the number of integers n with $1 \le n \le N$ and $x_n \in X_j$. Denote the number of elements of I not exceeding N by I_N. The difference $D(X_j, N) := A(X_j, N) - I_N \mu(X_j)$ is called the discrepancy of ω at (X_j, N). A sequence $\omega = \{\xi_i\}_{i\in I}$ is said to be uniformly distributed with respect to $\{X_j\}_{j\in J}$ if

$$\lim_{N\to\infty} I_N^{-1} D(X_j, N) = 0$$

for all $j \in J$. We shall distinguish two cases :

a) Static case : the study of sets ω such that $\{D(X_j, N)\}_{j\in J}$ for a fixed N is in some sense minimal. Here only the elements of ω which do not exceed N are of interest.

b) Dynamic case : the study of sets ω such that $\{D(X_j, N)\}_{\substack{j\in J\\ n\in\mathbb{N}}}$ is minimal in some sense.

We shall neglect the more general situation that the measure μ is not fixed, but that for every $i \in \mathbb{N}$ there is a measure μ_i such that X_j is μ_i-measurable for all i and j and $\mu_i(X) = 1$ for all i. Such problems occur, for example, in game theory.

If a result is stated without a reference, then it can be found in the book of Kuipers and Niederreiter [12].

2. - Uniform distribution modulo 1

Let $X = U$ be the real unit interval $[0,1)$ and let $J = (0,1]$. Let $\{X_b\}_{b\in J}$ be the set of all subintervals $X_b = [0,b)$ of X. Put $\mu(X_b) = b$ and $I = \mathbb{N}$. In this situation the static case has an ideal solution $\omega = \{\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}\}$ and is not interesting. The dynamic case coincides with the classical theory of uniform distribution modulo 1 initiated by Weyl and others in the beginning of our century. Put

$$D_N^* = \sup_{b\in J} |D(X_b, N)| .$$

Van der Corput posed the problem whether there exists a sequence $\omega = \{x_i\}_{i=1}^{\infty}$ for which $\{D_N^*\}_{N\in\mathbb{N}}$ is bounded. Mrs van Aardenne-Ehrenfest proved in 1945 that this is not the case and she gave a first quantitative result in 1949. This result was improved by Roth in 1954 and by Schmidt in 1972. Schmidt proved that for every sequence ω

$$\limsup_{N\to\infty} \frac{D_N^*}{\log N} > 0 . \tag{1}$$

This result is of the best possible order, since the sequence $\{n\sqrt{2}\}_{n\in\mathbb{N}}$ and also the so-called van der Corput-sequence satisfy

$$\limsup_{N\to\infty} \frac{D_N^*}{\log N} < \infty .$$

A number of people have tried to find bounds for the absolute constant

$$c^* = \inf_{\omega} \limsup_{N\to\infty} \frac{D_N^*}{\log N} .$$

The best bounds known at present were given by Béjian [6] and Faure [9], namely

$$.06 < c^* < .224 .$$

The lower bound is deduced by a refinement of Schmidt's argument. The upper bound is derived by constructing a low-discrepancy sequence which is a variant of the van der Corput-sequence. It follows from computations of Dupain, Sós and Ramshaw that the discrepancies of sequences $\{i\alpha\}_{i=1}^{\infty}$ for $\alpha\in\mathbb{R}$ are not that small (cf. [9, p. 144]).

Schmidt [21] showed that $\sup_N |D(X_b, N)|$ is unbounded for all but countably many b in J. The structure of such bounded discrepancy sets was studied by Tijdeman and Voorhoeve [31]. Schmidt [23, p. 40] derived also a lower bound for $|D(X_b, N)|$ valid for almost all b. Halász [11] and Tijdeman and Wagner [32] improved upon this bound. They showed that

$$\sup_N \frac{|D(X_b, N)|}{\log N} \geq \frac{1}{400}$$

for almost all b.

Numerous variants are possible. Many authors have studied the discrepancy function corresponding to all subintervals $X_{a,b} = [a, b)$ of X with $0 \leq a < b \leq 1$. Put

$$D_N = \sup_{0 \leq a < b \leq 1} |D(X_{a,b}, N)| .$$

Then $D_N^* \leq D_N \leq 2D_N^*$. Hence bounds for D_N^* imply bounds for D_N. The values of (a, b) for which $\{D(X_{a,b}, N)\}_{N \in \mathbb{N}}$ remains bounded were studied by Schmidt [22] and Shapiro [25]. One can also vary the definition of uniform distribution. For example, Rauzy [16] has shown that there exists a so-called completely uniformly distributed sequence such that

$$\limsup_{N \to \infty} \frac{D^*(N)}{\log N} \leq \frac{1}{\log 2} .$$

Another possible variation is the way of measuring discrepancies. In the next section we shall also consider the L_p-norm.

3. - Uniform distribution in a hypercube

Let $s \in \mathbb{N}$. Let X be the unit-hypercube U^s with $U = [0,1)$ and let $J = \overline{X} = \{b = (b_1, \ldots, b_s) : 0 \leq b_j \leq 1 \text{ for } j = 1, \ldots, s\}$. We consider the set $\{X_b\}_{b \in J}$ of all subintervals $X_b = \{(x_1, \ldots, x_s) : 0 \leq x_j < b_j \text{ for } j = 1, \ldots, s\}$ of X and put $\mu(X_b) = b_1 b_2 \ldots b_s$. It turns out that the s-dimensional dynamic case is closely related to the $(s+1)$-dimensional static case, cf. [17, pp. 77-78]. We may therefore restrict ourselves to the dynamic case. Put $I = \mathbb{N}$ and

$$D_N^{(s)} = \sup_{b \in J} |D(X_b, N)| .$$

It follows from a result of Roth [17] published in 1954 that for every sequence $\omega = \{\xi_i\}_{i=1}^{\infty}$ in X

$$(2) \qquad \limsup_{N\to\infty} D_N^{(s)}/(\log N)^{s/2} > 0 .$$

On the other hand, Halton constructed sequences ω in X such that

$$(3) \qquad \limsup_{N\to\infty} D_N^{(s)}/(\log N)^{s} < \infty .$$

It is widely believed that $(\log N)^s$ is closer to the optimal value than $(\log N)^{s/2}$, but this is only known for $s = 1$ (as has been discussed in the previous section). Recently Faure [10] improved upon the existing bounds for the constant which can be put at the right-hand side of (3) by constructing so-called Indian sequences. Like the sequences used by Halton these sequences are generalizations of the van der Corput-sequence, but for Indian sequences 2 is the base for all coordinate axes, whereas for the sequences introduced by Halton and Hammersley different prime bases are used for different axes. Indian sequences have properties which remind us of the proof of Roth of (2).

Another way of measuring discrepancies is by using L_p-norms. Put for $p > 0$ and for $\omega = \{\xi_i\}_{i=1}^{\infty}$ in X

$$\|D_N^{(s)}\|_p = \left(\int_X |D(X_b, N)|^p \, db\right)^{1/p} .$$

The above mentioned result of Roth gives even that for every $s \in \mathbb{N}$ and every ω

$$\limsup_{N\to\infty} \|D_N^{(s)}\|_2/(\log N)^{s/2} > 0 .$$

Davenport showed in 1956 that this result is best possible for $s = 1$. Schmidt [24] proved in 1977 that for every $p > 1$ and every ω

$$(4) \qquad \limsup_{N\to\infty} \|D_N^{(s)}\|_p/(\log N)^{s/2} > 0 ,$$

and, moreover,

$$\limsup_{N\to\infty} \|D_N^{(s)}\|_1 \Big/ \frac{\log\log N}{\log\log\log N} > 0 .$$

Halász [11] proved that (4) also holds if $p = s = 1$. On the other hand, it follows from recent work of Roth [19] and his student Chen [7] that for every s and $p > 0$ there exists a sequence ω such that

$$\limsup_{N\to\infty} \|D_N^{(s)}\|_p/(\log N)^{s/2} < \infty .$$

The proof makes uses of generalized Hammersley sequences and is very complicated. Thus (4) is best possible for all $p > 1$, $s \geq 1$.

There is no special reason to consider only blocks X_b for $b \in J$. It does not make much difference whether one studies blocks with edges parallel to the coordinate axes instead. For example, it follows from Schmidt's result (1) that for every set of N points in U^2

$$\sup_E |D(E,N)| / \log N > 0$$

if the supremum is taken over all rectangles with sides parallel to the coordinate axes. This result is best possible. However, if we take the supremum over <u>all</u> rectangles E in U^2, then, for any $\varepsilon > 0$,

$$\sup_E |D(E,N)| / N^{\frac{1}{4}-\varepsilon} > 0 .$$

This was proved by Schmidt in 1969. A recent result of Beck [2] implies that this result is almost best possible : There exists a set of N points in U^2 such that for every rectangle E in U^2

$$|D(E,N)| / N^{\frac{1}{4}} (\log N)^{\frac{1}{2}} < \infty .$$

Both Schmidt [20] and Beck [4] have proved a number of related results.

4. - <u>Uniform distribution in finite sets</u>

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let μ be a measure with $\mu(X) = 1$. Let $\{X_j\}_{j \in J}$ be a set of μ-measurable subsets of X. In the static case we want to estimate

$$D_N = \sup_{j \in J} |D_N(X_j)|$$

and in the dynamic case

$$D = \sup_{\substack{j \in J \\ n \in \mathbb{N}}} |D_N(X_j)| .$$

Since X is a finite set, we may assume without loss of generality that J is finite.

We shall first consider estimates in terms of the number $m = |J|$ of elements of J. A construction of sets $\{X_j\}$ by using Hadamardmatrices leads to a bound

$$D_N \gg \sqrt{m}$$

for every ω in X. By a so-called probabilistic method, on considering distribution of point sets in X put at random, Beck and Fiala [5] showed that for every $\{X_j\}_{j \in J}$ and $N \in \mathbb{N}$ there is an ω with

(5) $$D_N \ll \sqrt{m \log m} \ .$$

Spencer [28] has given a highly original proof that there is an ω such that

$$D \ll \sqrt{m} \log m \ .$$

In the special case that the sets X_j are pairwise disjoint much better bounds can be given. Then there exists an ω with

(6) $$D_N \leq 1 - \frac{1}{m+1} \ ,$$

and sets X_j can be constructed for which this bound is optimal. (A proof of this assertion is given in section 7.) The corresponding dynamic problem is the so-called chairman assignment problem posed by Niederreiter. Meijer [14] and Tijdeman [29, 30] have shown that there is an ω with

(7) $$D \leq 1 - \frac{1}{2m}$$

and that this result is best possible (cf. section 7). The algorithm given in [30] is closely related to the so-called Quota-method of Balinski and Young [1] introduced for solving the apportionment problem in the House of Representatives of the U.S.A.

It is also possible to give bounds in terms of the covering degree d. This is defined as the maximum of the number of sets X_j to which an element of X can belong. The case of pairwise disjoint sets X_j, discussed above, corresponds to $d=1$. Then, by (6), for every N there is an ω such that

$$D_N < 1 \ .$$

Assume $d \geq 2$. It follows from a nice result of Beck and Fiala [5] that for every N there is an ω such that

(8) $$D_N \leq d \ . \qquad \text{(cf. section 7)}$$

The dynamic case has not been studied yet. It is likely that the optimal bound for D is also (approximately) a constant power of d. If $d=1$, then, by (7), $D \leq 1$.

5. - Inhomogeneous approximation

Although it is beyond the scope of this paper, we want to state the basic lemma of result (5), since it is very interesting from a diophantine point of view

LEMMA (Beck and Fiala [5]). - Let $n \geq 5$. Let $v_i \in \mathbb{R}$, $u_{ij} \in \mathbb{R}$, $|u_{ij}| \leq 1$ for $i, j = 1, 2, \ldots, n$. Then there exist integers $x_1, x_2, \ldots, x_n$ with $|x_i - v_i| < 1$ $(i = 1, \ldots, n)$ such that

$$\max_{j=1,\ldots,n} \left| \sum_{i=1}^{n} u_{ij}(x_i - v_i) \right| \leq 4\sqrt{2n \log 2n} .$$

COROLLARY (Spencer [27]). - Let $n \geq 5$. Let $u_{ij} \in \mathbb{R}$ with $|u_{ij}| \leq 1$ for $i, j = 1, \ldots, n$. Then there exist integer $x_1, \ldots, x_n$ with $x_i \in \{-1, 1\}$ for $i = 1, \ldots, n$ such that

$$\max_{j=1,\ldots,n} \left| \sum_{i=1}^{n} u_{ij}\, x_i \right| \leq 8\sqrt{2n \log 2n} .$$

Proof. - Take $v_1 = v_2 = \ldots = v_n = \frac{1}{2}$ in the lemma and multiply by 2.

The proof of the lemma provides an algorithm for the construction of the numbers x_i. Spencer proved the corollary with other constants and has given game-theoretic variants of it.

6. - Uniform distribution in the set of integers

The classical concept of uniform distribution of integer sequences refers to the case $X = \mathbb{Z}$, $I = \mathbb{N}$ and $\{X_j\}_{j=0}^{m-1}$ are the residue classes modulo some positive integer m, each with measure $1/m$. Hence, a sequence $\omega = \{\xi_i\}_{i=1}^{\infty}$ of integers is uniformly distributed (mod. m) if each residue class mod. m contains $N/m + o(N)$ terms among $\xi_1, \xi_2, \ldots, \xi_N$. For example, the Fibonacci sequence is uniformly distributed mod. m if and only if m is a perfect power of 5 [13, 15]. A sequence is said to be uniformly distributed in $\mathbb{Z}$ if it is uniformly distributed modulo every positive integer. For example, the sequence $\{\lfloor n\alpha \rfloor\}_{n=1}^{\infty}$ is uniformly distributed in $\mathbb{Z}$ if and only if $\alpha \notin \mathbb{Q}$ or $\alpha = 1/m$ for some non-zero integer m. Here $\lfloor n\alpha \rfloor$ denotes the integer part of $n\alpha$. The study of uniformly distributed integer sequences is related to the study of normal numbers. It is clear that this case is an important, but very special example of uniform distribution in $\mathbb{Z}$. In fact, several results in additive and combinatorial number theory can be stated as results on discrepancy, but with other choices of J, $\{X_j\}_{j \in J}$ and μ. We give a typical example.

Let $X = \{1, 2, \dots, 2m\}$, $\mu(i) = 1/2m$ for $i = 1, \dots, 2m$ and $I = \{1, \dots, m\}$. Let $\{X_j\}_{j \in J}$ be the set of all arithmetic progressions with positive difference in X. The problem is to estimate

$$D_m = \inf_{\omega} \sup_{j} |D(X_j, m)| .$$

that is we wonder how well m integers in X can be selected such that almost half of the elements of each arithmetic progression belongs to this selection. Alternatively, one wants to find an optimal mapping $f : X \to \{-1, 1\}$ such that $|\Sigma f(x)|$ is small if x runs through any arithmetical progression in X. In 1964 Roth [18] proved that $D_m \gg m^{1/4}$ and he conjectured that for every $\varepsilon > 0$,

$$D_m \gg m^{\frac{1}{2} - \varepsilon} .$$

Spencer [26] proved the existence of sets $(\omega_m)_m$ such that

$$D_m \ll \sqrt{m} \, \frac{\log \log m}{\log m} .$$

In 1974, Sárközy [8, § 8] refuted the conjecture of Roth by constructing sets ω_m such that

$$D_m \ll (m \log m)^{\frac{1}{3}} .$$

Beck [3] has now replaced the right-hand side of this inequality by $m^{\frac{1}{4}} (\log m)^{5/2}$. This is almost best possible in view of the result of Roth.

7. - Proofs and remarks

We give a simple proof of (6), since we do not know an appropriate reference.

Let $X = \{x_1, \dots, x_n\}$ be a set with $\mu(X) = 1$ and let $\{X_j\}_{j=1}^m$ be a set of measurable subsets of X which are pairwise disjoint. Without loss of generality we may consider $X = \{X_o, X_1, \dots, X_m\}$, where $X_o = X \setminus (X_1 \cup \dots \cup X_m)$. Let $N \in \mathbb{N}$. Write $N\mu(X_j) = \lfloor N\mu(X_j) \rfloor + \theta_j$ for $j = 0, 1, \dots, m$. Compute $r = N - \sum_{j=0}^{m} \lfloor N\mu(X_j) \rfloor = \sum_{j=0}^{m} \theta_j$. This is an integer. If $r = 0$, then put $A(X_j, N) = N\mu(X_j)$ and hence $D(X_j, N) = 0$ for all j. If $r > 0$, then relabel $X_o, X_1, \dots, X_m$ in such a way that $\theta_o \geq \theta_1 \geq \dots \geq \theta_m$. Put $A(X_j, N) = \lfloor N\mu(X_j) \rfloor + 1$ for $j = 0, 1, \dots, r-1$ and $A(X_j, N) = \lfloor N\mu(X_j) \rfloor$ for $j = r, r+1, \dots, m$. We obtain $D(X_j, N) = 1 - \theta_j$ for $j = 0, 1, \dots, r-1$ and $D(X_j, N) = -\theta_j$ for $j = r, r+1, \dots, m$ and hence

$$\sum_{j=0}^{r-1} |D(X_j, N)| = \sum_{j=r}^{m} |D(X_j, N)| .$$

Furthermore, $1 \leq r \leq m$ and $\max_j |D(X_j, N)| = \max(|D(X_{r-1}, N)|\ |D(X_r, N)|)$. If $|D(X_{r-1}, N| > 1-1/(m+1)$, then $\theta_{r-1} < 1/(m+1)$ and therefore

$$1 - \frac{1}{m+1} < |D(X_{r-1}, N)| \leq \sum_{j=r}^{m} |D(X_j, N)| < (m-r+1)\frac{1}{m+1} \leq 1 - \frac{1}{m+1} \quad ,$$

a contradiction. If $|D(X_r, N)| > 1-1/(m+1)$, then $\theta_{r-1} \geq \theta_r > 1-1/(m+1)$ and $|D(X_j, N)| < 1/(m+1)$ for $j = 0, 1, \ldots, r-1$. Hence

$$1 - \frac{1}{m+1} < |D(X_r, N)| \leq \sum_{j=0}^{r-1} |D(X_j, N)| < r \cdot \frac{1}{m+1} \leq 1 - \frac{1}{m+1} \quad ,$$

again a contradiction. Thus $D_N \leq 1-1/(m+1)$. The value at the right hand side is attained if $N = 1$ and $\mu(X_o) = \mu(X_1) = \ldots = \mu(X_m) = 1/(m+1)$. For exactly one value of j we have $A(X_j, 1) = 1$ and hence $D(X_j, N) = 1-1/(m+1)$.

For the proof of (7) note that in this report we do not require $X_1 \cup X_2 \cup \ldots \cup X_m = X$ as was done in the papers of Meijer [14] and Tijdeman [29, 30]. We therefore define a complementary set X_o as in the previous paragraph and $m+1$ in this report corresponds to k in the quoted papers.

Inequality (8) needs a similar clarification. Beck and Fiala [5] prove the existence of a set ω in X such that

$$|A(X_j) - N\mu(X_j)| \leq d-1 \tag{9}$$

for all j, but it is not guaranteed that ω has exactly N elements. In order to prove (8) one can adjoin the equality

$$\sum_j \beta_j = 0$$

to the system of equalities considered in their proof. Thereby the covering degree is increased by 1. Since the new equality involves all variables, Case (b) in their proof cannot occur. By a suitable rounding off at the end we can secure that $A(X) = N\mu(X) = N$. Now (8) follows from (9) by replacing d by the new covering degree $d+1$.

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