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L. RODINO

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GEVREY HYPOELLIPTICITY FOR A CLASS OF OPERATORS WITH MULTIPLE CHARACTERISTICS

by L. RODINO (Università di Torino)

INTRODUCTION.

In this paper we prove a result of Gevrey regularity for a certain class of quasi-elliptic operators degenerate on a symplectic manifold; representative examples in $\mathbb{R}^2_{x,y}$ are given by

$$(0.1) \qquad P = D_x - r_1 x^h D_y^k \quad ,$$

$$(0.2) \qquad P = (D_x - r_1 x^h D_y^k)(D_x - r_2 x^h D_y^k) + \lambda x^{h-1} D_y^k \quad ,$$

where h , k are fixed positive integers and $\operatorname{Im} r_j \neq 0$ for $j = 1,2$, say $\operatorname{Im} r_1 < 0$ and $\operatorname{Im} r_2 > 0$.

The hypoellipticity of a related class of pseudo differential operators was discussed in Parenti-Rodino [8]; in particular, for P in (0.1) we have hypoellipticity if and only if one at least of the integers h , k is even, whereas P in (0.2) is hypoelliptic if and only if the parameter $\lambda \in \mathbb{C}$ avoids a certain discrete set of eigenvalues. The nature of the arguments in [8] was microlocal, and this allowed several geometric invariant applications (see Part II in [8]). Here, arguing from a local point of view, we shall limit ourselves to the case of a "flat" symplectic characteristic manifold and we shall consider only linear partial differential operators; in this situation we shall be able to prove a result of Gevrey regularity by combining some a priori estimates in [8] and a method of

Grushin, Section 5 of [4]. For P in (0.1), (0.2) it will come out that hypoellipticity implies Gevrey hypoellipticity with respect to suitable classes. Let us note that the above-mentioned argument of Grushin, which we generalize here in a trivial way, can be microlocalized, as we hope to show in future papers, so that it applies actually also to the pseudo differential case, for example to the operators which we obtain in (0.1), (0.2) by fixing $k = 1$ and allowing r_1, r_2, λ to be analytic pseudo differential operators of order zero (for operators of similar type the analytic hypoellipticity has been proved recently by Trèves [12], Tartakoff [11], Métivier [7], by means of other methods and under the additional assumption $h = 1$).

1. STATEMENT OF THE RESULT.

Let us first recall some definitions concerning Gevrey classes and quasi-elliptic operators. Write z for the real variable in $\mathbb{R}^n$ and let $M = (M_1,\dots,M_n)$ be a fixed n-tuple of positive rational numbers. Let Ω be an open subset of $\mathbb{R}^n$. We define $G^M(\Omega)$ to be the class of all $f \in C^\infty(\Omega)$ such that for every $K \subset\subset \Omega$

$$(1.1)\qquad \max_{z\in K} |D_z^\mu f(z)| \le C_K^{|\mu|+1} (\mu_1!)^{M_1} \dots (\mu_n!)^{M_n} ,$$

for all $\mu = (\mu_1,\dots,\mu_n) \in \mathbf{Z}_+^n$ and with a constant C_K depending only on f and K. When $M_j = 1$ for every j, $j = 1,\dots,n$, $G^M(\Omega)$ consists of all analytic functions in Ω. The G^M-singular support of a distribution $f \in \mathscr{D}'(\Omega)$, G^M-sing supp f, is defined in this way: $x_o \notin G^M$-sing supp f if and only if there exists a neighborhood $V \subset \Omega$ of x_o such that $f \in G^M(V)$. We say that a linear partial differential operator P with coefficients in $G^M(\Omega)$ is G^M-hypoelliptic in Ω if

$$(1.2) \qquad G^M\text{-}\underline{\text{sing supp}}\ Pf = G^M\text{-}\underline{\text{sing supp}}\ f\ ,\ \underline{\text{for every}}\quad f \in \mathscr{D}'(\Omega)\ .$$

Typical examples of G^M-hypoelliptic operators are the M-quasi-elliptic operators, i.e. the linear partial differential operators $P = \sum_{\langle\mu,M\rangle \le m} c_\mu(z)D_z^\mu$, $c_\mu \in G^M(\Omega)$, such that in Ω $\sum_{\langle\mu,M\rangle = m} c_\mu(z)\zeta^\mu \neq 0$ for $\zeta \neq 0$ (Volevich [13]; for the constant coefficients see Hörmander [5], Chapter 4, and Pini [9]; see also Zanghirati [14], where the classes $G^M(\Omega)$ are characterized through the iterations of such operators).

We shall consider here a class of operators P quasi-elliptic with respect to a suitable weight M , which degenerate on a flat symplectic submanifold of $\Omega \times \mathbb{R}^n_\zeta$. Fix ν , $1 \le \nu < n$, and split $z = (x,y)$, $x = (x_1=z_1,\dots,x_\nu=z_\nu) \in \mathbb{R}^\nu$, $y = (y_1=z_{\nu+1},\dots,y_{n-\nu}=z_n) \in R^{n-\nu}$. Let $L = (L_1,\dots,L_{n-\nu})$ be a given $(n-\nu)$-tuple of positive integers with $\min L_j = 1$ and let k be a given common multiple of the L_j's . Fix finally two positive integers h and m_1 , and write $m = m_1 k$. We define

$$(1.3) \qquad P = \sum_{(\alpha,\beta,\gamma)\in\mathfrak{M}} a_{\alpha\beta\gamma}(z)x^\gamma D_x^\alpha D_y^\beta \quad ,$$

where the set of indices $\mathfrak{M} \subset \mathbb{Z}_+^\nu \times \mathbb{Z}_+^{n-\nu} \times \mathbb{Z}_+^\nu$ is given by

$$(1.4) \qquad \mathfrak{M} = \Big\{(\alpha,\beta,\gamma)\ ;\ |\alpha|k + \langle\beta,L\rangle \le m\ ,\ hm \ge |\gamma|k \ge |\alpha|k + (1+h)\langle\beta,L\rangle - m$$
$$\underline{\text{and}}\quad \beta_j L_j/k \in \mathbb{Z}_+\quad \underline{\text{for all}}\quad j = 1,\dots,n-\nu \Big\}\ .$$

Consider the n-tuple

$$(1.5) \qquad M = (M_1,\dots,M_n)\ ,\ M_j = k\quad \underline{\text{for}}\quad 1 \le j \le \nu\quad ,\quad M_j = L_{j-\nu}\quad \underline{\text{for}}\ \nu+1 \le j \le n,$$

and suppose the coefficients $a_{\alpha\beta\gamma}$ are in $G^M(\Omega)$, where now Ω is in a neighborhood of the origin in $\mathbb{R}^n$. Introduce also the sets of indices

(1.6) $$\mathcal{M}_o = \left\{ (\alpha,\beta,\gamma) \in \mathcal{M} \, , |\gamma|k = |\alpha|k + (1+h) \, <\beta,L> - m \right\} ,$$

(1.7) $$\mathcal{M}_o^o = \left\{ (\alpha,\beta,\gamma) \in \mathcal{M}_o \, , \, |\alpha|k + <\beta,L> = m \right\} .$$

Writing ζ , ξ , η for the dual variables of z , x , y , respectively, we assume

(1.8) $$\sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_o^o} a_{\alpha\beta\gamma}(0)x^\gamma\xi^\alpha\eta^\beta \neq 0 \ \underline{\text{for}} \ |x| + |\zeta| \neq 0 \ , \text{ if } \eta \neq 0 \ .$$

Since $<\mu,M> = |\alpha|k + <\beta,L>$ with our choice of M and with $\mu = (\alpha,\beta)$, condition (1.8) implies the (micro) M-quasi-ellipticity of P outside the manifold $\{x=0, \xi=0\}$, for z in a small neighborhood of the origin in $\mathbb{R}^n$; then in view of Volevich [13] P is G^M-hypoelliptic for $x \neq 0$.

Consider now the operator with polynomial coefficients in $\mathbb{R}^\nu_x$

(1.9) $$\mathcal{L}(\eta,x,D_x) = \sum_{(\alpha,\beta,\gamma)\in\mathcal{M}_o} a_{\alpha\beta\gamma}(0)\eta^\beta x^\gamma D_x^\alpha$$

and assume

(1.10) $$\mathrm{Ker}\,\mathcal{L}(\eta,x,D_x) \cap \mathcal{S}(\mathbb{R}^\nu_x) = \{0\} \ \underline{\text{for all}} \ \eta \neq 0 \ .$$

Condition (1.10), joined with (1.8), guarantees the hypoellipticity of P in a full neighborhood of the origin (see Parenti-Rodino [8]), and it is also necessary for the hypoellipticity when the coefficients $a_{\alpha\beta\gamma}$ are constant (see Taniguchi [10]). In the case $\nu = 1$ it is possible to translate (1.10) into explicit conditions on the coefficients $a_{\alpha\beta\gamma}$ (Mascarello-Rodino [6], Sections 3, 4, 5).

The result which we shall prove here is the following:

<u>Theorem 1.1</u>: <u>Under the preceding hypotheses, and in particular under the conditions (1.8), (1.10) and the assumption</u> $a_{\alpha\beta\gamma} \in G^M(\Omega)$, M <u>as in (1.5), the operator</u> P <u>in (1.3) is</u> G^M<u>-hypoelliptic in a neighborhood of the origin</u>.

For example the models of the Introduction are $G^{(k,1)}$-hypoelliptic, for any h , if and only if they are hypoelliptic. When $k = 1$, $L_j = 1$ for every j , we recapture the result of Grushin [4]. In the case $m = 2$, $k = 1$ or $k = 2$, Theorem 1.1 intersects also some well known results about the analytic and Gevrey regularity of the second order operators (see Baouendi-Goulaouic [1], Derridj-Zuily [2], Durand [3] for a general study in this connection).

2. PROOF.

Applying the result of Volevich [13] in the region where $x \neq 0$ and observing that assumptions (1.8), (1.10) are invariant for small translations in the y variables, we are reduced to prove the following statement: if $f \in \mathcal{D}'(V)$ and $Pf \in G^M(V)$, for M as in (1.5) and for some neighborhood V of the origin in $\mathbb{R}^n$, $\bar{V} \subset\subset \Omega$, then f is in $G^M(V')$, for some other neighborhood $V' \subset V$. Actually, since we know from the results of [8] that P is hypoelliptic at the origin it will be not restrictive to argue under the additional assumption $f \in C^\infty(V)$.

We shall use the following estimate for L^2-norms: if the hypotheses of Theorem 1.1 are satisfied, there exists a neighborhood U of the origin in $\mathbb{R}^n$ and a constant $C > 0$ such that

$$\sum_{(\alpha,\beta,\gamma)\in\mathcal{M}} \|x^\gamma D_x^\alpha D_y^\beta u\| \leq C\|Pu\| \quad \underline{\text{for every}} \quad u \in C_0^\infty(U) \quad . \tag{2.1}$$

This inequality follows easily from the properties of continuity of the parametrix of P in [8] (see in particular the proof of Theorem 3.1 in the first part of the paper). Let us write for $\rho > 0$

$$B_\rho = \left\{ z = (x,y) \in \mathbb{R}^n \, , \, |x| < \rho^k \ \underline{\text{and}} \ \sum_{j=1}^{n-\nu} |y_j|^{2/L_j} < \rho^2 \right\} \quad . \tag{2.2}$$

By passing to a new system of coordinates we can assume without loss of generality

$U = B_2$ in (2.1) and, at the same time, $f \in C^\infty(V)$, $Pf \in G^M(V)$ for a neighborhood V of B_2. We shall prove that f is then in $G^M(B_1)$.
Let us begin with two technical lemmas.

Lemma 2.1: Let ρ, ρ_1 be given positive constants and let M be the weight in (1.5). There exists $\phi \in C_o^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \phi \subset B_{\rho_1+\rho}$, $\phi(z) = 1$ for $z \in B_{\rho_1}$, $0 \le \phi(z) \le 1$ for every $z \in \mathbb{R}^n$ and

(2.3) $$\max |D_z^\mu f(z)| \le c_\mu \rho^{-<\mu,M>}$$

where c_μ does not depend on ρ, ρ_1; moreover $D_x^\alpha \phi(z) = 0$ for $|x| < \rho_1^k$ if $|\alpha| \ne 0$.

Lemma 2.2: Let f be in $C^\infty(V)$ and fix $K \subset\subset V$. The following conditions are equivalent:

(2.4) There exists $A > 0$ such that for all $\mu \in \mathbb{Z}_+^n$

$$\max_{z \in K} |D_z^\mu f(z)| \le A^{|\mu|+1} (\mu_1!)^{M_1} \dots (\mu_n!)^{M_n} .$$

(2.5) There exists $A > 0$ such that for all $\mu \in \mathbb{Z}_+^n$

$$\max_{z \in K} |D_z^\mu f(z)| \le A(A <\mu,M>)^{<\mu,M>} .$$

(2.6) For every integer $t \ge 0$ write $I_t = \left\{ \mu \in \mathbb{Z}_+^n , <\mu,M> \le t \right\}$; there exists $A > 0$ such that for all $t \ge 0$

$$\max_{\mu \in I_t} \max_{z \in K} |D_z^\mu f(z)| \le A^{t+1} t! .$$

(2.7) Let (J_t) be a sequence of finite subsets of $\mathbb{Z}_+^n$ such that $I_{t-t_1} \subset J_t \subset I_{t+t_2}$, for suitable fixed integers t_1, $t_2 \ge 0$ and for every $t \ge t_1$. There exists $A > 0$ such that for all $t \ge 0$

$$\max_{\mu\in J_t}\ \max_{z\in K} |D_z^{\mu} f(z)| \le A^{t+1}\ t!\ .$$

(2.8) For (J_t) as in (2.7), there exists $A > 0$ such that for all $t \ge 0$

$$\max_{\mu\in J_t} \|D_z^{\mu} f\|_{L^2(K)} \le A^{t+1}\ t!\ .$$

The construction of ϕ in Lemma 2.1 is standard (cf. Lemma 1.6 in [14], for example). As for Lemma 2.2, one gets easily from Stirling's formula that (2.4)$\longleftrightarrow$(2.5) and (2.5)$\longleftrightarrow$(2.6); the equivalence (2.6)$\longleftrightarrow$(2.7) is obvious and (2.7)$\longleftrightarrow$(2.8) is a consequence of the embedding theorems of Sobolef (the statement is valid in this case for any n-tuple M of rational positive numbers). In the sequel we shall apply Lemma 2.2 with M as in (1.5) and with the following choice of the sets J_t :

$$(2.9)\qquad \begin{cases} J_t = J_t^1 \cup J_t^2 \ , \\ J_t^1 = \{\mu = (\alpha,\beta),\ 2|\alpha|k + \langle\beta,L\rangle \le t \ \text{ and } \ |\alpha|k \le (1+h)m\} \ , \\ J_t^2 = \{\mu = (\alpha,\beta),\ \mu \in I_{t-(1+h)m} \ \text{ and } \ |\alpha|k \ge (1+h)m\} \ ; \end{cases}$$

since now $I_t = \{\mu = (\alpha,\beta),\ |\alpha|k + \langle\beta,L\rangle \le t\}$ we have $I_{t-(1+h)m} \subset J_t \subset I_t$ for $t \ge (1+h)m$ and the assumption on J_t in (2.7) is satisfied. Using Lemma 2.2 with $K = B_2$ and J_t as in (2.9), we obtain from the hypotheses $Pf \in G^M(V)$, $a_{\alpha\beta\gamma} \in G^M(V)$ that for a suitable constant A and for all $t \ge 0$

$$(2.10)\qquad \max_{\mu\in J_t} \|D_z^{\mu} Pf\|_{L^2(B_2)} \le A^{t+1}\ t!\ ,$$

$$(2.11)\qquad \max_{\mu\in I_t}\ \max_{z\in B_2} |D_z^{\mu}\, a_{\alpha\beta\gamma}(z)| \le A^{t+1}\ t!\ .$$

Set now for $f \in C^\infty(V)$ and $1 \le \sigma \le 2$

$$(2.12) \qquad |||f,\sigma||| = \sum_{(\alpha,\beta,\gamma)\in\mathfrak{M}} \|x^\gamma D_x^\alpha D_y^\beta f\|_{L^2(B_\sigma)} ,$$

$$(2.13) \qquad |||f,\sigma|||_t = \max_{\mu\in J_t} |||D_z^\mu f,\sigma||| .$$

Take $\rho > 0$ and $\rho_1 > 1$, with $\rho_1 + \rho < 2$, and let ϕ be defined according to Lemma 2.1. From the estimate (2.1) we obtain for any $\mu \in \mathbb{Z}_+^n$

$$(2.14) \qquad |||D_z^\mu f,\rho_1||| \leq |||\phi D_z^\mu f,2||| \leq C\|P(\phi D_z^\mu f)\| .$$

Then writing

$$(2.15) \qquad \|P(\phi D_z^\mu f)\| \leq \|\phi D_z^\mu Pf\| + \|[P,\phi]D_z^\mu f\| + \|\phi[P,D_z^\mu]f\| ,$$

we have for every $t \geq 0$

$$(2.16) \qquad |||f,\rho_1|||_t \leq C \max_{\mu\in J_t} \left\{ \|D_z^\mu Pf\|_{L^2(B_2)} + \|[P,\phi]D_z^\mu f\| + \|[P,D_z^\mu]f\|_{L^2(B_{\rho_1+\rho})} \right\} .$$

The two last terms in the right-hand side of (2.16) will be estimated by means of the following lemmas; in the statements t_o represents a large integer, which will be determined in the proofs.

<u>Lemma 2.3</u>. For every $\mu \in J_t$, $t \geq t_o$, we have

$$(2.17) \qquad \|[P,\phi]D_z^\mu f\| \leq C \sum_{j=1}^{m} \rho^{-j} |||f,\rho_1+\rho|||_{t-j} ,$$

where the constant C does not depend on ρ , ρ_1 , t .

<u>Lemma 2.4</u>. There exists $C > 0$ such that for every $\mu \in J_t$, $t \geq t_o$:

$$(2.18) \qquad \|[P,D_z^\mu]f\|_{L^2(B_{\rho_1+\rho})} \leq C \sum_{j=1}^{t} A^j \frac{t!}{(t-j)!} |||f,\rho_1+\rho|||_{t-j} .$$

Applying the lemmas and the hypothesis (2.10), from (2.16) we deduce the inequality

$$(2.19) \qquad |||f,\rho_1|||_t \le C \left\{ A^{t+1}\, t! + \sum_{j=1}^{m} \rho^{-j} |||f,\rho_1+\rho|||_{t-j} + \sum_{j=1}^{t} A^j \frac{t!}{(t-j)!} |||f,\rho_1+\rho|||_{t-j} \right\} ,$$

where the constant C does not depend on ρ , ρ_1 , t and we assume $t \ge t_o$. Let now s be any fixed integer, $s \ge t_o$, and define for $t \le s-t_o$:

$$(2.20) \qquad \omega_{s,t} = s^{-t-t_o} |||f,2-(t+1)/s|||_t .$$

If we set $\rho = 1/s$, $\rho_1 = 2-t/s$, from (2.19) we obtain for a new constant C_1 :

$$(2.21) \qquad \omega_{s,t} \le C_1 \left\{ A^t + \sum_{j=1}^{m} \omega_{s,t-j} + \sum_{j=1}^{t} A^j\, \omega_{s,t-j} \right\} .$$

It is now easy to prove that there exists a constant B independent of s and t , $B > 1$, such that

$$(2.22) \qquad \omega_{s,t} \le (2B)^{t+1} \quad \text{for} \quad t \le s-t_o .$$

In fact, (2.22) is trivially satisfied by a suitable B if $t \le t_o$, and it can be proved to be valid in general for a large B arguing by induction and using (2.21) (cf. [4], Section 5). In particular, if we set in (2.22) $t = s-t_o$ we obtain that for some constant C_2 and for $s \ge t_o$

$$(2.23) \qquad |||f,2-(s-t_o+1)/s|||_{s-t_o} \le (C_2 s)^s .$$

From (2.23), from Lemma 2.2, from Stirling's formula and from the obvious estimates

$$(2.24) \qquad \max_{\mu \in J_t} \|D_z^\mu f\|_{L^2(B_1)} \le |||f,1|||_t$$

we get finally the conclusion $f \in G^M(B_1)$.

It remains to prove Lemma 2.3 and Lemma 2.4. Of course the definition of J_t in (2.9) will play an essential role here; we shall use in particular the following two properties of the sets J_t :

(2.25) if $(\alpha+\tilde\alpha,\beta+\tilde\beta) \in J_t$ and $\langle\bar\beta,L\rangle \le |\tilde\alpha|k + \langle\tilde\beta,L\rangle$ then $(\alpha,\beta+\bar\beta) \in J_{t-|\tilde\alpha|k - \langle\tilde\beta,L\rangle + \langle\bar\beta,L\rangle}$;

(2.26) if $|\alpha+\tilde\alpha|k \le (1+h)m$, $(\alpha+\tilde\alpha,\beta) \in J_t$ and $\langle\tilde\beta,L\rangle \le |\tilde\alpha|k$ then $(\alpha,\beta+\tilde\beta) \in J_{t-|\tilde\alpha|k}$.

Proof of Lemma 2.3. The expression $[P,\phi]D_z^\mu f$ in (2.17) is a linear combination of a finite number of terms of the form

$$x^\gamma(D_z^{\mu''}\phi)D_z^{\mu'+\mu}f \quad , \tag{2.27}$$

where $(\mu'+\mu'',\gamma)\in \mathfrak{M}$, $\mu'' \neq 0$, and $\mu = (\alpha_1,\beta_1) \in J_t$. Split $\mu' = (\alpha',\beta')$, $\mu'' = (\alpha'',\beta'')$ and assume first $\alpha'' = 0$. If $\mu \in I_{m+(n-\nu)k}$ and if t_o in the statement is sufficiently large the norm of the function (2.27) can be easily estimated by means of the term $\rho^{-m}|||f,\rho_1+\rho|||_{t-m}$ in the right-hand side of (2.17). Therefore we may assume without loss of generality $|\alpha_1|k + \langle\beta_1,L\rangle \ge m+(n-\nu)k$; in this case we can write (α_1,β_1) in the form $(\alpha_1'+\alpha_1'',\beta_1'+\beta_1'')$ and β' in the form (β_2',β_2'') , with $\langle\beta_2',L\rangle < (n-\nu)k$, in such a way that $|\alpha_1''|k + \langle\beta_1'',L\rangle = \langle\beta''+\beta_2',L\rangle$. Regrouping the derivatives in (2.27), we obtain

$$x^\gamma(D_y^{\beta''}\phi)D_x^{\alpha'+\alpha_1''}D_y^{\beta_1''+\beta_2''}(D_x^{\alpha_1'}D_y^{\beta_1'+\beta_2'}f) \quad , \tag{2.28}$$

where $(\alpha_1',\beta_1'+\beta_2') \in J_{t-j}$ with $j = \langle\beta'',L\rangle$, in view of (2.25). On the other hand, if we choose β_1'' , β_2'' suitably, then the components of $\beta_1''+\beta_2''$ satisfy

$$(2.29)\qquad (\beta_1''+\beta_2'')_q\, L_q/k \in \mathbb{Z}_+ \ ,\ q = 1,\ldots,n-\nu \quad .$$

Therefore $(\alpha'+\alpha_1'',\beta_1''+\beta_2'',\gamma) \in \mathfrak{M}$ and, in view of (2.3), the norm of the function in (2.28) is estimated by the term in the right-hand side of (2.17) at which $j = \langle\beta'',L\rangle$.

If $\alpha'' \neq 0$, then in (2.27) $D_z^{\mu''}\phi = 0$ for $|x| < 1$, according to Lemma 2.1. Thus in all the estimates one can assume that the power of γ in (2.27) is maximal, i.e. $|\gamma|k = hm$, and repeating the preceding arguments we find easily that for $|\alpha_1|k + \langle\beta_1,L\rangle$ sufficiently large the norm of (2.27) is estimated by the term of (2.17) in which $j = |\alpha''|k + \langle\beta'',L\rangle$.

<u>Proof of Lemma 2.4</u>. The expression $[P,D_z^{\mu} f]$ in (2.18) consists of the sum of the terms

$$(2.30)\qquad (D_x^{\alpha''} D_y^{\beta''} a_{\alpha\beta\gamma})(D_x^{\alpha'''} x^{\gamma}) D_x^{\alpha} D_y^{\beta}(D_x^{\alpha'} D_y^{\beta'} f) \quad ,$$

where $(\alpha'+\alpha''+\alpha''',\ \beta'+\beta'') = (\alpha_1,\beta_1) = \mu \in J_t$, $|\alpha'''|\, k \leq hm$, and $(\alpha,\beta,\gamma) \in \mathfrak{M}$. If $|\alpha''|k + \langle\beta'',L\rangle = t_2$ and $|\alpha'''|\, k = t_3$, then their number does not exceed $c^j t!/\ t_2!(t-j)!$, where $j = t_2 + t_3$ and c is a suitable constant. Then, in view of (2.11), it will be sufficient to check that for $j \neq 0$

$$(2.31)\qquad \|(D_x^{\alpha'''} x^{\gamma}) D_x^{\alpha+\alpha'} D_y^{\beta+\beta'} f\|_{L^2(B_{\rho_1+\rho})} \leq c|||f,\rho_1+\rho|||_{t-j} \quad .$$

The expression $D_x^{\alpha'''} x^{\gamma}$ is nonzero only in the case when each component of γ does not exceed the corresponding component of α''' ; hence $\gamma' = \gamma-\alpha''' \in \mathbb{Z}_+^{\nu}$ is well defined. Since $D_x^{\alpha'''} x^{\gamma}$ is proportional to $x^{\gamma'}$, we are reduced to prove that every triple of multi-indices $(\alpha+\alpha',\beta+\beta',\gamma')$ can be written in the form

$$(2.32)\qquad (\alpha+\alpha',\beta+\beta',\gamma') = (\alpha_2,\beta_2,\gamma') + (\alpha_3,\beta_3,0) \quad ,$$

where

$$(2.33) \qquad (\alpha_2,\beta_2,\gamma') \in \mathfrak{M} \quad \text{and} \quad (\alpha_3,\beta_3) \in J_{t-j} \quad .$$

Assume first $|\alpha'|k \geq \langle\beta,L\rangle$; then we can write $\alpha+\alpha'$ in the form $\alpha_2+\alpha_3$, where $|\alpha_2|k = |\alpha|k + \langle\beta,L\rangle$ (this is possible since $\langle\beta,L\rangle$ is a multiple of k). If we take $\beta_2 = 0$ and $\beta_3 = \beta+\beta'$, then the relation (2.32) holds and clearly $(\alpha_2,\beta_2,\gamma') \in \mathfrak{M}$, whereas $(\alpha_3,\beta_3) \in J_{t-j}$ follows from (2.25).
Assume now $|\alpha'|k < \langle\beta,L\rangle$ and $|\alpha'''| k \leq \langle\beta,L\rangle$; then we can write $\beta+\beta'$ in the form of a sum $\beta_2+\beta_3$, where $\langle\beta_2,L\rangle = \langle\beta,L\rangle - |\alpha'''| k$ and the components of β_2 satisfy:

$$(2.34) \qquad (\beta_2)_q \, L_q/k \in \mathbb{Z}_+ \ , \ q = 1,\dots,n-\nu \quad .$$

Let us take $\alpha_2 = \alpha$, $\alpha_3 = \alpha$. Then the identity (2.32) is valid and clearly $(\alpha_2,\beta_2,\gamma') \in \mathfrak{M}$. On the other hand we have in this case $|\alpha'+\alpha'''| k \leq (1+h)m$; then we may apply (2.26) and from $(\alpha'+\alpha''',\beta') \in J_{t-t_2}$ we deduce $(\alpha_3,\beta_3) \in J_{t-j}$.
Finally, if $|\alpha'|k < \langle\beta,L\rangle < |\alpha'''| k$ we set $(\alpha_2,\beta_2) = (\alpha,0)$, $(\alpha_3,\beta_3) =$ $= (\alpha',\beta+\beta')$, and the relations (2.32), (2.33) follow easily from the preceding arguments.

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Luigi RODINO
Istituto di Analisi Matematica
Università di Torino
Via Carlo Alberto, 10
I-10123 Torino
Italy