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J. PERSSON

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SINGULAR HOLOMORPHIC SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH HOLOMORPHIC COEFFICIENTS AND NONANALYTIC SOLUTIONS OF EQUATIONS WITH ANALYTIC COEFFICIENTS

by J. PERSSON (University of Tromsø)

1. INTRODUCTION.

Let $P(x,D)$ be a linear partial differential operator of order $m > 0$ with coefficients analytic in a neighbourhood of the origin in $\mathbb{R}^n$.
Let $H = \{x;\ x_1 = 0,\ x \in \mathbb{R}^n\}$ and let $k \geq m$ be an integer. Is it then possible to find a C^∞ solution $u \not\equiv 0$ of $P(x,D)u = 0$ such that $D^\xi u = 0,\ x \in H$, all ξ or to find a C^k solution u which is not in C^{k+1} and with $D_1^j u = 0 \quad 0 \leq j \leq k$ for $x \in H$. In this paper one finds a long range of conditions on $P(x,D)$ garanteeing that one or both kinds of solutions exist. These conditions also show that the solutions are analytic outside H. A global version for operators with constant coefficients shows that Hörmander's nullsolutions, [7, Theorem 3.2] or [8, Theorem 5.2.2, p. 121], can be chosen analytic outside the initial hyperplane. This is one main feature of the paper. Before going into more details we give the notation.

We shall use the standard notation of [8] with some exceptions. Let $n > 1$ be an integer and let $z = (z_1,\dots,z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j$, x_j and y_j real, $1 \leq j \leq n$. We let $D_j = (\partial/\partial x_j - i\partial/\partial y_j)/2$ or $D_j = \partial/\partial x_j$. The meaning of D_j will always be defined by the context. Let $s \in \{1,3,4,\dots,n\}$ be a fixed integer. Let $z' = (z_1,z_3,z_4,\dots,z_s)$ and let $z'' = (z_3,z_4,\dots,z_s)$. The corresponding notation will be used for multi-indices, the operator $D = (D_1,\dots,D_n)$

and x . Thus $D''^{\alpha''} = D_3^{\alpha_3} D_4^{\alpha_4} \ldots D_s^{\alpha_s}$. We let $\xi d = \xi_1 d_1 + \ldots + \xi_n d_n$, $d \in \mathbb{R}^n$. We also let $j^+ = \max(0,j)$.

The paper is disposed as follows. In this section we first state the theorems without proofs. Then at the end of the section the theorems are related to other results in the field. All constructions of singular solutions in this paper is based on an idea which goes back to Goursat [6, pp. 303-308] when he showed non-uniqueness in certain characteristic Cauchy problems. Here we solve our Goursat problems by successive approximations. We use the technique developed in Persson [16][17] and [18]. The corresponding lemmas modified to our present needs are stated and proved in Section 2. In Section 3 one constructs holomorphic functions which have a restriction to $\mathbb{R}^n$ which is singular or flat on $x' = 0$. In Section 4 one shows how to choose a coordinate system for each proof of the theorems stated in the introduction. After this choice the rest of the proofs is rather easy and with only minor differences from one theorem to another. Theorem 1.1 is proved in detail in Section 5. The rest of the theorems are proved in Section 6. In Section 7 we indicate how other singular solutions could be constructed without formulating any theorems. This shows that it is possible to make a unified approach to a rather vast range of problems.

The theorems run as follows.

Theorem 1.1. Let $P(z,D)$ be a linear partial differential operator of order $m > 0$ with holomorphic coefficients in a neighbourhood of the origin in $\mathbb{C}^n$. Let l, $0 < l \leq m$ be an integer and let $\beta = (m-1,1,0,\ldots,0)$ be a multi-index. It is assumed that $P(z,D)$ is given by

$$(1.1) \qquad P(z,D) = D^\beta + \sum_{\substack{\alpha \neq \beta \\ |\alpha| \leq m}} a_\alpha D^\alpha .$$

If

$$(1.2) \qquad a_\alpha(z) = O(|z''|^{|\alpha''|}) , \quad \text{if } \alpha \neq \beta ,$$

and if

(1.3) $\quad a_\alpha = 0$, if $|\alpha| = m$, and $\alpha_1 > m - 1$,

then there exist a neighbourhood Ω of the origin in $\mathbb{R}^n$ and a solution $u \in C^\infty(\Omega)$ of $P(x,D)u = 0$ such that $0 \in \operatorname{supp} u$, $D^\xi u(x) = 0$ if $x \in \Omega$ with $x_1 = 0$ or $x'' = 0$, all ξ , and u is analytic for such $x \in \Omega$ that both $x_1 \neq 0$ and $x'' \neq 0$.

Theorem 1.2. Let $P(z,D)$ be given by (1.1). If

(1.4) $\quad a_\alpha(z) = O(|z_1|^{(\alpha_1-m+1)^+}|z''|^{|\alpha''|})$, if $|\alpha| = m$,

(1.5) $\quad a_\alpha(z) = 0$, if $\alpha_2 > 1$,

and

(1.6) $\quad a_\alpha = O(|z''|^{|\alpha''|})$, if $|\alpha| < m$, and $\alpha_2 = 1$,

then the conclusion of Theorem 1.1 is true.

Theorem 1.3. Let $P(z,D)$ be given by (1.1). If

(1.7) $\quad a_\alpha(z) = O(|z_1|^{(\alpha_1-m+1)^+}|z''|^{|\alpha''|})$ all α ,

then the conclusion of Theorem 1.1 is true. Furthermore to each integer $k \geq m$ there exists a $u \in C^k(\Omega)$, $u \notin C^{k+1}(\Omega)$ such that $P(x,D)u = 0$, $u = O(|x_1|^{k+1/2}|x''|^{k+1/2})$, $0 \in \operatorname{supp} u$ and u is analytic when both $x_1 \neq 0$ and $x'' \neq 0$.

Theorem 1.4. Let $P(z,D)$ be given by (1.1). If

(1.8) $\quad a_\alpha(z) = O(|z'|^{\alpha_1-m+1+|\alpha''|})$, all α ,

and

(1.9) $\quad a_\alpha(z) = 0$, if $\alpha_1 < m - 1$,

then there are a neighbourhood Ω of the origin in $\mathbb{R}^n$ and a function $u \in C^\infty(\Omega)$ such that $0 \in \operatorname{supp} u$, $P(x,D)u = 0$, $D^\xi u(x) = 0$ if $x \in \Omega$ and $x' = 0$, all ξ , and u is analytic when $x' \neq 0$. Furthermore to each integer $k \geq m$ there exists a $u \in C^k(\Omega)$, $u \notin C^{k+1}(\Omega)$, such that $P(x,D)u = 0$, $u = O(|x'|^{k+1/2})$,

$0 \in \operatorname{supp} u$ and u is analytic when $x' \neq 0$.

Theorem 1.5. Let $P(z,D)$ given by (1.1) have constant coefficients such that

(1.10) $\quad a_\alpha = 0,$ if $|\alpha| = m,$ and $\alpha_1 > m - 1$.

Then there is a $u \in C^\infty(\mathbb{R}^n)$ such that $P(D)u = 0$, $0 \in \operatorname{supp} u$, $D^\xi u(x) = 0,\ x_1 = 0,\ x \in \mathbb{R}^n$, all ξ . Furthermore u is analytic when $x_1 \neq 0$.

We give some examples on operators fulfilling the hypothesis of one theorem but not fulfilling the hypothesis of any of the other theorems among Theorem 1.1 to Theorem 1.4. Let $n = s = 3$.

$D_1^2 D_2^2 + D_2^4 + z_3^3 D_3^3 + D_1^3$ Theorem 1.1,

$D_1^2 D_2^2 + z_1^2 D_1^4 + z_1 z_3 D_1^3 D_3 + z_3 D_2^2 D_3 + D_2 D_3^2 + D_1^3$ Theorem 1.2,

$D_1^2 D_2^2 + z_1 D_1^3 D_2 + D_2^4 + z_3^2 D_1^2 D_3^2 + z_1 D_1^3$ Theorem 1.3,

$D_2^2 + (z_3^2 + z_1^2)(D_1^2 + D_3^2)$ Theorem 1.4.

It is obvious that an operator fulfilling the hypothesis of one of these theorems is not analytic-hypoelliptic. If it fulfils the hypothesis of Theorem 1.3 or Theorem 1.4 it is not hypoelliptic. It is obvious from the proofs that one could lower the regularity of the coefficients in the z_1 variable in Theorem 1.1 to some Gevrey class if one also allows for this in the solution. If there is no differentiation in the z_1 variable in the successive approximations in the existence proof then one could lower the regularity down to continuity in the z_1 variable if one at the same time adjusts the corresponding class of solutions.

We start by commenting the results for $s = 1$. Then $x' = x_1$ and the theorems above all give the existence of nullsolutions in C^∞ or in C^k . The solutions are analytic for $x_1 \neq 0$. As we have already mentioned the idea in the proofs goes back to Goursat. Malgrange pointed out that Goursat's method works equally well for the simply characteristic Cauchy problem in $\mathbb{R}^n$ when the principal part has real coefficients. See [13] or Hörmander [8, Theorem 5.2.1, p. 120]. The same result is proved by Mizohata [15] by another method. Theorem 1.3 can be seen as a direct generalization of [8, Theorem 5.2.1]. De Paris [5, Proposition 2] and later Komatsu [11] and [12] have other generalizations of this result starting from the Mizohata paper. They both assume that the principal part of the operator can be factorized into powers of simply characteristic operators and a noncharacteristic factor. The result by De Paris is a special case of Theorem 1.3. Komatsu's results could be considered as special cases of Theorem 1.3 or as an easy extension of Theorem 1.1 when Komatsu's idea is known. Indeed Komatsu uses a complex coordinate transformation such that the operator in the new coordinates has the form of the operator in Theorem 1.1. In these coordinates we make the construction in the proof of the theorem. It then turns out that the restriction of the holomorphic solution to the original $\mathbb{R}^n$ has the wanted properties. We have not incorporated this result since we want to stress that in our hypothesis there is no factorization of the operator.

The first one to construct a C^∞ nullsolution of a characteristic Cauchy problem seems to be Tihonov [30]. In 1935 he gave a series solution of $D_1u = D_2^2u$ which is of Gevrey class less than 2 in the x_1 variable and which is analytic in the x_2 variable. Afterwards Täcklind in 1936 refined Tihonov's result as to the regularity of the possible nullsolutions of the heat equation. See [29]. Täcklind's results have been extended to the general normal characteristic Cauchy problem for operators with constant coefficients by Persson [27].
Hörmander proved the existence of a C^∞ nullsolution of the general characteristic Cauchy problem for operators with constant coefficients [8, Theorem 5.2.2, p. 121]. Hörmander's proof is based on the use of the Fourier transformation so there is no immediate generalization to the variable coefficient case. In 1972 Persson [19], see also [20] and [21], showed how to overcome this difficulty. Persson proves that his earlier result on the Goursat problem in Gevrey classes in [16], [17], and

[18] are indeed slightly more general. This observation does not require a new proof and it gives a theorem from which the existence of nullsolutions follows immediately. We like to stress again that neither in [19] nor in this paper there is a condition that our Cauchy problem is simply characteristic or that the principal part of the operator can be factorized as for instance in [5] and [11].

The result in [19] has been generalized to the nonlinear case by Bronšteĭn [3]. Bronšteĭn also uses the solution of a Goursat problem and the observation from [19]. A special case of [19] is given by Bronšteĭn and Fridlender [4].

In all theorems above one has constant multiplicity of the characteristic initial hypersurface. When this multiplicity varies lower order terms of the operator may play a very decisive role. See Treves [31] and Birkeland and Persson [2]. For examples of nullsolutions when the principal part of the operator vanishes on the initial hypersurface see Hörmander [9].

As to the analyticity of the solution u for $x_1 \neq 0$ one notices that the only difference between the proof in [19] and the proof of Theorem 1.1 and Theorem 1.5 is that one here has noticed that the Gevrey estimates are valued in a complex neighbourhood of $\mathbb{R}^n \setminus \{0\}$. This observation has been used earlier in the construction of singular solutions of equations with holomorphic coefficients by Persson in [22], [23], and [24]. Indeed this was the impetus of extending an old result presented in a talk at the University of Genoa in June 1974 by the author. In this talk Theorem 1.2 and the C^∞ part of Theorem 1.3 were exposed with $s = 1$.

The case $s \geq 3$ has been treated by Baouendi, Treves and Zachmanoglou [1]. The part of [1] which treats operators covered by some of our theorems corresponds to the case when $\{x;\ x' = 0\}$ is contained in a simply characteristic hypersurface. Then their results are generally stronger except for the first order operators covered by Theorem 1.4. In this case the result is practically the same. The only difference is that $u(x) \neq 0$ when $x' \neq 0$ is not included in the conclusion of Theorem 1.4. The technique used in [1] is an adaptation of the technique used by Komatsu in [9]. See also Zachmanoglou [33].

Section 7 gives some comments on the possibility of constructing holomorphic solutions of $P(z,D)u = 0$ with a singularity in $z_1 = 0$. Here we suppose that $s = 1$ and that $P(z,D)$ fulfils the hypothesis of some of the theorems above. No theorems are formulated but there is a strong evidence that our theorems all have a singular complex version as has Theorem 1.1 and Theorem 1.5, see Persson [23], [25] and [24]. We have been informed that the result [23, Theorem 4.1] is contained or obtainable by the results in Kashiwara and Schapira [10] and Schapira [28]. Since our methods are considerably more elementary they may have their own value from this point of view. Anyhow the question arises whether the technique with exponential majorization is an optimal tool. Could it be that the variant of the Nagumo method presented in Yamanaka and Persson [32] or its abstract Goursat problem version in Miyake [4] would give better results. Again we like to stress that in the results indicated in Section 7, the cited results from [19], [20], [21], [23], [24] and [25] and our theorems above there is no hypothesis of factorization of the principal part of the operator. Nor is the initial hypersurface necessarily simply characteristic.

2. EXPONENTIAL MAJORIZATION LEMMAS.

In this section the lemmas needed for the exponential majorization are stated and also proved if not known before. The first one is used in connection with Leibniz' formula. For a proof see Birkeland and Persson [2, Lemma 5.1]. Here and in the following we let $\xi d^{\xi d-1} = 1$ for $\xi = 0$.

Lemma 2.1. Let $d = (d_1,\dots,d_n) \in \mathbb{R}^n$, $d_j \geq 1$, $1 \leq j \leq n$. Let ν and ξ be multi-indices with nonnegative components. Then for all n, d, and ξ

$$(2.1)\qquad \sum_{\nu\leq\xi}\binom{\xi}{\nu}\ (\xi-\nu)d^{(\xi-\nu)d-1}\nu d^{\nu d-1}\xi d^{1-\xi d} \leq 4\ ,$$

Here $\nu \leq \xi$ denotes that $\nu_j \leq \xi_j$, $1 \leq j \leq n$.

The next lemma is the principle of exponential majorization extended to functions of a complex variable.

Lemma 2.2. Let M, K, and r_1 be positive constants. Let $f(z)$ be continuous in $\{z;\ |z| \leq r_1,\ z \in \mathbb{C}\}$. Let

$$\int_0^z f(t)\,dt$$

denote radial integration from the origin to z in $\mathbb{C}$. If

$$(2.2) \qquad |f(z)| \leq M \exp(K|z|),\ |z| \leq r_1 \quad ,$$

then

$$(2.3) \qquad \left|\int_0^z f(t)\,dt\right| \leq K^{-1} M \exp(K|z|),\ |z| \leq r_1 \ .$$

Proof.

$$\left|\int_0^z f(t)\,dt\right| \leq \int_0^{|z|} M e^{Kt}\,dt \leq MK^{-1}(e^{K|z|} - 1) \leq MK^{-1}e^{K|z|} \quad .$$

The lemma is proved.

Remark. Let C be a double cone with vertex at $z = 0$ in $\mathbb{C}$. It is obvious that one may replace the set $|z| \leq r_1$ in the lemma by the set $|z| \leq r_1$ and $z \in C$. In the following we shall use Lemma 2.2 modified or not modified without referring explicitly to it.

Now we give seven variants of exponential majorization. The first one is used in the proof of Theorem 1.1.

Lemma 2.3. Let m, l, and β be defined as in Theorem 1.1. Let M, r, r_1, r_2, a, and t' be positive constants such that $r < 1$, $a = r^{-1}e^2$, and $r_1 \leq a^{-1}$. Define

$$(2.4) \qquad \Omega' = \{z;\ z \in \mathbb{C}^n,\ |z_1|+|z_2| \leq r_1,\ |z_3|+\ldots+|z_n| \leq r_2,\ |y_1| \leq t'|z_1|,$$
$$|y_j| \leq t'|z''|,\ 3 \leq j \leq s\} \quad .$$

Let $d = (1+1/m,1,\dots,1) \in \mathbb{R}^n$. Let g be continuous in Ω' and let it be holomorphic in the inner points of Ω' such that $(D^\xi g)$ is extendable to a continuous function in Ω' for all ξ. Let

$$H_1(z,\xi) = \exp(-|z_1|^{-2m}-|z"|^{-2m}+a(|z_1|+|z_2|)(\xi d+1)) \quad .$$

If

$$(2.5) \qquad |D^\xi g| \le M|z"|^{-|\xi"|}r^{-|\xi|}\xi d^{\xi d-1}H_1(z,\xi),\ z \in \Omega',\ z_1 \neq 0,\ z" \neq 0,\quad \text{all } \xi \quad .$$

Then

$$(2.6) \qquad |D^{\xi+\alpha-\beta}g| \le M|z"|^{-|\xi"|-|\alpha"|}r^{-|\xi|}\xi d^{\xi d-1}H_1(z,\xi),\ z \in \Omega',\ z_1 \neq 0,\ z" \neq 0,$$

$$\text{all } \xi \text{ , if } |\alpha| = m \text{ and } \alpha_1 \le m-1 \text{ or if } |\alpha| \le m-1 \text{ ,}$$

where D_j^{-1} denotes integration in the z_j variable from the origin to z_j radially with all other variables fixed.

<u>Proof</u>. There are uniquely defined multi-indices μ and η with nonnegative components such that $\xi + \alpha - \beta = \eta - \mu$ where $\mu_j = 0$ if $\eta_j > 0$ and $\eta_j = 0$ if $\mu_j > 0$. We shall estimate

$$A = |D^{\xi+\alpha-\beta}g| = |D^{-\mu+\eta}g| \quad .$$

It follows from (2.5) that

$$A \le M|z"|^{-|\xi"|-|\alpha"|}r^{-|\eta|}\eta d^{\eta d-1}(a(\eta d+1))^{-|\mu|}H_1(z,\eta) \quad .$$

Now we notice that $|z_1| + |z_2| \le r_1 \le a^{-1}$ with $a = e^2/r$. So

$$A \le M|z"|^{-|\xi"|-|\alpha"|}r^{-|\xi|}\xi d^{\xi d-1}H_1(z,\xi)K$$

with

$$K = r^{-|\eta|+|\xi|+|\mu|}e^{-2|\mu|}(1+(\eta d-\xi d)/\xi d)^{\xi d-1}\eta d^{\eta d-\xi d-|\mu|}\exp(\eta d-\xi d) \quad .$$

Since $\alpha_1 \le \beta_1$ if $|\alpha| = |\beta| = m$ and $d_1 = 1 + 1/m$ one always has $|\eta| - |\xi| - |\mu| \le 0$ and $\eta d - \xi d - |\mu| \le 0$. Now $r < 1$ and $ar_1 < 1$ so $K < 1$. The lemma is proved.

The next lemma is used in the proof of Theorem 1.2.

Lemma 2.4. Let the hypothesis be as in Lemma 2.3 except that now $d = (1,\dots,1)$ and

(2.7) $$|D^{\xi}g| \le M|z_1|^{-\xi_1}|z''|^{-|\xi''|}r^{-\xi d}\xi d^{\xi d-1}H_2(z,\xi),\ z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,\ \text{all}\ \xi\ \text{with}\ \xi_2 = 0\ ,$$

where

$$H_2(z,\xi) = \exp((-|z_1|^{-2m}-|z''|^{-2m})(1-a|z_2|) + a(\xi d+1)(|z_1|+|z_2|))\ .$$

Then

(2.8) $$|D^{\xi+\alpha-\beta}g| \le M|z_1|^{-(\xi_1+\alpha_1-\beta_1)^+}|z''|^{-|\xi''+\alpha''|}r^{-\xi d}\xi d^{\xi d-1}H_2(z,\xi),$$
$$z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,\ \text{all}\ \xi\ \text{with}\ \xi_2 = 0\ \text{if}\ |\alpha| \le m\ ,$$

and

(2.9) $$|D^{\xi+\alpha-\beta}g| \le M|z_1|^{-\xi_1}|z''|^{-|\xi''|}r^{-\xi d}\xi d^{\xi d-1}H_2(z,\xi),\ z \in \Omega'\ .$$
$$z_1 \neq 0,\ z'' \neq 0,\ \text{all}\ \xi\ \text{with}\ \xi_2 = 0,\ \text{if}\ |\alpha| < m\ \text{and}\ \alpha_2 < 1\ .$$

Proof. Just as in the proof of Lemma 2.3 it here follows from (2.7) that

$$A \le |D^{\xi+\alpha-\beta}g| \le M|z_1|^{-\eta_1}|z''|^{-|\eta''|}r^{-\eta d}\eta d^{\eta d-1} \times$$
$$\times\ |D^{-\mu}H_2(z,\eta)| \le M|z_1|^{-(\xi_1+\alpha_1-\beta_1)^+}|z''|^{-|\xi''+\alpha''|}r^{-\xi d}\xi d^{\xi d-1}H_2(z,\xi)K\ ,$$

where for $|\alpha| \le m$ $K \le 1$. Here we have used $d = (1,\dots,1)$ and

$$|D_1^{-1}H_2(z,\eta)| \le \exp(-(1-a|z_2|)(|z_1|^{-2m} + |z''|^{-2m}) + (1+\eta d)|z_2|a) \times$$
$$\times \int_0^{|z_1|} \exp((1+\eta d)ta)dt\ .$$

The latter inequality is true since $a|z_2| < 1$ when $|z_2| \le r_1$. If $|\alpha| < m$ and $\alpha_2 < 1$ then one has at least one integration in the z_2 variable and

$$|D^{-\mu}H_2(z,\eta)| \le ((|z_1|^{-2m} + |z''|^{-2m} + (1+\eta d))a)^{-\mu_2}(a+a\eta d)^{-\mu_1}H_2(z,\eta) \le$$
$$\le (a+a\eta d)^{1-|\mu|}H_2(z,\eta)(\min(|z_1|^{2m},|z''|^{2m}))\ .$$

Here $\eta d - \xi d \le |\mu| - 1$. This gives (2.9). Lemma 2.4 is proved.

The lemma below is used in the proof of Theorem 1.3.

Lemma 2.5. Let the hypothesis be as in Lemma 2.3 except that now $d = (1,\dots,1)$ and

$$(2.10)\qquad |D^{\xi}g| \le M|z_1|^{-\xi_1-\beta_1}|z''|^{-|\xi''|}r^{-\xi d}\xi d^{\xi d-1}H_3(z,\xi),\ z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,$$

all ξ,

where

$$H_3(z,\xi) = \exp(-|z_1|^{-2m}-|z''|^{-2m} + a(1+\xi d)(|z_1|+|z_2|)) .$$

Then

$$(2.11)\qquad |D^{\xi+\alpha-\beta}g| \le M|z_1|^{-(\xi_1+\alpha_1-\beta_1)^+-\beta_1}|z''|^{-|\xi''+\alpha''|}r^{-\xi d}\xi d^{\xi d-1}H_3(z,\xi),\ z \in \Omega',$$

$z_1 \neq 0$, $z'' \neq 0$, all ξ, if $|\alpha| \le m$.

The proof of (2.11) is practically the same as that of (2.8).

If one does not integrate in the z_1 variable in the successive approximations then the following lemma can be useful.

Lemma 2.6. Let the hypothesis be as in Lemma 2.3 except that here $d = (1,\dots,1)$, $\Omega' = \{z;\ |y_j| \le t'|z'|,\ j = 1,3,\dots,s,\ |z_1| + |z_2| \le r_1,\ |z_3| +\dots+ |z_n| \le r_2\}$, and

$$(2.12)\qquad |D^{\xi}g| \le M|z'|^{-|\xi'|-\beta_1}r^{-\xi d}\xi d^{\xi d-1}H_4(z,\xi),\ z \in \Omega',\ z' \neq 0,\ \text{all } \xi,\ \text{if}$$

$|\alpha| \le m$,

where

$$H_4(z,\xi) = \exp(-|z'|^{-2m} + a(1+\xi d)|z_2|) .$$

Then

$$(2.13)\qquad |D^{\xi+\alpha-\beta}g| \le M|z'|^{-|\xi'+\alpha'|}r^{-\xi d}\xi d^{\xi d-1}H_4(z,\xi),\ z \in \Omega',\ z' \neq 0,\ \text{all } \xi,\ \text{if}$$

$|\alpha| \le m$ and $\alpha_1 \ge m-1$.

The proof is practically the same as that of (2.8) in Lemma 2.4.

Lemma 2.7. Let the hypothesis be as in Lemma 2.5 except that now g can be extended to a function in $C^k(\Omega')$ where k is an integer and $k \geq m$ and that the following is true

$$(2.14) \qquad |D^\xi g| \leq M|z_1|^{-\xi_1-\beta_1+k+1/2}|z''|^{-|\xi''|+k+1/2}r^{-\xi d}\xi d^{\xi d-1} \times$$
$$\times \exp(a(1+\xi d)(|z_1|+|z_2|)),\ z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,\ \text{all } \xi .$$

Then

$$(2.15) \qquad |D^{\xi+\alpha-\beta}g| \leq M|z_1|^{-(\xi_1+\alpha_1-\beta_1)^+-\beta_1+k+1/2}|z''|^{-|\xi''+\alpha''|+k+1/2}r^{-\xi d} \times$$
$$\times \xi d^{\xi d-1}\exp(a(1+\xi d)(|z_1|+|z_2|)),\ z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,\ \text{all } \xi,\ \text{if } |\alpha| \leq m$$

The proof of this lemma and that of Lemma 2.5 are almost identical.

Lemma 2.8. Let the hypothesis be as in Lemma 2.6 except that now

$$(2.16) \qquad |D^\xi g| \leq M|z'|^{-|\xi'|+k+1/2-m+l}r^{-\xi d}\xi d^{\xi d-1}\exp(a(1+\xi d)(|z_1|+|z_2|),\ z \in \Omega',$$
$$z' \neq 0,\ \text{all } \xi .$$

Then

$$(2.15) \qquad |D^{\xi+\alpha-\beta}g| \leq M|z'|^{-|\xi'+\alpha'|+k+1/2}r^{-\xi d}\xi d^{\xi d-1} \times \exp(a(1+\xi d)(|z_1|+|z_2|)),$$
$$z \in \Omega',\ z' \neq 0,\ \text{all } \xi,\ \text{if } |\alpha| \leq m \text{ and } \alpha_1 \geq m - l .$$

A minor adjustment of the proof of (2.8) gives the proof of this lemma.

Lemma 2.9. Let m, l, and β be defined as in Theorem 1.1. Let $r_1 > 0$, $r_2 > 0$, and $t' > 0$ be constants. Let $\Omega' = \{z;\ z \in \mathbb{C}^n,\ |y_1| \leq t'|z_1|,\ |z_1| + |z_2| \leq r_1,\ |z_3| +\ldots+ |z_n|<r_2\}$. Let g be a function holomorphic in the inner points of Ω' such that to each choice of $r > 0$ with $re^2r_1 \leq 1$ there is a constant C_r fulfilling

$$(2.16) \qquad |D^\xi g| \leq C_r(1+r\xi d)^{\xi d} \exp(-|z_1|^{-2m}+(1+r\xi d)(|z_1|+|z_2|)e^2),\ z \in \Omega',\ z_1 \neq 0,$$
$$\text{all } \xi .$$

with $d = (1+1/m,1,\ldots,1)$. Then one has

(2.17) $\quad |D^{\xi+\alpha-\beta}g| \le C_r(1+r\xi d)^{\xi d}\exp(-|z_1|^{-2m}+(1+r\xi d)(|z_1|+|z_2|)e^2),\ z \in \Omega',$

$z_1 \neq 0$, all ξ, if $|\alpha| = m$ and $\alpha_1 \le \beta_1$, or $|\alpha| < m$.

Proof. Just as in the proof of Lemma 2.3 using the same notation one sees that $\eta d - \xi d - |\mu| \le 0$ and $|\eta| - |\xi| - |\mu| \le 0$. One finds that (2.17) is true with C_r replaced by KC_r where

$$K = (1+r\xi d)^{-\xi d}(1+r\eta d)^{\eta d-|\mu|}e^{-2|\mu|+\eta d-\xi d} < 1 .$$

The lemma is proved. See also [18, Lemma].

3. FUNCTIONS WITH SINGULARITIES.

In this section we construct functions with singularities in $\mathbb{C}^n$. They are then used in the construction of solutions of the equation $P(z,D)u = 0$. Afterwards the restriction of these solutions to $\mathbb{R}^n$ gives the solutions wanted in the conclusions of the theorems in Section 1.

Let $m > 0$ be an integer. Let $z \in \mathbb{C}^n$.

We shall estimate the derivatives of

$$g(z) = \exp(-4(\sum_{j=1}^{n} z_j^2)^{-m}) \ , \ z \in \Omega_{t'}, \ , \tag{3.1}$$

where

$$\Omega_{t'} = \{z;\ |y_k| \le t'|z|,\ k = 1,\dots,n\} \tag{3.2}$$

for some small fixed $t' > 0$ to be defined later on.

One sees that

$$\sum_{j=1}^{n} z_j^2 = \sum_{j=1}^{n}(x_j+iy_j)^2 = \sum_{j=1}^{n}(x_j^2-y_j^2+2ix_jy_j) = |z|^2 - 2|y|^2 + \sum_{j=1}^{n}2ix_jy_j = A + iB$$

with A and B real. One sees that

$$|z|^2 \ge A \ge |z|^2(1-18t^2n) \quad\text{and}\quad |B| \le 6nt|z|^2,\ z \in \Omega_{3t} . \tag{3.3}$$

We choose t so small that $t < (12n)^{-1}$. Then $|B| < A$. We notice that

$$(A + iB)^{-m} = (A - iB)^{m}(A^2 + B^2)^{-m} .$$

So from (3.3) one has

$$\begin{aligned}&\operatorname{Re}\ (A+iB)^{-m} \geq (A^m-2^m A^{m-1}|B|)(A^2+B^2)^{-m} \geq \\ &\geq |z|^{-2m}(1-2^m 6nt(1-18t^2n)^{-1})(1+(6nt/(1-18t^2n))^2)^{-1} .\end{aligned}$$

It follows that we can choose a fixed t such that

$$\operatorname{Re}\ (A+iB)^{-m} \geq 3|z|^{-2m}/4,\ z \in \Omega_{3t} . \tag{3.5}$$

We estimate $D^{\xi}g$ when $z \in \Omega_t$. We integrate along

$$|\zeta_j - z_j| = t|z|,\ j = 1,\dots,n,\ z \in \Omega_t,\ z \neq 0, \tag{3.6}$$

in the Cauchy integral formula. Then $(1 + tn)|z| \geq |\zeta| \geq (1 - tn)|z|$ and

$$|\operatorname{Im} \zeta_j| \leq |\operatorname{Im} \zeta_j - y_j| + |y_j| \leq 2t|z|,\ z \in \Omega_t .$$

We choose $t < 1$ so small that $\zeta \in \Omega_{3t}$ and also so small that for $z \in \Omega_t$ the inequality of (3.5) is true with z replaced by ζ in the left member. This and the Cauchy formula then give

$$|D^{\xi}g| \leq \xi!(t|z|)^{-|\xi|}\exp(-2|z|^{-2m}),\ z \in \Omega_t,\ z \neq 0,\ \text{all}\ \xi . \tag{3.7}$$

Now let $g(z) = \exp((-4z_1^{-2m} - 4(\sum_{j=3}^{s} z_j^2)^{-m})$. Then the argument above gives

$$\begin{aligned}&|D^{\xi}g| \leq |z_1|^{-\xi_1}|z"|^{-|\xi"|}t^{-|\xi|}\xi!\exp(-2|z_1|^{-2m}-2|z"|^{-2m}), \\ &z \in \Omega"\ \text{with both}\ z_1 \neq 0\ \text{and}\ z" \neq 0,\ \text{all}\ \xi .\end{aligned} \tag{3.8}$$

Here $\Omega"$ is defined by

$$\Omega" = \{z;\ |y_1| \leq t|z_1|,\ |y_j| \leq t|z"|,\ j = 3,\dots,s\} . \tag{3.9}$$

One notices that the function

$$h(s) = s^{-j}\exp(-s^{-2m}),\ s > 0 ,$$

has its maximum at $s = (2m/j)^{1/2m}$ and that its value at this point is $j^{j/2m}(2me)^{-j/2m}$. Let $r = t$ and let $d = (1+1/2m,1,\dots,1)$. It follows from

(3.8) that for some $C > 0$

$$(3.10) \qquad |D^\xi g| \le C|z''|^{-|\xi''|} r^{-\xi_d} \xi_d^{\xi_d - 1} \exp(-|z_1|^{-2m} - |z''|^{-2m}),$$

$$z \in \Omega'' \text{ with both } z_1 \neq 0 \text{ and } z'' \neq 0 , \text{ all } \xi .$$

Let $k \ge m \ge 0$ be integers. Let

$$(3.11) \qquad g(z) = (\sum_{j=1}^{n} z_j^2)^{(2k+1)/4}, \ z \in \Omega_t, \ z \neq 0 .$$

We choose the branch in (3.11) such that $g(z)$ is real and positive when $z = x \neq 0$. Then g is holomorphic in the inner points of $\Omega_{t'}$, defined in (3.2) when t' is a small fixed number. Let $t = t'/4n$. We use Cauchy's integral formula and integrate along the circles of (3.6). Instead of (3.7) we now get

$$(3.12) \qquad |D^\xi g| \le C\xi! \ t^{-|\xi|} |z|^{-|\xi|+k+1/2}, \ z \in \Omega_t, \ z \neq 0, \ \text{all } \xi ,$$

for some C depending on t. In the same way we get

$$(3.13) \qquad |D^\xi g| \le C^2 \xi! \ t^{-|\xi|} |z_1|^{-\xi_1+k+1/2} |z''|^{-|\xi''|+k+1/2}, \ z \in \Omega_t ,$$

$$z_1 \neq 0, \ z'' \neq 0, \ \text{all } \xi ,$$

if

$$g(z) = (z_1^2)^{(2k+1)/4} (\sum_{j=3}^{s} z_j^2)^{(2k+1)/4}$$

and if C is chosen as in (3.12).

Let $\tilde{z} = (\tilde{z}_1, z_2, \ldots, z_n)$ be a point in the universal covering $\tilde{\Omega}$ of $\{z;\ z \in \mathbf{C}^n,\ z_1 \neq 0\}$. Let $g(\tilde{z})$ be a function holomorphic on $\tilde{\Omega}$. For each compact subset K of $\tilde{\Omega}$ there are constants M, r, and t such that

$$(3.14) \qquad |D^\xi g(\tilde{z})| \le Mt^{-\xi_1} r^{-|\xi|} \xi!, \ \tilde{z} \in K, \ \text{all } \xi .$$

Here $2t = \min_{\tilde{z} \in K} |z_1|$ and $M = \sup_{\tilde{z} \in K_t} |g(\tilde{z})|$, where K_t is the set of all points $\tilde{z} \in \tilde{\Omega}$ whose projection on $\{z;\ z_1 \neq 0,\ z \in \mathbf{C}^n\}$ has a distance not greater than t to the projection of K down to the same set and with $|\arg \tilde{z}_1| \le 2\pi + \max_{\tilde{\zeta} \in K} |\arg \tilde{\zeta}_1|$. Then the Cauchy formula gives

(3.14). It is obvious that for each fixed K there are an M and an r such that

$$(3.15)\qquad |D^{\xi}g(\tilde{z})| \leq M|z_1|^{-\xi_1} r^{-|\xi|}\xi! \ , \ z \in K, \quad \text{all} \quad \xi \ .$$

It is clear that for each K with common bounded argument and projection down to $\{z;\ z_1 \neq 0,\ |z_j| \leq 1,\ j = 1,\dots,n\}$ there is an M such that (3.15) is true with a common r independent of K. This is used in Persson [20], [21], and [22] to get

$$(3.16)\qquad |D^{\xi}g(\tilde{z})| \leq Cr^{-|\xi|}\xi d^{\xi d-1}, \ \tilde{z} \in K, \quad \text{all} \quad \xi \ .$$

for $d = (d_1,1,\dots,1)$ with $d_1 > 1$. Here C depends on K and r is independent of K as long as it is as the K just described.

4. DILATATIONS OF COORDINATES.

The proofs of the theorems in the introduction are all very similar after that the results of Section 2 have been established. The central point is the solution of a Goursat problem by successive approximations. We want to show that the mapping used in the successive approximations is in a certain sense a contraction mapping. This is achieved by performing the successive approximations in a new coordinate system.

We shall use the coordinate transformation

$$(4.1)\qquad z_1' = tt_1z_1,\ z_2' = tt_2z_2\ ,\ z_j' = z_j,\ j = 3,\dots,n\ .$$

Here we have suspended the general meaning of the primes for a moment. Let $D_j' = \partial/\partial z_j'$, $j = 1,\dots,n$. One notices that

$$\left(D^{\xi}u = t^{\xi_1+\xi_2}t_1^{\xi_1}t_2^{\xi_2}D'^{\xi}u\right) \ .$$

After multiplying $P(z,D)u = 0$ by $t^{-m}t_1^{-m+1}t_2^{-1}$ one gets

$$D'^{\beta}u + \sum_{\alpha \neq \beta} a_{\alpha}'(z')D'^{\alpha}u = 0$$

where

$$a'_\alpha(z') = t^{\alpha_1+\alpha_2-m} t_1^{\alpha_1-m+1} t_2^{\alpha_2-1} a_\alpha(z) \ . \tag{4.3}$$

If $P(z,D)$ satisfies the hypothesis of Theorem 1.1 the Cauchy formula and (1.2) give that there are positive constants r, r' and M_α such that

$$|D^\xi a_\alpha| \le M_\alpha r^{-|\xi|} \xi_d^{\xi_d - 1} |z''|^{(|\alpha''|-|\xi''|)^+}, \ |z| \le r', \text{ all } \xi \tag{4.4}$$

in the original coordinates. From (4.3) it follows that after deleting the primes in the new coordinate system

$$|D^\xi a_\alpha| \le M_\alpha t^{\alpha_1+\alpha_2-m-\xi_1-\xi_2} t_1^{\alpha_1-m+1-\xi_1} t_2^{\alpha_2-1-\xi_2} |z''|^{(|\alpha''|-|\xi''|)^+} \times \tag{4.5}$$
$$\times\, r^{-|\xi|} \xi_d^{\xi_d-1}, \ |z| \le r', \text{ all } \xi \ ,$$

if $t \ge 1$, $t_1 \ge 1$, and $t_2 \ge 1$. This we assume from now on. Let

$$M'_\alpha = t^{\alpha_1+\alpha_2-m} t_1^{\alpha_1-m+1} t_2^{\alpha_2-1} M_\alpha \ ,$$

We want to prove that after deleting the primes of M'_α (4.4) is true together with

$$\sum_{\alpha \neq \beta} M_\alpha < 2^{-3} \ . \tag{4.6}$$

for some fixed choice of t, t_1, t_2 . We choose $M_\alpha = 0$ if $|\alpha| = m$ and $\alpha_1 > m-1$. Then we choose t_2 so big that

$$\sum_{\substack{|\alpha|=m \\ \alpha_1 \ge m-1 \\ \alpha \neq \beta}} M_\alpha < 2^{-5} \ , \ t \ge 1, \ t_1 \ge 1 \ . \tag{4.7}$$

For such a fixed t_2 we now choose a t_1 such that

$$\sum_{\substack{|\alpha|=m \\ \alpha_1 < m-1}} M_\alpha \le 2^{-5} \ , \ t \ge 1 \ . \tag{4.8}$$

At last for these fixed t_1 and t_2 one chooses t so big that

$$\sum_{|\alpha|<m} M_\alpha \le 2^{-4} \ . \tag{4.9}$$

Then one has (4.6) in the new coordinate system.

Let d and $P(z,D)$ fulfill the hypothesis of Theorem 1.2. Then (1.4), (1.5), (1.6) and the Cauchy formula show that

$$(4.10)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}|z_1|^{(\alpha_1-m+1-\xi_1)^+}|z''|^{(|\alpha''|-|\xi''|)^+}r^{-\xi d}\xi d^{\xi d-1}, \ |z| \le r' ,$$
if $|\alpha| = m$ and $\alpha_2 \le 1$,

$$(4.11)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}r^{-\xi d}\xi d^{\xi d-1}, \ |z| \le r', \text{ all } \xi , \text{ if } |\alpha| < m \text{ and } \alpha_2 < 1 ,$$

and

$$(4.12)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}|z''|^{(|\alpha''|-|\xi''|)^+}r^{-\xi d}\xi d^{\xi d-1}, \ |z| \le r', \text{ all } \xi , \text{ if}$$
$|\alpha| < m$ and $\alpha_2 = 1$.

With the transformation (4.1) it follows from (4.10), (4.11) and (4.12) that after deleting the primes

$$(4.13)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}t^{\alpha_1+\alpha_2-m-(\alpha_1-m+1)^+}t_1^{\alpha_1-m+1-(\alpha_1-m+1)^+}t_2^{\alpha_2-1}r^{-\xi d}\xi d^{\xi d-1} \times$$
$$\times |z_1|^{(\alpha_1-m+1-\xi_1)^+}|z''|^{(|\alpha''|-|\xi''|)^+}, \text{ all } \xi , \text{ if } |\alpha| = m \text{ and } \alpha_2 \le 1,$$

$$(4.14)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}t^{\alpha_1+\alpha_2-m}t_1^{\alpha_1-m+1}t_2^{\alpha_2-1}r^{-\xi d}\xi d^{\xi d-1}, \ |z| \le r', \text{ all } \xi ,$$
if $|\alpha| < m$ and $\alpha_2 < 1$,

and

$$(4.15)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}t^{\alpha_1+\alpha_2-m}t_1^{\alpha_1-m+1}t_2^{\alpha_2-1}|z''|^{(|\alpha''|-|\xi''|)^+}r^{-\xi d}\xi d^{\xi d-1} ,$$
$|z| \le r'$, all ξ , if $|\alpha| < m$ and $\alpha_2 = 1$.

It is clear from (4.13), (4.14) and (4.15) that after deleting the primes in the new coordinate system (4.10), (4.11) and (4.12) are true with new constants M_{α} . By choosing t_2 big then t_1 big and at last t big one achieves that (4.7), (4.8) and (4.9) are true and thus also (4.6).

With d and $P(z,D)$ from Theorem 1.3 one gets from the Cauchy formula and (4.7) that for some M , r and r'

$$(4.16)\qquad |D^{\xi}a_{\alpha}| \le M_{\alpha}|z_1|^{(\alpha_1-m+1-\xi_1)^+}|z''|^{(|\alpha''|-|\xi''|)^+}r^{-\xi d}\xi d^{\xi d-1},$$
$|z| \le r'$, all ξ , if $|\alpha| \le m$.

In the coordinate system given by (4.1) one notices that the inequality of (4.13) is true for all α after that the primes have been deleted. Then by a suitable choice of t_2, t_1, and t we can find new constants M_α such that (4.16) and (4.6) are true in the new coordinate system.

Let $d = (1,\dots,1)$ and let $P(z,D)$ fulfill the hypothesis of Theorem 1.4. Then the Cauchy formula, and (1.8) give that

$$(4.17)\qquad |D^\xi a_\alpha| \le M_\alpha |z'|^{(\alpha_1-m+1+|\alpha''|-\xi_1-|\xi''|)^+} r^{-\xi d}\xi d^{\xi d-1}, \quad |z| \le r', \text{ all } \xi, \quad \text{if } |\alpha| \le m \text{ and } \alpha_1 \ge m-1,$$

for some constants M_α, r and r'.

Let $M_\alpha = 0$ if $\alpha_1 < m-1$. In (4.1) let $t = t_1 = 1$. From (4.17) one gets

$$(4.18)\qquad |D^\xi a_\alpha| \le t_2^{\alpha_2-1} |z'|^{(\alpha_1-m+1+|\alpha''|-\xi_1-|\xi''|)^+} r^{-\xi d}\xi d^{\xi d-1}, \quad |z| \le r', \quad \text{all } \xi, \text{ if } |\alpha| \le m \text{ and } \alpha_1 \ge m-1 .$$

in the new coordinate system after that the primes have been deleted. It follows the inequality (4.17) is true with new M_α and that these M_α can be chosen such that (4.7) and (4.9) and thus also (4.6) are true for some big t_2 in (4.1).

Let $P(z,D)$ fulfil the hypothesis of Theorem 1.5. Then the transformation used in the Theorem 1.1 case shows that in some coordinate system of this kind

$$(4.19)\qquad \sum_{\alpha\neq\beta} |a_\alpha| < 1/2 .$$

5. PROOF OF THEOREM 1.1.

Let $P(z,D)$ satisfy the hypothesis of Theorem 1.1. We assume that we have chosen a coordinate system such that (4.4) and (4.6) are true. We choose a new r, $0 < r < 1$, such that with this r both (4.4) and (3.8) are true. Then we choose $r_1 = \min(r'/2, re^{-2}/2)$ and $r_2 = r_1$. Let Ω' be as in Lemma 2.3 and let

(5.1) $\quad f = P(z,D)g$.

Then

$$|D^{\xi}f| = |\sum_{\alpha}\sum_{\nu\leq\xi}\binom{\xi}{\nu}(D^{\xi-\nu}a_{\alpha})(D^{\nu+\alpha}g)| \leq \sum_{\alpha}\sum_{\nu\leq\xi}\binom{\xi}{\nu}M_{\alpha}Cr^{-|\alpha|} \times$$

$$\times |z''|^{-|\xi''|}r^{-|\xi|}(\xi-\nu)d^{(\xi-\nu)d-1}(\nu+\alpha)^{(\nu+\alpha)d-1} \times \exp(-|z_1|^{-2m} - |z''|^{-2m}),$$

$$z \in \Omega',\ z_1 \neq 0,\ z'' \neq 0,\ \text{all}\ \xi .$$

Now we use Lemma 2.1 and the fact that there is a constant c such that $r^{-|\xi|}(\nu d+\alpha d)^{\alpha d} \leq c(r/2)^{-|\xi|}$, $\nu \leq \xi$ and $((\nu d+\alpha d)/\nu d)^{\nu d-1} \leq e^{2m}$, $\nu \leq \xi$, are true for all ξ . From this and from (4.6) it follows that

(5.2) $$|D^{\xi}f| \leq e^{2m}Ccr^{-m}|z''|^{-|\xi''|}(r/2)^{-|\xi|}\xi d^{\xi d-1}H_1(z,\xi),\ z \in \Omega',\ z_1 \neq 0,$$

$$z'' \neq 0,\ \text{all}\ \xi .$$

Here $H_1(z,\xi)$ is defined in Lemma 2.3.

We now write r instead of $r/2$. Still we can use Lemma 2.3 in Ω' because of our choice of r_1 . We let $v^0 = f$ and we let

(5.3) $$v^{j+1} = -\sum_{\alpha\neq\beta}a_{\alpha}D^{\alpha-\beta}v^j,\ j = 0,1,2,\dots .$$

We notice that (5.2) says that with our new r and with a new C

(5.4) $$|D^{\xi}v^j| \leq C2^{-j}|z''|^{-|\xi''|}r^{-|\xi|}\xi d^{\xi d-1}H_1(z,\xi),\ z \in \Omega',\ z_1 \neq 0,$$

$$z'' \neq 0,\ \text{all}\ \xi,\ j = 0,\ 1,\ 2,\dots ,$$

is true for $j = 0$. Then if it is true for $j = p$ (5.3) gives

$$|D^{\xi}v^{p+1}| = |\sum_{\alpha\neq\beta}D^{\xi}(a_{\alpha}D^{\alpha-\beta}v^p)| = |\sum_{\alpha\neq\beta}\sum_{\nu\leq\xi}\binom{\xi}{\nu}(D^{\xi-\nu}a_{\alpha})(D^{\nu+\alpha-\beta}v^p)| .$$

Now (5.4), (4.4) and Lemma 2.3 give

$$|D^{\xi}v^{p+1}| \leq C2^{-p}\sum_{\alpha\neq\beta}\sum_{\nu\leq\xi}M_{\alpha}\binom{\xi}{\nu}|z''|^{|\xi''|}r^{-|\xi|}(\xi-\nu)d^{(\xi-\nu)d-1} \times \nu d^{\nu d-1}H_1(z,\xi) .$$

Now we use Lemma 2.1 and (4.6). Then we see that (5.4) is true for $j = p + 1$ too. Thus it is always true.

Let $v = \sum_{j=0}^{\infty}v^j$. Then v like all its derivatives is represented by a series

which converges uniformly on all compact subsets of inner points of Ω'. It is obvious that v solves $v + \sum_{\alpha \neq \beta} D^{\alpha-\beta} v = f$. Let $u = g - D^{-\beta} v$. It is obvious that $P(z,D)u = 0$ in Ω' when all derivatives of v and g are extended to continuous functions on Ω'. The restriction of u to $\mathbb{R}^n$ is then in C^∞ and u is analytic when both $x_1 \neq 0$ and $x'' \neq 0$. One notices that $D^{-\beta}v(x) = 0$ if $x_2 = 0$ but that $g(x) \neq 0$ when both $x_1 \neq 0$ and $x'' \neq 0$. So $0 \in \operatorname{supp} u$. One further notices that $D^\xi u(x) = 0$ for all ξ, if $x_1 = 0$ or $x'' = 0$.
The proof of Theorem 1.1 is complete.

6. THE PROOFS OF THEOREM 1.2 TO THEOREM 1.5.

The proofs of Theorem 1.2 to Theorem 1.5 are very similar to the proof of Theorem 1.1. As to Theorem 1.2 one chooses a coordinate system such that (4.10), (4.11), (4.12), and (4.6) are true. Then one chooses a g fulfilling (3.8) in the inner points of Ω''. One lets $f = P(z,D)g$ and finds that for some r (2.7) is fulfilled when g is replaced by f. At the same time the conditions (4.10), (4.11), (4.12) and (4.6) remain true with the new r. Then one chooses $r_1 = r_2 = \min(r'/2, r/2a)$. The procedure of Section 5 is then repeated. We construct a solution of $P(z,D)D^{-\beta}v = f$ by successive approximations. To prove the convergence we now use Lemma 2.4. We do not write down the proof.

For the first part of the proof of Theorem 1.3 we use Lemma 2.5 instead of Lemma 2.3 from the proof of Theorem 1.1. Otherwise the proofs are identical. We just use (3.8) instead of (3.10). For the second part we use the g of (3.13) and Lemma 2.7. It is clear from the estimates that u have all the wanted properties. The proof is not written down.

Theorem 1.4 has a proof where one chooses coordinates such that (4.17) and (4.6) are true. One chooses g such that (3.7) or (3.12) are true with z

replaced by z' in the estimate and with $\Omega_t = \{z;\ |y_j| \le t|z'|,\ j = 1,3,\dots,s\}$. Then one uses Lemma 2.6 or Lemma 2.8. We do not write down the proof.

For the proof of Theorem 1.5 we choose a coordinate system such that (4.19) is true. We use g with the estimate (3.10) for $s = 1$. We fix an arbitrary constant r_1 and let $r_2 = r_1$. Then we take an $r < r_1 e^{-2}$, let $f = P(z,D)g$ and solve $P(z,D)D^{-\beta}v = f$ with this arbitrary r_1. Use Lemma 2.9 in the estimation of the successive approximations. The solution is unique and r_1 is arbitrary. So the global solution exists. Then one just repeats the argument of the last part of the proof of Theorem 1.1 to get the conclusion of Theorem 1.5.

7. SINGULAR SOLUTIONS IN $\mathbb{C}^n$.

We have seen that the nullsolutions in Theorem 1.1 to Theorem 1.5 are restrictions of singular solutions of $P(z,D)u = 0$ in $\mathbb{C}^n$. Let $s = 1$. Then one sees that the result in Persson [23, Theorem 4.1] applies to an operator fulfilling the hypothesis of Theorem 1.1. See also Persson [25, Theorem 1.2].

If one adopts the procedure in Persson [22] one can get a singular solution in $\tilde{\Omega} = \{\tilde{z};\ z_1 \neq 0,\ |z_j| \le r',\ j = 1,\dots,n\}$. Here $\tilde{z}$ is any but a fixed point in $\tilde{\Omega}$ with projection z down to $\Omega = \{z;\ z_1 \neq 0,\ |z_j| \le r',\ j = 1,\dots,n\}$. For the corresponding global result see Persson [24, Theorem 1] which corresponds to Theorem 1.5. We think that some small modification of the procedure in [22] in the Theorem 1.2 or Theorem 1.3 case would work equally well. The operator from Theorem 1.4 will not cause problems since there is no integration in the z_1 variable in the successive approximations in the existence proof. We do not go into more details here.

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Jan PERSSON,
Institutt for matematiske realfag, Breivika,
Universitetet i Tromsø,
Postboks 953,
N-9000 Tromsø,
Norway.