W. ABRAMCZUK

A remark on ellipticity of systems of linear partial differential equations with constant coefficients

Astérisque, tome 89-90 (1981), p. 117-128
<http://www.numdam.org/item?id=AST_1981__89-90__117_0>

A REMARK ON ELLIPTICITY OF SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

by W. ABRAMCZUK (University of Stockholm)

INTRODUCTION.

It is well known (and easy to prove) that a linear partial differential operator with constant coefficients, $P(D)$, is elliptic and has order N if and only if it is a bounded operator with closed range when it acts between the spaces $H_o^m(\Omega)$ and $H_o^{m-N}(\Omega)$, where $H_o^k(\Omega)$ denotes the closure of $C_o^\infty(\Omega)$ in the norm $|\phi|_k = \sum_{|\alpha|\leqq k} |D^\alpha \phi|_{L^2}$, m is an integer such that $m-N \geqq 0$ and Ω is a bounded open subset of R^n.

Here this result is (partially) extended to systems of linear partial differential operators with constant coefficients and to more general spaces of distributions.

Theorem 1 below is a rather straightforward generalisation of the considerations in 10.6 of Hörmander [1]. The first part of Theorem 2 is an easy consequence of the coercivity results in Smith [1] and the second part was inspired by a counter example in Eskin and Shamir [1].

NOTATION AND DEFINITIONS

To measure the regularity of distributions we use the spaces $L_s^p = L_s^p(R^n)$,

$1 < p < \infty$, $s \in R$, of Bessel potentials of L^p functions (Calderon [1]): $u \in L^p_s$ if u is a temperate distribution and $(1 + |\xi|^2)^{s/2} . \hat{u}$ is the Fourier transform of a L^p function denoted here by $J^{-s}u$. We let J^{-s} transport the L^p norm to L^p_s: $|u|_{L^p_s} = |J^{-s}u|_{L^p}$. When u is a test function this can be made more explicit:

$$|u|_{L^p_s} = |(2\pi)^{-n} \int (1 + |\xi|^2)^{s/2} \hat{u}(\xi) e^{ix\xi} d\xi |_{L^p} .$$

When Ω is an open subset of R^n we let $L^p_{s,\bar{\Omega}}$ denote the distributions in L^p_s supported in $\bar{\Omega}$ and we put $L^p_s(\Omega) = L^p_s / L^p_{s,R^n\setminus\bar{\Omega}}$ which we think of as the restriction of L^p_s to Ω . For technical reasons we assume in what follows that Ω is also bounded and convex.

When $r = (r_1,\dots,r_K) \in R^K$ we denote the product space $L^p_{r_1} \times \dots \times L^p_{r_K}$ by L^p_r , the space $L^p_{r_1,\bar{\Omega}} \times \dots \times L^p_{r_K,\bar{\Omega}}$ by $L^p_{r,\bar{\Omega}}$, etc.

By $P(D)$ we denote a matrix of linear differential operators with constant coefficients: $P(D) = (P_{jk}(D))$, $j = 1,\dots,J$, $k = 1,\dots,K$, and by ${}^tP(D)$ the transpose of $P(-D)$.

<u>Definition 1</u>: The operator $P(D)$ is <u>determined</u> if $P(D)u = 0$ has no non-trivial solutions with compact support (i.e. $P(D):\mathscr{E}'^K \longrightarrow \mathscr{E}'^J$ is injective).

<u>Definition 2</u>: Let r_k and s_j, $k = 1,\dots,K$, $j = 1,\dots,J$, be real numbers such that $r_k - s_j$ are non-negative integers. We call the operator $P(D)$ <u>(r_k-s_j)-elliptic</u> if

i) $\deg P_{jk} \leqq r_k-s_j$

ii) $\operatorname{rank} (\overset{\circ}{P}_{jk}(\xi)) = K$ if $0 \neq \xi \in R^n$;

here $\overset{\circ}{P}_{jk}$ denotes the part in P_{jk} of degree r_k-s_j .

If i) and

ii)' rank $(\overset{\circ}{P}_{jk}(\zeta)) = K$ if $0 \neq \zeta \in C^n$

are satisfied we call $P(D)$ <u>(r_k-s_j)-very strongly elliptic</u>.

This definition of (r_k-s_j)-ellipticity was given in Douglis and Nirenberg [1]. See also Hörmander [1], Ch.X. Systems (r_k-s_j)-v.s. elliptic in a similar sense were studied in Smith [1].

<u>Remark</u>: It is easy to see that (r_k-s_j)-ellipticity implies the usual one defined, for example, in terms of the characteristic variety of $P(D)$ and that (r_k-s_j)-v.s. ellipticity implies that the characteristic variety is discrete. The converse is obviously not true and it is an open problem whether an elliptic $P(D)$ (a $P(D)$ with discrete characteristic variety) becomes (r_k-s_j)-elliptic ((r_k-s_j)-v.s. elliptic) when multiplied by a non-singular $K \times K$ - matrix with differential operator entries.

<u>Definition 3</u>: Let $1 \leqq m < n$. Consider $R^n = R^m + R^{n-m}$ and write $x = (x',x'')$, $D = (D',D'')$ with the obvious meaning. We say that $P(D)$ is of <u>tensor product type</u> if $P(D) = (P^1(D'),P^2(D'')) = (P_1(D'),\ldots,P_I(D'),P_{I+1}(D''),\ldots,P_J(D''))$ is a row matrix with all polynomials P_j, $1 \leqq j \leqq J$, homogeneous of degree $N > 0$ with no non-trivial relations (i.e. every relation $\Sigma P_j Q_j = 0$, Q_j polynomials, is of the form $P_j P_k - P_k P_j = 0$).

THEOREMS

Let $r = (r_k)_{k=1,\ldots,K}$, $s = (s_j)_{j=1,\ldots,J}$. Consider the condition

(*) $\quad P(D): L^p_{r,\overline{\Omega}} \longrightarrow L^p_{s,\overline{\Omega}}$ is bounded with closed range.

<u>Theorem 1</u>: For determined $P(D)$ (*) and (r_k-s_j)-ellipticity are equivalent.

<u>Theorem 2</u>: (*) is implied by (r_k-s_j)-v.s. ellipticity of ${}^tP(D)$. The converse is true (at least) when $P(D)$ is of tensor product type and $\Omega = \Omega' \times \Omega''$, where Ω' and Ω'' are open, bounded and convex sets of R^m and R^{n-m} respectively.

<u>Proof of Theorem 1</u>: We first prove that <u>(*) implies (r_k-s_j)-ellipticity</u>: (*) and the injectivity of $P(D)$ give the estimate

(1) $$C^{-1}\cdot \sum_j \Big| \sum_k P_{jk}(D)u_k \Big|_{L^p_{s_j}} \leqq \sum_k |u_k|_{L^p_{r_k}} \leqq C \cdot \sum_j \Big| \sum_k P_{jk}(D)u_k \Big|_{L^p_{s_j}}$$

for some constant $C > 0$ and all $u_k \in C_o^\infty(\Omega)$. If we in the first inequality put all u_k but one equal to zero we get

(2) $$|P_{jk}(D)u|_{L^p_{s_j}} \leqq C \cdot |u|_{L^p_{r_k}}$$

for all $u \in C_o^\infty(\Omega)$, $1 \leqq j \leqq J$, $1 \leqq k \leqq K$. Putting $u(x) = e^{i\lambda x\eta}\cdot \phi(x)$ with $\lambda > 0$, $0 \neq \eta \in R^n$, $0 \neq \phi \in C_o^\infty(\Omega)$ and using Lemma A7 from the Appendix one easily checks that (2) implies $\deg P_{jk} \leqq r_k - s_j$.

We now show that $\operatorname{rank}\,(\mathring{P}_{jk}(\eta)) < K$ for some $0 \neq \eta \in R^n$ violates the second estimate in (1):

if $\operatorname{rank}\,(\mathring{P}_{jk}(\eta)) < K$ then, for some $0 \neq (a_1,\ldots,a_K) \in C^K$, $\Sigma_k \mathring{P}_{jk}(\eta)a_k = 0$, $j = 1,\ldots,J$, or, since $\mathring{P}_{jk}$ are homogeneous of degree $r_k - s_j$,

(3) $$\sum_k \mathring{P}_{jk}(\lambda\eta)\cdot\lambda^{-r_k}\cdot a_k = 0 \qquad j = 1,\ldots,J .$$

Now put in the second estimate in (1) $u_k^\lambda(x) = \lambda^{-r_k}\cdot a_k \cdot e^{i\lambda x\eta}\cdot\phi(x)$, $0 \neq \phi \in C_o(\Omega)$. By lemma A7, as $\lambda \to \infty$,

(4) $$\Sigma_k |u_k^\lambda|_{L^p_{r_k}} = \Sigma_k \lambda^{-r_k}\cdot|a_k|\cdot|e^{i\lambda x\eta}\cdot\phi|_{L^p_{r_k}} \longrightarrow \Sigma_k |a_k|\cdot|\phi|_{L^p} > 0 .$$

At the same time it is easy to see that

(5) $$\Sigma_k |P_{jk}(D)u_k^\lambda|_{L^p_{s_j}} = 0(\lambda^{-1}) \quad \text{as} \quad \lambda \to \infty,\ j = 1,\ldots,J .$$

Namely, in

$$|\Sigma_k P_{jk}(D)u_k^\lambda|_{L^p_{s_j}} \leqq |\Sigma_k \mathring{P}_{jk}(D)u_k^\lambda|_{L^p_{s_j}} + |\Sigma_k (P_{jk} - \mathring{P}_{jk})(D)u_k^\lambda|_{L^p_{s_j}}$$

the second term of the right hand side is $O(\lambda^{-1})$ by Lemma A7, and as for the first term, observe that

$$\Sigma_k \overset{\circ}{P}_{jk}(D)u_k^\lambda = \Sigma_k \overset{\circ}{P}_{jk}(\lambda\eta)\cdot\lambda^{-r_k}\cdot a_k\cdot e^{i\lambda x\eta} + \Sigma_k\Sigma_l\ \lambda^{-r_k}\cdot\ Q_{jk}^l(\lambda\eta)\ e^{i\lambda x\eta}\cdot\psi_{lk}$$

for some homogeneous polynomials Q_{jk}^l of degree l, $1 \leqq l < r_k - s_j$, and some $\psi_{lk} \in C_o^\infty(\Omega)$, and then use (3) and Lemma A7.

This ends the proof of the first part of Theorem 1 since (4) and (5) clearly contradict (1).

We now prove that <u>(r_k-s_j)-ellipticity implies (*)</u>: that $P(D)$ in (*) is bounded is trivially clear because each $P_{jk}(D)\colon L^p_{r_k,\overline{\Omega}} \to L^p_{s_j,\overline{\Omega}}$ is bounded if $\deg P_{jk} \leqq r_k - s_j$.

To show that $P(D)$ has closed range we first observe that the set $(P(D)\mathscr{E}'^K_{\overline{\Omega}}) \cap L^p_{s,\overline{\Omega}}$ is closed in $L^p_{s,\overline{\Omega}}$ (it is well known that $P(D)\mathscr{E}'^K_{\overline{\Omega}}$ is closed in $\mathscr{E}'^J_{\overline{\Omega}}$ and the topology of $L^p_{s,\overline{\Omega}}$ is stronger than that of $\mathscr{E}'^J_{\overline{\Omega}}$) and then we prove that $(P(D)\mathscr{E}'^K_{\overline{\Omega}}) \cap L^p_s = P(D)L^p_{r,\overline{\Omega}}$.

<u>Proposition</u>: If $u_k \in \mathscr{E}'$, $k = 1,\dots,K$, $\Sigma_k P_{jk}(D)u_k = f_j \in L^p_{s_j}$, $j = 1,\dots,J$, and $P(D)$ is (r_k-s_j)-elliptic, then $u_k \in L^p_{r_k}$.

<u>Proof</u>: First we show how we can reduce the proof to the case of a square system, then we prove that case.

Denote by Δ the polynomial $\sum_{k=1}^n x_k^2$ and the corresponding differential operator: $\Delta = \Delta(D) = \sum_{k=1}^n D_k^2$. Let N be some integer $\geqq \max_{j,k}(r_k-s_j)$. From $\Sigma_k P_{jk}(D)u_k = f_j$, $j = 1,\dots,J$, we get

$$\Sigma_j \overline{P}_{jl}(D)\Delta^{N-(r_l-s_j)}\ \Sigma_k P_{jk}(D)u_k = \Sigma_j \overline{P}_{jl}(D)\Delta^{N-(r_l-s_j)} f_j,\quad l = 1,\dots,K\quad ,$$

where $\overline{P}_{jk}$ denotes the polynomial obtained by complex conjugation of the coefficients of P_{jk}. After changing the order of summation and putting

$Q_{lk} = \Sigma_j \overline{P}_{jl} P_{jk} \Delta^{N-(r_l - s_j)}$ and $\phi_l = \Sigma_j \overline{P}_{jl}(D) \Delta^{N-(r_l-s_j)} f_j$ we see that u_k, $k = 1,\dots,K$, satisfy a square system of differential equations:

(6) $$\Sigma_k Q_{lk}(D) u_k = \phi_l, \quad l = 1,\dots,K \ .$$

This system is $(2N+r_k-r_l)$-elliptic: $\deg Q_{lk} \leqq \deg (\overline{P}_{jl} P_{jk} \Delta^{N-(r_l-s_j)}) \leqq (r_l-s_j)+(r_k-s_j)+2N-2(r_l-s_j) = 2N+r_k-r_l$. Denote by $\mathring{Q}_{lk}$ the part in Q_{lk} of degree $2N+r_k-r_l$ and observe that $\mathring{Q}_{lk} = \Sigma\, \overline{\mathring{P}}_{jl} \mathring{P}_{jk} \Delta^{N-(r_l-s_j)}$. Now, if rank $(\mathring{Q}_{lk}(\xi)) < K$ for some $0 \neq \xi \in R^n$, then for some $0 \neq (a_1,\dots,a_K) \in C^n$

$$\Sigma_k \Sigma_j \overline{\mathring{P}}_{jl}(\xi) \mathring{P}_{jk}(\xi) |\xi|^{2(N-(r_l-s_j))} \cdot a_k = 0 \, , \quad l = 1,\dots,K \ .$$

Putting $\xi = \lambda\cdot\eta$, $\lambda > 0$, $|\eta| = 1$, this gives

(7) $$\Sigma_k \Sigma_j \overline{\mathring{P}}_{jl}(\eta) P_{jk}(\eta) \cdot (\lambda^{r_k} \cdot a_k) = 0 \ , \qquad l = 1,\dots,K \ .$$

On the other hand it is easy to show that if rank $(\mathring{P}_{jk}(\eta)) = K$ then also rank $(\Sigma_j \overline{\mathring{P}}_{jl}(\eta) \mathring{P}_{jk}(\eta)) = K$ what clearly contradicts (7) thus proving that rank $(Q_{lk}(\xi)) = K$ if $0 \neq \xi \in R^n$.

Observe now that ϕ_l in (6) are in $L^p_{r_l-2N}$ and so, if the Proposition is true in the special case when $K = J$, it follows that

$$u_k \in L^p_{r_l-2N+(2N+r_k-r_l)} = L^p_{r_k}$$

and the Proposition is true in the general case.

So assume from now on that $K = J$. Let $\phi \in C_o^\infty(R^n)$ be $= 1$ on the (real) zeroes of $\det (P_{jk})$. This is possible since the set is bounded. Using the matrix notation we then have

$$\hat{u} = (\phi+(1-\phi)P^{-1}P)\cdot\hat{u} = \phi\cdot\hat{u} + (1-\phi)P^{-1}f\hat{\ } \ ,$$

or, denoting by $({}^{co}P_{kj})$ the matrix formed by the cofactors in (P_{jk}),

$$\hat{u}_k = \phi\cdot\hat{u}_k + \Sigma_j(1-\phi)\cdot\frac{{}^{co}P_{kj}}{\det P}\cdot f_j\hat{\ }, \quad k = 1,\dots,K \quad . \tag{8}$$

Now put $m_s(\xi) = (1+|\xi|^2)^{s/2}$. It follows from (8) that

$$m_{r_k}\cdot\hat{u}_k = \phi\cdot m_{r_k}\cdot\hat{u}_k + \Sigma_j(1-\phi)\,\frac{{}^{co}P_{kj}}{\det P}\cdot m_{r_k-s_j}\cdot m_{s_j}\cdot\hat{f}_j\ , \quad k = 1,\dots,K \quad . \tag{9}$$

The first term in each of the sums of (9) is a C_o^∞ function and therefore its inverse Fourier transform is in every L^p, $1 \leqq p \leqq \infty$. The inverse Fourier transform of every other term is also in L^p, $1 < p < \infty$, because the functions

$$(1-\phi)\,\frac{{}^{co}P_{kj}}{\det P}\,m_{r_k-s_j}$$

are easily seen to be multipliers on L^p, $1 < p < \infty$.

This ends the proof of the Proposition and thus of Theorem 1.

Proof of Theorem 2. (r_k-s_j)-v.s. ellipticity of ${}^tP(D)$ implies (*): $\deg p_{jk} = \deg ({}^tP)_{kj} \leqq r_k-s_j$ $(= -s_j-(-r_k))$ and so $P(D)$ is bounded. To see that $P(D)$ has closed range it is enough to see that the adjoint of $P(D)$, ${}^tP(D): L^q_{-s}(\Omega) \longrightarrow L^q_{-r}(\Omega)$, $1/p + 1/q = 1$, has closed range. Now, it is a well known fact that, for any ${}^tP(D)$, ${}^tP(D)\mathscr{D}'(\Omega)^J$ is closed in $\mathscr{D}'(\Omega)^K$ and since $L^q_{-r}(\Omega) \subset \mathscr{D}'(\Omega)^K$ topologically, $({}^tP(D)\mathscr{D}'(\Omega)^J) \cap L^q_{-r}(\Omega)$ is closed in $L^q_{-r}(\Omega)$. But $({}^tP(D)\mathscr{D}'(\Omega)^J) \cap L^q_{-r}(\Omega) = {}^tP(D)L^q_{-s}(\Omega)$ when ${}^tP(D)$ is (r_k-s_j)-v.s. elliptic by Theorem 8.15 in Smith [1].

(*) implies (r_k-s_j)-v.s. ellipticity of ${}^tP(D)$ when $P(D)$ is of tensor product type and $\Omega = \Omega' \times \Omega''$: by duality this amounts to proving the following assertion:

Let $P(D): u \to (P^1(D')u, P^2(D'')u) = (P_1(D')u,\dots,P_I(D')u,P_{I+1}(D'')u,\dots,P_J(D'')u)$ as an operator from $L^p_s(\Omega)$ to $L^p_{s-N}(\Omega)^J$ have closed range for some $s \in R$,

$1 < p < \infty$. Then the polynomials P_j, $j = 1,\dots,J$, have no common complex non-trivial zero.

Now the proof goes as follows: for any $g \in C_o(\Omega')$, $P^1(D')g \neq 0$, and $h \in L^p_{s-N}(\Omega'')$, $P^2(D'')h = 0$, put $f = (P^1(D')g \otimes h, 0)$, i.e. $f_j = P_j(D')g \otimes h$ if $1 \leqq j \leqq I$ and $f_j = 0$ if $j > I$. Using Lemma A4 of the Appendix, one easily checks that f is in the closure of the range of $P(D)$. Assume now that $f = P(D)u$ for some $u \in L^p_s(\Omega)$. Since also $f = P(D)(g \otimes h)$, we must have

$$(10) \qquad u = g \otimes h + v$$

for some $v \in \mathscr{D}'(\Omega)$, $P(D)v = 0$. Let $\phi_1 \in C^\infty_o(\Omega')$ separate g and the kernel of $P^1(D')$ in $\mathscr{D}'(\Omega')$: $(g,\phi_1) = 1$ and $(v_1,\phi_1) = 0$ if $P^1(D')v_1 = 0$. Apply to (10) the operator T_{ϕ_1} of Lemma A5: it follows that $h = T_{\phi_1}(u)$ and is thus in $L^p_s(\Omega'')$. In this way the assumption that the range of $P(D)$ is closed leads to the implication: if $h \in L^p_{s-N}(\Omega'')$ and $P^2(D'')h = 0$, then $h \in L^p_s(\Omega'')$. By Lemma A6 it now follows that the polynomials P_j, $I+1 \leqq j \leqq J$, have no common non-trivial complex zero. If we let P^1 and P^2 change roles we see that the same is also true about P_j , $1 \leqq j \leqq I$. Theorem 2 is proved.

Example: Let $P(D)$ in the proof of the second part of Theorem 2 be the $\bar\partial$ operator and let Ω be a polydisc in $C^n \cong R^{2n}$. The result is that $\bar\partial u = f$ cannot, in general, be solved with gain of one derivative in the L^p_s-space meaning in a polydisc in C^n , $n > 1$.

APPENDIX

We first introduce some additional notation. The set of exponential polynomials in R^n is denoted by $EXP(R^n) = EXP$. Given two matrices of polynomials P and Q , we say that Q is a compatibility matrix for P if the rows of Q

generate the module of relations between the rows of P. We call a matrix of polynomials <u>homogeneous</u> if all its elements are homogeneous of the same degree. By Φ_P we denote the set of solutions to $P(D)u = 0$ in a space Φ of (tuples of) distributions; when Φ is a cartesian product of K copies of some space Ψ, we write Ψ_P instead of $(\Psi^K)_P$. For $\phi \in C_0^\infty(R^n)$ and $\varepsilon > 0$ define $\phi_\varepsilon(x) = \phi(\varepsilon x)$ and $\phi^\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$. For a distribution u define u_ε by $(u_\varepsilon,\phi) = (u,\phi^\varepsilon)$, $\phi \in C_0(R^n)$. Note that when u is a tempered distribution we have $u_\varepsilon\hat{} = \hat{u}^\varepsilon$.

<u>Lemma A1</u>: For $1 < p < \infty$ and $s \in R$ $f \to f_\varepsilon$ is a bounded operator in L^p_s and $|f_\varepsilon - f|_{L^p_s} \to 0$ as $\varepsilon \to 1$.

<u>Lemma A2</u>: Let $\phi \in C_0^\infty(R^n)$, $\phi \geqq 0$, $\int\phi = 1$, $1 < p < \infty$, $s \in R$. Then $f \to \phi^\delta * f$ is a bounded operator in L^p_s and $|f - \phi^\delta * f|_{L^p_s} \to 0$ as $\delta \to 0$.

These lemmas are common knowledge when s is a non-negative integer; for the general case see Abramczuk [1].

<u>Lemma A3</u>: For a homogeneous matrix Q (the restriction to Ω of) EXP_Q is dense in $L^p_s(\Omega)_Q$ <u>(in the topology of $L^p_s(\Omega)$)</u>.

<u>Proof</u>: Consider the inclusions: $EXP_Q|_\Omega \subset \bigcup_{\Omega\subset\subset\Omega'} C^\infty(\Omega')_Q|_\Omega \subset L^p_s(\Omega)_Q$. The range of the first one is dense in the $C^\infty(\bar{\Omega})$-topology by the known (local) density results. We show that the range of the second inclusion is dense if Q is homogeneous: given $u \in L^p_s(\Omega)_Q$ it is clear that $Q(D)u_\varepsilon = \varepsilon^{\deg Q}\cdot(Q(D)u)_\varepsilon$ so $u_\varepsilon \in L^p_s(\varepsilon^{-1}\cdot\Omega)_Q$ and $\Omega \subset\subset \varepsilon^{-1}\cdot\Omega$ if $\varepsilon < 1$ and $0 \in \Omega$ what can be assumed without loss of generality. With ϕ like in Lemma A2 $u_\varepsilon * \phi^\delta \in C^\infty(\Omega_{\varepsilon,\delta})_Q$ for some $\Omega_{\varepsilon,\delta} \supset\supset \Omega$ if δ is small enough. The proof ends by using the two preceding lemmas on

$$|u_\varepsilon * \phi^\delta - u|_{L^p_s(\Omega)} \leqq |u_\varepsilon * \phi^\delta - u_\varepsilon|_{L^p_s} + |u_\varepsilon - u|_{L^p_s} .$$

<u>Lemma A4</u>: Let $P(D): L^p_r(\Omega) \to L^p_s(\Omega)$, $r \in R^K$, $s \in R^J$. If P has a homogeneous compatibility matrix Q then the range of $P(D)$ is dense in $L^p_s(\Omega)_Q$.

Proof: $EXP_Q|_\Omega = P(D)(EXP^K|_\Omega) \subset P(D)L^p_r(\Omega) \subset L^p_s(\Omega)_Q$ and use Lemma A3.

Lemma A5: Let $\Omega = \Omega' \times \Omega''$ be like in Theorem 2. For $\phi_1 \in C^\infty_o(\Omega')$ and $u \in \mathscr{D}'(\Omega)$ let $T_{\phi_1}(u)$ be the linear functional on $C^\infty_o(\Omega'')$ defined by $T_{\phi_1}(u): \phi_2 \to (u, \phi_1 \otimes \phi_2)$. Then

i) the operator $u \to T_{\phi_1}(u)$ maps $L^p_s(\Omega)$ into $L^p_s(\Omega'')$

ii) if $P(D): u \to (P^1(D')u, P^2(D'')u)$ and ϕ_1 vanishes on $\mathscr{D}'(\Omega')_{P^1}$ then T_{ϕ_1} vanishes on $\mathscr{D}'(\Omega)_P$.

Proof: We only prove ii): it is easily seen that $\phi_1 = \Sigma_j P_j(-D')\psi_{1_j}$ for some $\psi_{1_j} \in C^\infty_o(\Omega')$. But then, if $v \in \mathscr{D}'(\Omega)_P$, $(v, \phi_1 \otimes \phi_2) = (v, (\Sigma_j P_j(-D')\psi_{1_j}) \otimes \phi_2) = \Sigma_j (v, P_j(-D')(\psi_{1_j} \otimes \phi_2)) = \Sigma_j (P_j(D')v, \psi_{1_j} \otimes \phi_2) = 0$.

Lemma A6: Let $1 < p < \infty$ and $s \in R^K$. If for some K-tuple of positive integers $N = (N_1,\dots,N_K)$ every solution to $P(D)u = 0$ in $L^p_s(\Omega)$ is actually in $L^p_{s+N}(\Omega)$ then the linear space of distribution solutions to $P(D)u = 0$ in Ω is finite dimensional.

Proof: The assumption implies that $L^p_s(\Omega)_P \subset \bigcap_r L^p_r(\Omega) = C^\infty(\overline{\Omega})^K$. Now $L^p_s(\Omega)_P$ is closed in $L^p_s(\Omega)$ and in the stronger topology of $C^\infty(\overline{\Omega})^K$. Hence $L^p_s(\Omega)_P$ is a Fréchet space in two comparable topologies. By the closed graph theorem, the two topologies coincide. One of these is a Banach space topology and the other is a Montel space topology and it is known that these coincide only on finite dimensional spaces. Hence $\dim L^p_s(\Omega)_P < \infty$ and $\mathscr{D}'(\Omega)_P = L^p_s(\Omega)_P$ by a density argument.

Lemma A7: Let $\phi \in C^\infty_o(R^n)$, $\lambda > 0$, $\eta \in R^n$, $s \in R$, $1 < p < \infty$. Let P be a polynomial of degree m with principal part P_m . Then

$$\lim_{\lambda\to\infty} \lambda^{-(s+m)} |P(D)(e^{i\lambda x\eta}\cdot\phi)|_{L^p_s} = |P_m(\eta)|\cdot|\eta|^s\cdot|\phi|_{L^p} .$$

Proof: Consider first the case $m = 0$:

$$|e^{i\lambda x\eta}\cdot\phi|_{L^p_s} = |(2\pi)^{-n}\int \hat{\phi}(\xi-\lambda\eta)(1+|\xi|^2)^{s/2}e^{ix\xi}d\xi|_{L^p} = |(2\pi)^{-n}\int \hat{\phi}(\xi)\, e^{ix\xi}\cdot$$

$$\cdot(1+|\xi+\lambda\eta|^2)^{s/2}d\xi|_{L^p} = \lambda^s|(2\pi)^{-n}\int \hat{\phi}(\xi)\, e^{ix\xi}\, (\lambda^{-2}+|\lambda^{-1}\xi+\eta|^2)^{s/2}d\xi|_{L^p} \ .$$

Now multiply by λ^{-s} and let $\lambda \to \infty$. If $m \neq 0$ $P(D)(e^{i\lambda x\eta}\cdot\phi) =$

$\sum_\alpha P^{(\alpha)}(\lambda\eta)\cdot e^{i\lambda x\eta}\cdot\frac{D^\alpha\phi}{\alpha!} = \lambda^m\, P_m(\eta)\, e^{i\lambda x\eta} + \sum_{1\leq j<m} \lambda^j\cdot Q_j(\eta)\cdot e^{i\lambda x\eta}\cdot\psi_j$ for some homogeneous polynomials Q_j , $\deg Q_j = j$, and test functions ψ_j . After multiplication of the last equality by $\lambda^{-(s+m)}$ the L^p_s-norm of the first term has the limit $|P_m(\eta)|\cdot|\eta|^s\cdot|\phi|_{L^p}$ as $\lambda \to \infty$ by the previous case and the second term $\to 0$.

R E F E R E N C E S

ABRAMCZUK, W.: [1] On the range of linear partial differential operators in Lebesgue spaces of distributions, Research report, University of Stockholm, 1980.

CALDERON, A.P.: [1] Lebesgue spaces of differentiable functions and distributions, Proc. Symp. on Pure Math., Partial Differential Equations, 33-49 (1961).

DOUGLIS, A. and NIRENBERG, L.: [1] Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8, 503-538 (1955).

ESKIN, M. and SHAMIR, E.: [1] Elliptic overdetermined differential systems in convex cones, Journal d'Analyse Mathématique, vol. 32 (1977).

HÖRMANDER, L.: [1] Linear partial differential operators, Springer, 1969.

HÖRMANDER, L.: [2] An introduction to complex analysis in several variables, North Holland, 1973.

PALAMODOV, V.P.: [1] Linear differential operators with constant coefficients, Moscow, 1967.

SMITH, K.T.: [1] Formulas to represent functions by their derivatives, Math. Ann. 188, 53-77 (1970).

Wojciech ABRAMCZUK,
Department of Mathematics,
University of Stockholm,
Box 6701,
S-113 85 Stockholm,
Sweden.