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CONSTRUCTIVE INVARIANT THEORY

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1. Let k be an algebraically closed field, char k = 0.

We denote by V an n-dimensional coordinate linear space (of columns) over k, by Mat_n the space of all $n\times n$-matrices with its coefficients in k and by GL_n the subgroup of all nondegenerate matrices in Mat_n. We use the notation (a_{ij}) for an element of Mat_n: this is the matrix with the coefficient a_{ij} situated in its i-th row and j-th column, $1 \leqslant i,j \leqslant n$. Let us denote also by x_i, $1 \leqslant i \leqslant n$, and resp. by x_{ij}, $1 \leqslant i,j \leqslant n$, the coordinate functions on V, resp. Mat_n, with respect to the canonical basis

(i.e. $x_i\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right) = a_i$ and $x_{ij}((a_{pq})) = a_{ij}$).

Let G be a reductive algebraic subgroup of GL_n. The group GL_n (and hence also G) acts in a natural way on V (by means of a multiplication of a matrix by a column). This action defines an action of GL_n on an algebra $k[V] = k[x_1,\dots,x_n]$ of all polynomial functions on V. Let $k[V]^G$ be the subalgebra of G-invariants. This is a finitely generated graded subalgebra.

The problem of constructive invariant theory is to find explicitly a minimal (i.e. noneliminatable) system of homogeneous generators of $k[V]^G$. This means, from the theoretical point of view, that we must find such an algorithm that provides in principle a specified system by means of a finite number of effectively feasible operations (of course, from a practical point of view, a question arises immediately about the amount of all needed calculations, but it is already another side of the problem that is connected with an improvement of the corresponding estimates or with a choice of a more effective algorithm but not with the existence of the algorithm itself). Solving this problem one assumes that G itself is "known", i.e. that one can solve in an effective way certain questions about its structure. Thus, the central concern mainly of this paper is with the case of a connected semisimple group G and we assume that *the following data are known:*

a) *the Lie algebra* Lie G of G (we consider Lie G as a linear subspace in Mat_n and it will be sufficient to know for instance the set of matrices that is a basis of this subspace),

b) *a maximal torus* T of G (we consider T as an image of a standard torus $k^{*r} = \underbrace{k^* \times \ldots \times k^*}_{r}$ under a known homomorphism

$$(1) \qquad \phi\colon k^{*r} \to GL_n,\ \phi((a_1,\ldots,a_r)) = (f_{ij}(a_1,\ldots,a_r)),$$

where $f_{ij}(t_1,\ldots,t_r)$, $1 \leqslant i,j \leqslant n$, are known rational functions of the canonical coordinate function t_s, $1 \leqslant s \leqslant r$, on k^{*r}, i.e. $t_s((a_1,\ldots,a_r)) = a_s$).

For example, a typical situation occurs in the invariant theory when G is an image of a known standard group G' under a known homomorphism (say, $G' = SL_m$ and G is its image under a natural representation in a space of forms of a given degree in m variables). Clearly, if one knows the Lie algebra and a maximal torus of G', then one also knows a) and b) for G itself.

Now, an algorithm for finding explicitly a minimal system of homogeneous generators of $k[V]^G$ being given, one can in principle calculate such a constant $M \in \mathbb{N}$, that $k[V]^G$ is generated by a subspace

$$(2) \qquad \bigoplus_{d=0}^{M} k[V]_d^G$$

(here and further R_d is a space of all homogeneous elements of degree d of an arbitrary $\mathbb{N}$-graded k-algebra R). On the contrary, if it is known that for a certain $M \in \mathbb{N}$ the subspace (2) generates the algebra $k[V]^G$, one can in principle find explicitly a minimal system of homogeneous generators of this algebra. Let us show how it can be done.

First of all one can explicitly describe the subspace $k[V]_d^G$ of $k[V]_d$ for any concrete d. It is as follows. If G is finite then $k[V]_d^G$ is clearly a linear span of $\frac{1}{|G|} \sum_{g \in G} g.f$, as f ranges over all monomials in $x_1, \ldots, x_n$ of degree d. If G is connected then $k[V]_d = \{f \in k[V]_d \mid (\mathrm{Lie}\ G)f = 0\}$ with respect to the natural action of Lie G by means of differentiations of the algebra k[V]. Therefore, the question in this case is reduced to solving the corresponding system of linear equations. In general, the question can be reduced to the above two cases.

Let now d_1 be such a minimal integer, $0 < d_1 \leqslant M$, that $k[V]_{d_1}^G \neq 0$. Let us take a basis $f_1, \ldots, f_q$ of $k[V]_{d_1}^G$.
Let d_2 be such a minimal integer, $d_1 < d_2 \leqslant M$, that $k[V]_{d_2}^G$ is not a linear span of all possible monomials in $f_1, \ldots, f_q$ that lie in $k[V]_{d_2}^G$. Let $g_1, \ldots, g_p$ be a basis of an arbitrary linear supplement to this linear span in $k[V]_{d_2}^G$. Analogously, let d_3 be such a minimal integer, $d_2 < d_3 \leqslant M$, that $k[V]_{d_3}^G$ is not a linear span of all possible monomials in $f_1, \ldots, f_q, g_1, \ldots, g_p$ that lie in $k[V]_{d_3}^G$, and let $h_1, \ldots, h_s$ be a basis of an arbitrary linear supplement to this linear span in $k[V]_{d_3}^G$ and so forth.
This process will stop after a finite number of steps and it is not difficult to see that a finite set of polynomials $f_1, \ldots, f_q, g_1, \ldots, g_p, h_1, \ldots, h_s, \ldots$ is in fact a minimal system of homogeneous generators of the algebra $k[V]^G$ and that an arbitrary system of such type is obtained by this procedure (it also follows

from this that a minimal system of homogeneous generators of $k[V]^G$ is defined, in a sense, in a unique way; specifically, the number of such generators is well defined and equal to $\dim I_+/I_+^2$, I_+ being the ideal in $k[V]^G$ generated by the homogeneous elements of strictly positive degree).

So we see that the problem of the constructive invariant theory is in fact equivalent to the problem of the finding of a number M. The problem of the case in which G is finite has already been solved by E. Noëther, [10], [17], who proved that one can take $M = |G|$ (and hence $\dim I_+/I_+^2 < \binom{|G|+n}{n}$). The problem is by no means so satisfactorily solved when G is a connected group. Of course, there are a lot of such concrete groups G when a minimal system of homogeneous generators of $k[V]^G$ is explicitly described. Nevertheless, it seems that in its general setting the problem was considered only by Hilbert in [4] and (following Hilbert's idea) in [2] (in this paper G is an image of SL_m under a homogeneous polynomial homomorphism).

Hilbert gives two similar approaches to the problem. The first one reduces it to "a solution of an elementary problem from the arithmetic theory of algebraic functions"; more precisely, to the Kronecker method for the constructing of a basis of the ingegral closure of a finitely generated domain in its field of fractions. The second one is a modification of the first which makes it possible to avoid a direct reference to the Kronecker method because of some specific properties of the algebras and fields under consideration. Hilbert tried to avoid a reference to the Kronecker method because he apparently understood that it gives only the general strategy (reducing the problem of finding of the estimates of the degrees of the certain equations, calculating of its discriminants etc.), but leaves the question of an explicit calculation of M somewhat vague. As a matter of fact the same is related to his second approach and, as a result, a constant M was not in fact explicitly calculated (Hilbert refers, but only in passing, to his second approach when he remarks that "a closer examination also provides an upper bound for the weights of the invariants in a complete system which depends only on n"). All these circumstances were, I believe, the reason why the authors of [2] preserving Hilbert's main idea tried to change the final part of his general strategy, replacing the appeal to a method to describe the integral quantities (in such a form or another) by a reference to the Nullstellensatz (but again the constant M was not explicitly calculated in [2]).

Several aims will be pursued in this lecture. First, it will be shown how "the vagueness" related to the Kronecker method and its modifications by means of the recent achievements of the invariant theory (the theorem of Hochster and Roberts) can be avoided. Second, it will be shown how to generalize the main idea of Hilbert's paper [4] for the case of an arbitrary connected semisimple group G. It is to be hoped that this generalization will clarify to a certain extent the somewhat unexpected (and at a first glance accidental) role which the function $(a_{ij}) \mapsto \det(a_{ij})$ plays in Hilbert's theory. In the case under consideration an explicit estimate M can be obtained. This estimate is fantastically large and its radical improvement is apparently connected with the principal change in the approach itself. To be more precise, it will be shown that *the degrees of a minimal system of homogeneous generators of* $k[V]^G$ *are not greater than the number*

$$(3) \qquad M = m \cdot \left(\begin{array}{l} \text{minimal positive integer divisible} \\ \text{by each } a \in \mathbb{Z}, \\ \\ 0 < a \leqslant \dfrac{2^{r+s} n^{s+1} (n-1)^{s-r} t^r (s+1)!}{3^s \left(\left(\frac{s-r}{2} \right)! \right)^2} \end{array} \right)$$

where $s = \dim G$, $r = \operatorname{rk} G$, m *is the transcendence degree over* k *of* $k[V]^G$ *and* t *is the maximum of* $|m_\ell|$ *taken over all* $1 \leqslant \ell \leqslant r$ *and all the monomials* $t_1^{m_1} \ldots t_r^{m_r}$ *that occur in the polynomials* $f_{ij}(t_1, \ldots, t_r)$, $1 \leqslant i, j \leqslant n$, *of* (1). For instance, if G is the image of SL_m under its natural representation in a space of forms of degree p in m variables, one can take $t = p$. The reader will see that (3) can be slightly improved by means of more thorougness in a number of places, but this was not attempted because it does not provide a principal improvement.

Third, an approach will be discussed, as suggested in [2], and it will be shown that in reality it *does not* provide a solution to the problem (unfortunately in the abstract of the present lecture [1] it was pointed out that Hilbert's approach is the same as in [2], but as a matter of fact this is not the case and it would not be correct to say that Hilbert's general strategy does not lead to the solution, although it however flawed by a certain vagueness). Interestingly, the error in [2] is connected with the certain ingenious properties of $k[V]^G$. Several intriguing problems arise in this context.

These and other questions are discussed in nn. 7,8.

The following notations are fixed throughout:

$k[X]$ — the algebra of regular functions of an algebraic variety X,

k^* — multiplicative group of k,

$\mathbb{N}$ additive semigroup of nonnegative integers,

$k[b_1,\ldots,b_d]$, resp. $k(b_1,\ldots,b_d)$ the algebra, resp. the field, generated by $b_1,\ldots,b_d$ over k,

G a connected semisimple algebraic group (if it is not otherwise stated).

2. As it was proved by Hochster and Roberts, [6], $k[V]^G$ is a Cohen-Macaulay algebra. This is equivalent to saying that if $\theta_1,\ldots,\theta_m$ is a homogeneous system of parameters, i.e. $k[V]^G$ is integral over

$$P = k[\theta_1,\ldots,\theta_m], \tag{4}$$

then $k[V]^G$ is a free module over P. Let $\eta_1,\ldots,\eta_\ell$ be a homogeneous system of generators of this module,

$$k[V]^G = P\eta_1 \oplus \ldots \oplus P\eta_\ell. \tag{5}$$

We assume that

$$\begin{aligned} &\deg \theta_i = d_i,\ d_1 \leqslant \ldots \leqslant d_m, \\ &\deg \eta_j = e_j,\ 0 = e_1 \leqslant \ldots \leqslant e_\ell. \end{aligned} \tag{6}$$

It follows from (4), (5) and (6) that the Poincaré series of $k[V]^G$ is

$$F(k[V]^G,x) = \sum_{d=0}^{\infty} \dim k[V]^G_d x^d = \frac{\sum_{i=1}^{\ell} x^{e_i}}{\sum_{j=1}^{m} (1-x^{d_j})}. \tag{7}$$

A classical formula of Molien-Weyl makes it possible to calculate this rational function when $k = \mathbb{C}$. More precisely, let K be a maximal compact subgroup of G such that $K \cap T$ is a maximal torus of K. Let $\alpha_1,\ldots,\alpha_{s-r}$ be the set of the roots of K with respect to $K \cap T$ (we consider a root as a function on $K \cap T$) and W is the Weyl group of K. Then, [18],

$$(8) \qquad F(k[V]^G,x) = \frac{1}{|W|}\int_{K\cap T} \frac{(1-\alpha_1(g))\ldots(1-\alpha_{s-r}(g))}{\det(1-xg)}\,d\mu(g), \quad (|x| < 1),$$

where $d\mu(g)$ is the normalized Haar-measure. The integral on the right side of (8) reduces to integrals over the unit circumference in C and can be calculated in principle by means of residues. In the special cases (say, when G is the image of SL_2 under an irreducible representation) other formulas are also known to calculate the Poincaré series, cf. [11], [12]).

G being a connected semisimple, it follows that $k[V]^G$ is Gorenstein, see [6]. This means (and in fact is equivalent because of Stanley's theorem, [15]) that $F(k[V]^G,x)$ satisfies the following functional equation

$$(9) \qquad F(k[V]^G,x^{-1}) = (-1)^m x^q F(k[V]^G,x),$$

where $q \in \mathbb{Z}$.

Using (7) we see that (9) is equivalent to

$$(10) \qquad x^{d_1+\ldots+d_m} \sum_{j=1}^{\ell} x^{-e_j} = x^q \sum_{j=1}^{\ell} x^{e_j}$$

and it follows from (6) that (10) is equivalent to

$$(11) \qquad d_1+\ldots+d_m-e_{\ell-i+1} = q+e_i, \quad i=1,\ldots,\ell.$$

Specifically it follows from (11) that

$$(12) \qquad e_\ell = d_1 + \ldots + d_m - q$$

and

$$(13) \qquad e_i + e_{\ell-i+1} = e_\ell, \quad i = 1,\ldots,\ell.$$

In as much as we can in principle effectively find $F(k[V]^G,x)$, the number q is known. We need in reality to know only q, but not $F(k[V]^G,x)$ itself. Several facts are known *a priori* about q (for instance, always $q \geqslant 0$, [8]). We shall discuss this question in n.7.

It follows from (2) and (3) that $\theta_1,\ldots,\theta_m,\eta_1,\ldots,\eta_\ell$ is a homogeneous system of generators of the algebra $k[V]^G$.
We conclude now from (6) and (12) that one can take M in (2) to be equal to

$$\max(d_1+\ldots+d_m-q,d_m) \tag{14}$$

But it is not generally the case that d_i, $1 \leqslant i \leqslant m$, are known. One can not derive an upper boundary for these numbers directly from $F(k[V]^G,x)$ (for example, one can take $\theta_1^{s_1},\ldots,\theta_m^{s_m}$ instead of $\theta_1,\ldots,\theta_m$ for arbitrary $s_i \in \mathbb{Z}$, $s_i > 0$, $1 \leqslant i \leqslant m$, and this set is again a homogeneous system of parameters of $k[V]^G$). So the problem arises to find an *a priori* upper estimate of numbers $d_1,\ldots,d_m$.

If it is known for instance that

$$d_m \leqslant D \tag{15}$$

then one can take M in (2) to be equal to

$$\max(mD-q,D) \leqslant mD \tag{16}$$

(It should be noted here that m is equal to the order of the pole of $F(k[V]^G,x)$ at $x = 1$, hence one can in principle effectively find m. Another way to calculate m follows from the equality $m = n-\dim Gv = n-s+\dim\{X \in \mathrm{Lie}\ G \mid Xv = 0\}$, where v is a point of V in general position).

Our concern now will be to find an upper estimate of the degrees of some homogeneous systems of parameters of $k[V]^G$. As a matter of fact this problem was solved by Hilbert in [4], when G is an image of SL_d under a homogeneous polynomial representation.

3. We shall now show how to find D in general, by developing Hilbert's idea.

The general plan is the following.
Our starting point will be the following important theorem (for the first time proved by Hilbert in [4]):

> Let Q be any finitely generated graded k-subalgebra of $k[V]^G$. Then $k[V]^G$ is integral over Q if and only if $\{v \in V \mid f(v)=0$ for every $f \in \bigoplus_{d=1}^{\infty} k[V]^G_d\} = \{v \in V \mid h(v)=0$ for every $h \in \bigoplus_{d=1}^{\infty} Q_d\}$.

Denote by N the variety of zeros in V of the homogeneous elements of $k[V]^G$ of strictly positive degree.
It is known that

$$N = \{v \in V \mid 0 \in \overline{Gv}\} \tag{17}$$

(we use bar to denote the closure in V). The points from $V-N$ are called *semi-stable*.

Let us assume for a moment that we can find such an integer $C > 0$ that

(18) *for every semi-stable point* $v \in V$, *there is a homogeneous polynomial* $f_v \in k[V]^G$ *of degree* $\leqslant C$ *such that* $f_v(v) \neq 0$.

Let A be the k-subalgebra of $k[V]^G$ generated by the subspace $\bigoplus_{d=0}^{C} k[V]_d^G$. The variety of zeros in V of the homogeneous elements of A of strictly positive degree is exactly N. Let $h_1,\ldots,h_q$ be a minimal homogeneous system of generators of A. We can in principle effectively find such a system by means of the procedure described in n.1. Clearly, $\deg h_i \leqslant C$, $1 \leqslant i \leqslant q$. Let $\tilde{C}$ be the minimal positive integer divisible by each $\deg h_i$, $1 \leqslant i \leqslant q$. We have

$$(19) \qquad \tilde{C} \leqslant \begin{pmatrix} \text{minimal positive integer} \\ \text{divisible by each } a \in \mathbb{Z}, \\ 0 < a \leqslant C \end{pmatrix}.$$

We have also $N = \{v \in V \mid h_i(v) = 0,\ 1 \leqslant i \leqslant q\} = \{v \in V \mid h_i^{\tilde{C}/\deg h_i}(v) = 0,\ 1 \leqslant i \leqslant q\}$. Hence $k[V]^G$ is integral over $B = k[h_1^{\tilde{C}/\deg h_1},\ldots,h_q^{\tilde{C}/\deg h_q}]$. It follows from the equality $\deg h_i^{\tilde{C}/\deg h_i} = \tilde{C}$, $1 \leqslant i \leqslant q$, and from the homogeneous variant of E. Noëther's normalization lemma, [19], that there exists a homogeneous system of parameters of B which are k-linear combinations of $h_i^{\tilde{C}/\deg h_i}$, $1 \leqslant i \leqslant q$. Clearly this will automatically be a homogeneous system of parameters of $k[V]^G$. This proves that one can take D in (15) to be equal to $\tilde{C}$ on (more roughly but withoug finding of h_i, $1 \leqslant i \leqslant q$) to the right side of (19).

Therefore we now have the problem of finding C with the property (18).

4. We proceed to the solution of this problem. Further we shall identify k^* with the subgroup of the scalar matrices in GL_n. Let us consider the subgroup $H = k^*G$. This is an

(s+1)-dimensional connected reductive group with a onedimensional center. Let us consider also its closure $\overline{H}$ in Mat_n. The group H acts on Mat_n by means of left multiplication and H and $\overline{H}$ are invariant under this action. Specifically, H and $\overline{H}$ are invariant under k^* and hence are cones. H acts also on V being a subgroup of GL_n.

Let $\overline{x}_{ij}$ be the restriction of x_{ij} to $\overline{H}$. The algebra $k[\overline{H}]$ is generated by $\overline{x}_{ij}$, $1 \leqslant i,j \leqslant n$,

(20) $$k[\overline{H}] = k[\ldots,\overline{x}_{ij},\ldots].$$

and is $\mathbb{N}$-graded ($\deg \overline{x}_{ij} = 1$, $1 \leqslant i,j \leqslant n$).

For a point $v \in V$ we shall consider its orbit Hv and the closure $\overline{Hv}$ of this orbit in V. Both Hv and $\overline{Hv}$ are cones, i.e. invariant under k^*. Let $\overline{x}_i$, $1 \leqslant i \leqslant n$, be the restriction of x_i on $\overline{Hv}$. The algebra $k[\overline{Hv}]$ is generated by $\overline{x}_i$, $1 \leqslant i \leqslant n$,

(21) $$k[\overline{Hv}] = k[\overline{x}_1,\ldots,\overline{x}_n]$$

and is $\mathbb{N}$-graded ($\deg \overline{x}_i = 1$, $1 \leqslant i \leqslant n$). It should be noted here that the structure of $\mathbb{N}$-graded algebras in these cases is defined by weight decompositions of the underlying algebras with respect to the action of torus k^*, i.e. a function f is homogeneous of the weight $d \in \mathbb{N}$ iff $f(tx) = t^d f(x)$ for each $t \in k^*$ and x.

Proposition 1. A point $v \in V$ *is semi-stable if and only if* $k[\overline{Hv}]^G \neq k$.

Proof. Let v be a semi-stable point. Then $0 \notin \overline{Gv}$, see (17). Hence, 0 and $\overline{Gv}$ are disjointed closed G-invariant sets in $\overline{Hv}$. But such sets are separated by G-invariants, [9], and therefore $k[\overline{Hv}]^G \neq k$.

On the contrary, let $k[\overline{Hv}]^G = k$. Clearly $k[\overline{Hv}]^G$ is a graded subalgebra of $k[\overline{Hv}]$. Let $f \in k[\overline{Hv}]^G$ be an element of strictly positive degree. Being homogeneous f has the properties $f(0) = 0$ and $f(tgv) = t^{\deg f} f(gv) = t^{\deg f} f(v)$. Therefore, $f(v) \neq 0$ because Hv is dense in $\overline{Hv}$. We see now that $\overline{Gv}$ lies in the closed set $\{u \in \overline{Hv} \mid f(u) = f(v)\} \not\ni 0$. The proposition is proved.

Let us consider the morphism

$$\alpha: \overline{H} \to \overline{Hv} \tag{22}$$

defined by formula

$$\alpha(h) = hv \quad \text{(multiplication of a matrix by a column)}$$

This morphism commutes with the action of H. Clearly the corresponding comorphism of the $\mathbb{N}$-graded algebras

$$\alpha^*: k[\overline{Hv}] \to k[\overline{H}]$$

is a degree-preserving monomorphism. If $x_i(v) = a_i$, $1 \leqslant i \leqslant n$, then

$$\alpha^*(\overline{x}_i) = a_i\overline{x}_{i1} + \ldots + a_n\overline{x}_{in}.$$

It follows now from (21) that

(23) $\alpha^*(k[\overline{Hv}])$ *is an* H-*invariant graded subalgebra of* $k[\overline{H}]$ *generated by* n *homogeneous elements* $a_1\overline{x}_{i1}+\ldots+a_n\overline{x}_{in}$, $1 \leqslant i \leqslant n$, *of degree* 1

We derive from Proposition 1 the following

Proposition 2. A point $v \in V$ *is semi-stable if and only if*

$$\alpha^*(k[\overline{Hv}]) \cap k[\overline{H}]^G \neq k. \tag{24}$$

The advantage of this formulation is that one can say a lot about the structure and properties of $k[\overline{H}]^G$.

More precisely, let f be a rational character of H, i.e. a homomorphism $H \to k^*$. We consider f as a regular function on H and a rational function on $\overline{H}$. This function is G-invariant because G is semisimple. The set of all such characters of H that are regular on $\overline{H}$ is a semigroup S (with an unity) in the (multiplicative) character group of H. Its elements are homogeneous elements of $k[\overline{H}]^G$.

Proposition 3. One has:

1) $k[\overline{H}]^G$ *is the vector space spanned by* S *over* k,

2) $\dim k[\overline{H}]^G_d \leqslant 1$ *for each* $d \in \mathbb{N}$,

3) $k[\overline{H}]^G \neq k$,

4) S *is a finitely generated semigroup and* $f \mapsto \deg f$ *is an isomorphism of* S *with a certain subsemigroup of* $\mathbb{N}$.

Proof. 1) Let f be a nonzero homogeneous element of $k[H]^G$, $\deg f = d$. We have $f(tg) = t^d f(g) = t^d f(1_n)$ for every $t \in k^*$, $g \in G$ (here 1_n is the unity of the group G). Therefore $f(1_n) \neq 0$ because $H = k^*G$ is dense in $\overline{H}$. Multiplying f by a suitable constant we can assume $f(1_n) = 1$. Hence $f(tg) = t^d$, i.e. $f \in S$.

2) Let f and h be nonzero homogeneous elements of $k[\overline{H}]^G$ and $\deg f = \deg h$. These elements are the semi-invariants of the torus k^* of the same weight. Hence, the rational function f/h is an H-invariant. Therefore, $f/h \in k$.

3) Clearly, the function det: $(a_{ij}) \mapsto \det(a_{ij})$ is a nonconstant homogeneous element of $k[\overline{H}]^G$ of the degree n.

4) $k[\overline{H}]^G$ is finitely generated because G is reductive. It follows from 1) and 2) that S is finitely generated.
The algebra $k[H]$ is $\mathbb{N}$-graded and the elements of S are homogeneous.

Hence we have deg $f \in \mathbb{N}$, $f \in S$, and 4) follows.
The proposition is proved.

It should be noted here that if H is known to be normal, then we can describe semigroup S more precisely.

Proposition 4. If $\overline{H}$ *is normal then* S *is a semigroup with one generator. This generator in its turn is a generator of the character group of* H.

Proof. The character group of H is isomorphic to $\mathbb{Z}$ because the center of H is onedimensional. Let us take one of the (two possible) generators of this group, say Φ, that is uniquely defined by the property: $\det = \Phi^d$ for a certain $d > o$. But $\det \in k[\overline{H}]$ and therefore Φ is integral over $k[\overline{H}]$. Hence, $\Phi \in S$ because of normality. We see now that $\Phi^h \in S$, $h = 0,1,2,\ldots$.
We have $\Phi(0) = 0$ hence $\Phi^h \notin S$ for $h < 0$.
The proposition is proved.

Remark. It is easy to understand that Φ is given by the formula $\Phi(tg) = t^{|k^* \cap G|}$, $t \in k^*$, $g \in G$. Therefore $\deg \Phi = |k^* \cap G|$. Specifically, if the centre of G is trivial, then $\Phi(tg) = t$.

In general, we know nothing about the normality of $\overline{H}$. Therefore, we have to use in our effective constructions only the elements that we definitely know to lie in S and these elements are $\det^d$, $d \in \mathbb{N}$. But they are sufficient for our purposes.

The condition (24) is equivalent to the condition

$$(25) \qquad \alpha^*(k[\overline{Hv}]) \cap S \neq 1.$$

How can one know when (25) is fulfilled? The answer is given by

Proposition 5. Let F *be a graded* H*-invariant* k*-subalgebra of* $k[\overline{H}]$, $k \subset F$. *Then the following properties are equivalent:*

1) $F \cap S \neq 1$,

2) *there exists* $f \in S$, $f \neq 1$, *that is integral over* F,

3) *every* $h \in S$ *is integral over* F,

4) $k[\overline{H}]^G$ *is integral over* F.

Proof. 2) $\Leftrightarrow$ 3). It follows from 4) of Proposition 3 that $f^{\deg h} = h^{\deg f}$. Function f being integral over F, hence $f^{\deg h}$ is also integral over F. Therefore, $h^{\deg f}$ is integral over F and it follows from $\deg f > 0$ that h itself is also.

3) $\Leftrightarrow$ 4) follows from 4) of Proposition 3.

1) $\Leftrightarrow$ 2) Let

$$(26) \qquad f^d + f^{d-1}b_1 + \ldots + fb_{d-1} + b_d = 0,\ b_i \in F,\ 1 \leqslant i \leqslant d,$$

be an equation of integral dependence. We can assume that b_i, $1 \leqslant i \leqslant n$, are homogeneous because f is homogeneous. Now, we apply Reynolds operator $\natural$ of averaging over G to the both sides of (26), [9]. It follows from the G-invariant of f that we shall obtain

$$(27) \qquad f^d + f^{d-1}b_1^{\natural} + \ldots + fb_{d-1}^{\natural} + b_d^{\natural} = 0,$$

where $b_i^{\natural} \in k[\overline{H}]^G$, $1 \leqslant i \leqslant d$. Clearly F is invariant under $\natural$, therefore $b_i^{\natural} \in F \cap k[\overline{H}]^G$, $1 \leqslant i \leqslant d$. We have also

$$(28) \qquad \deg b_i^{\natural} = i \deg f \leqslant d \deg f,\ 1 \leqslant i \leqslant d.$$

It follows from (27), (28) and from the condition $f \neq 1$ that there exists at least one i such that $b_i^{\natural}$ is a nonconstant homogeneous G-invariant. According to 1) and 2) of Proposition 3 this element multiplied by a suitable constant from k lies in S. Hence $F \cap S \neq 1$.
The proposition is proved.

Corollary 1. $F \cap k[\overline{H}]^G \neq k$ *if and only if* det *is integral over* F.

From (28) and the proof of Proposition 5 may be drawn the following

Corollary 2. If $f \in S$, $f \neq 1$, *is integral over* F *then there exists a nonconstant homogeneous* G*-invariant element in* F *of degree* $\leqslant d \deg f$, *where* d *is the minimal degree of the equations of integral dependence satisfied by* f *over* F.

Applying to the case $F = \alpha^*(k[\overline{Hv}])$ and using (25), (23) and Proposition 2 we obtain

Proposition 6. A point $v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in V$ *is semi-stable if and only if the function* $\det \in k[\overline{H}]$ *is integral over the graded subalgebra* A_v *of* $k[\overline{H}]$ *that is generated by the elements* $a_1\overline{x}_{i1} + \ldots + a_n\overline{x}_{in}$, $1 \leqslant i \leqslant n$, *of degree* 1. *If* d_0 *is the minimal degree of the equations of integral dependence satisfied by* det *over* A_v, *then there exists a nonconstant homogeneous polynomial of degree* $\leqslant d_0 n$ *in* $k[V]^G$ *which do not vanish at* v.

Therefore we shall be able to find C satisfying (18) if a number will be explicitly known that is not less than the minimal degree of the equations of integral dependence satisfied by det over A_v for every semi-stable point v. We shall now show how to find such a number.

5. Let $v \in V$ be a semi-stable point. There exists a homogeneous system of parameters of A_v, say $b_1,\ldots,b_h$, such that $\deg b_i = 1$, $1 \leqslant i \leqslant h$ (because A_v is generated by a system of homogeneous elements of degree 1). Function det is integral over A_v, hence also over $B_v = k[b_1,\ldots,b_h]$. Therefore the minimal degree of the equations of integral dependence satisfied by det over A_v is not

greater than the same for det over B_v. But B_v is integraly closed because of the algebraical independence of $b_1,\dots,b_h$. Therefore, the minimal equation of det over $k(b_1,\dots,b_h)$ has its coefficients in B_v. Complete the sequence $b_1,\dots,b_h$ with $s+1-h$ elements $b_{h+1},\dots,b_{s+1}$, taken from among the $\overline{x}_{ij}$, $1 \leqslant i,j \leqslant n$ such that $b_1,\dots,b_{s+1}$ are algebraically independent over k (it is possible because of (20)). It is easy to see that the minimal polynomial of det over $k(b_1,\dots,b_h)$ is the same as over $k(b_1,\dots,b_{s+1})$. Therefore, it will be sufficient for our purposes to find an upper boundary (that does not depend on v) of the degree of the minimal polynomial of det over $k(b_1,\dots,b_{s+1})$. One can do it as follows.

Let us consider for each $p \in \mathbb{N}$ the set of all monomials in $b_1,\dots,b_{s+1}$, det of degree p

(29) $$b_1^{i_1}\dots b_{s+1}^{i_{s+1}}\,\det^i,\quad i_1 + \dots + i_{s+1} + in = p.$$

If it happens that for a certain $p = p_0$ these monomials are linearly dependent over k then one can say that the degree of the minimal polynomial of det over $k(b_1,\dots,b_{s+1})$ is not greater than p_0/n. Indeed, if for $p = p_0$ a nontrivial linear combination

(30) $$\sum_{i_1+\dots+i_{s+1}+ni=p} \alpha_{i_1\dots i_{s+1} i}\; b_1^{i_1}\dots b_{s+1}^{i_{s+1}}\,\det^i,$$

where $\alpha_{i_1\dots i_{s+1}i} \in k$, is equal to zero, then there exists at least one nonzero $\alpha_{i_1\dots i_{s+1}i}$ with $i \neq 0$ (because $b_1,\dots,b_{s+1}$ are algebraically independent). It follows from (29) that $i \leqslant p_0/n$ for every $\alpha_{i_1\dots i_{s+1}i}$. Therefore it follows from (30) that det is a root of an equation of degree $\leqslant p_0/n$ with its coefficients in $k(b_1,\dots,b_{s+1})$.

So we now have the problem of finding p_0. Technically it will be more conveniet for us to consider only those of (29) for which

(31) $\quad p = nc \quad$ and $\quad i_1 = nj_1, \ldots, i_{s+1} = nj_{s+1}$

where $c, j_1, \ldots, j_{s+1}$ are integers. We shall find such $c = c_0$ that the monomials (29) with property (31) are linearly dependent over k. Hence the degree of the minimal polynomial of det over $k(b_1, \ldots, b_{s+1})$ will not be greater than c_0.

6. Let N^+ and N^- be the pair of "opposite" maximal nilpotent subalgebras of Lie G that correspond to a choice of positive and negative roots with respect to T. Let $Y_1, \ldots, Y_d$ and $Z_1, \ldots, Z_d$ be the bases of the linear spaces N^+ and N^-, $d = (s-r)/2$, and $y_1, \ldots, y_d$ and $z_1, \ldots, z_d$ the coordinate functions on these spaces with respect to these bases.

Let us consider the variety $N^+ \times k^{*r} \times N^-$. One can identify in a natural way the functions $y_1, \ldots, y_d$, $z_1, \ldots, z_d$, t_1, $t_1^{-1}, \ldots, t_r$, t_r^{-1} with the regular functions on this variety. Clearly $k[N^+ \times k^{*r} \times N^-]$ is generated by these functions.

For an arbitrary nilpotent matrix $X \in \mathrm{Mat}_n$ its exponential exp X is the matrix

(32) $$1_n + \frac{1}{1!} X + \frac{1}{2!} X^2 + \ldots + \frac{1}{(n-1)!} X^{n-1},$$

because $X^p = 0$ when $p \geqslant n$.

It is well known, [7], that the morphism

$$\beta \colon N^+ \times k^{*r} \times N^- \to \overline{H}$$

given by the formula (cf. (1))

$$\beta(X^+, (a_1, \ldots, a_r), X^{-1}) = \exp X^+ (f_{ij}(a_1, \ldots, a_r)) \exp X^-$$

is an isomorphism of $N^+ \times k^{*r} \times N^-$ with an open set of G ("big cell").

Consider now the following matrices (with its coefficients in $k[N^+ \times k^{*r} \times N^-]$)

$$Y = \sum_{i=1}^{d} y_i Y_i \quad \text{and} \quad Z = \sum_{i=1}^{d} z_i Z_i \tag{33}$$

Let exp Y and exp Z be the results of the substitution of Y and resp. Z instead of X in (32). Then

$$\left| \begin{array}{l} \beta^*(\bar{x}_{ij}) \textit{ is the coefficient of the matrix} \\ \qquad \exp Y \, (f_{pq}(t_1,\ldots,t_r)) \exp Z \\ \textit{situated in its } i\textit{-th row and } j\textit{-th column} \end{array} \right. \tag{34}$$

It follows now from (34), (33), (32) and the definition of the number t (cf. (1) and (3)) that

$$\left| \begin{array}{l} \beta^*(\bar{x}_{ij}) \textit{ is a } k\textit{-linear combination of the monomials} \\ \qquad y_1^{r_1}\cdots y_d^{r_d} z_1^{q_1}\cdots z_d^{q_d} t_1^{m_1}\cdots t_r^{m_r} \\ \textit{where } r_i, q_i \in \mathbb{N},\ 1 \leqslant i \leqslant d,\ m_j \in \mathbb{Z},\ 1 \leqslant j \leqslant r, \textit{ and} \\ \qquad 0 \leqslant r_1+\cdots+r_d \leqslant n-1, \\ \qquad 0 \leqslant q_1+\cdots+q_d \leqslant n-1, \\ \qquad \max\limits_{1 \leqslant i \leqslant r} |m_i| \leqslant t \quad . \end{array} \right. \tag{35}$$

Now we note that the function (30) is homogeneous and hence is equal to zero if and only if its restriction to G vanishes. In its turn it happens if and only if the image of this restriction under the comorphism β^*, i.e. the function

$$\sum_{i_1+\cdots+i_{s+1}+ni=p} \alpha_{i_1\ldots i_{s+1}i} \, \beta^*(b_1^{i_1}\cdots b_{s+1}^{i_{s+1}} \det{}^i), \tag{36}$$

vanishes.

The function $b_1^{i_1}\cdots b_{s+1}^{i_{s+1}} \det^i$, $i_1+\cdots+i_{s+1}+ni = p$, is homogeneous of degree p ,therefore it follows form (35) that

(37) *the function* (36) *is a linear combination of the monomials*

$$y_1^{r_1}\dots y_d^{r_d} z_1^{q_1}\dots z_d^{q_d} t_1^{m_1}\dots t_r^{m_r}$$

where $r_i, q_i \in \mathbb{N}$, $1 \leqslant i \leqslant d$, $m_j \in \mathbb{Z}$, $1 \leqslant j \leqslant r$, *and*

$$0 \leqslant r_1+\dots+r_d \leqslant p(n-1),$$

$$0 \leqslant q_1+\dots+q_d \leqslant p(n-1),$$

$$\max_{1\leqslant i\leqslant r} |m_i| \leqslant pt$$

and the coefficients of this linear combination are the k-*linear combinations of the element* $\alpha_{i_1\dots i_{s+1}i}$, $i_1+\dots+i_{s+1}+ni = p$.

We need to remember here that if $\delta_1,\dots,\delta_h$ are the variables then

(38) *the total number of the monomials* $\delta_1^{s_1}\dots\delta_h^{s_h}$, $s_i \in \mathbb{N}$, $1 \leqslant i \leqslant h$, *is equal to* $\binom{k+h-1}{k}$ *if* $s_1+\dots+s_h = k$ *and equal to* $\binom{k+h}{k}$ *if* $s_1+\dots+s_h \leqslant k$.

It easily follows from (38) that the total number of the monomials in $y_1,\dots,y_d$, $z_1,\dots,z_d$, $t_1,\dots,t_r$ that are described in (37) is not greater than

$$\binom{d+p(n-1)}{d}^2 (2pt+1)^r.$$

On the other hand assume now that the conditions (31) are fulfilled. It follows then from (38) that for a given c the total number of the coefficients $\alpha_{i_1\dots i_{s+1}i}$ in (30) is equal to

$$\binom{s+c+1}{c}.$$

In view of (37) we see that if for a certain c one has an inequality

$$(39) \qquad \binom{d+cn(n-1)}{d}^2 (2cnt+1)^r < \binom{s+c+1}{c}$$

then the monomials (29) (with the conditions (31) to be fulfilled) are linearly dependent over k. Let us show that (39) is fulfilled for

$$(40) \qquad c = \left[\frac{2^{s+r} n^s (n-1)^{s-r} t^r (s+1)!}{3^s\left(\left(\frac{s-r}{2}\right)!\right)}\right]$$

Indeed, we have $d = (s-r)/2 \leqslant n(n-1)/2$ and

$$\binom{d+cn(n-1)}{d}^2 (2cnt+1)^r \leqslant \frac{(cn(n-1)+d)^{2d}}{(d!)^2} (2cnt+1)^r =$$

$$= \frac{(c + \frac{d}{n(n-1)})^{s-r} n^{s-r} (n-1)^{s-r}}{\left(\left(\frac{s-r}{2}\right)!\right)^2} \quad (c + \frac{1}{2nt})^r (2nt)^r \leqslant$$

$$\leqslant (c + \tfrac{1}{2})^s \frac{n^s (n-1)^{s-r} 2^r t^r}{\left(\left(\frac{s-r}{2}\right)!\right)^2} .$$

Also $\binom{s+c+1}{c} > \frac{c^{s+1}}{(s+1)!}$. Hence (39) is definitely fulfilled if

$$(c+\tfrac{1}{2})^s \frac{n^s (n-1)^{s-r} 2^r t^r}{\left(\left(\frac{s-r}{2}\right)!\right)} \leqslant \frac{c^{s+1}}{(s+1)!} \text{ or equivalently if}$$

$$\frac{n^s (n-1)^{s-r} 2^r t^r (s+1)!}{\left(\left(\frac{s-r}{2}\right)!\right)^2} \leqslant c\left(\frac{1}{1+\frac{1}{2c}}\right)^s .$$

This inequality in its turn is fulfilled if c is given by the formula (40) because $(1+\frac{1}{2c}) \geqslant \frac{3}{2}$.

Summing up all the results of nn. 1-6 we obtain the following

Theorem. Let $G \subset GL_n(V)$ *be a connected semisimple algebraic group,* $s = \dim G$, $r = \operatorname{rk} G$. *Let* m *be the transcendence degree of* $k[V]^G$ *over* k *and* t *be a constant defined in* (3). *Then the degrees of the elements of a minimal homogeneous system of generators of the algebra* $k[V]^G$ *are not greater than the number*

$$m \cdot \left(\begin{array}{l} \text{minimal positive integer divisible} \\ \text{by each } a \in \mathbb{Z}, \\ \\ 0 < a \leqslant \dfrac{2^{r+s} n^{s+1} (n-1)^{s-r} t^r (s+1)!}{3^s \left(\left(\frac{s-r}{2} \right)! \right)^2} \end{array} \right)$$

7. This section contains several remarks about the number q in (9).

We know that $F(k[V]^G, x)$ has a pole of order m at $x = 1$. Therefore the Laurent expansion about $x = 1$ begins

$$F(k[V]^G, x) = \frac{\gamma}{(1-x)^m} + \frac{\tau}{(1-x)^{m-1}} + \dots \, . \tag{41}$$

It follows from (7) and (41) that

$$\gamma = \lim_{x \to 1} (1-x)^m \frac{\sum_{i=1}^{\ell} x^{e_i}}{\prod_{j=1}^{m} (1-x^{d_j})} = \frac{\ell}{\prod_{j=1}^{m} d_j}, \tag{42}$$

$$(43)\qquad \tau = -\lim_{x\to 1}\frac{d}{dx}\frac{\sum_{i=1}^{\ell} x^{e_i}}{\prod_{j=1}^{m}(1-x^{d_j})}(1-x^m) = -\lim_{x\to 1}\frac{d}{dx}\frac{\sum_{i=1}^{\ell} x^{e_i}}{\prod_{j=1}^{m}(1+x+\ldots+x^{d_j-1})} =$$

$$= \frac{-\sum_{i=1}^{\ell} e_i \prod_{j=1}^{m} d_j + \ell\sum_{j=1}^{m}\left(\frac{d_j(d_j-1)}{2}\prod_{i\neq j} d_i\right)}{\prod_{j=1}^{m} d_j^2} =$$

$$= \frac{\ell\sum_{j=1}^{m}(d_j-1) - 2\sum_{i=1}^{\ell} e_i}{2\prod_{j=1}^{m} d_j}\;.$$

Therefore

$$(44)\qquad \frac{2\tau}{\gamma} = \sum_{j=1}^{m}(d_j-1) - \frac{2}{\ell}\sum_{i=1}^{\ell} e_i .$$

Adding up the equalities (11) over all $i=1,\ldots,\ell$ we shall obtain

$$(45)\qquad \sum_{j=1}^{m} d_j - q = \frac{2}{\ell}\sum_{i=1}^{\ell} e_i .$$

Now it follows from (45) and (44) that

$$\frac{2\tau}{\gamma} = q-m$$

or in another form

$$(46)\qquad q = \frac{2\tau}{\gamma} + \dim V - (\dim Gv,\ v\in V \text{ is a point in general position})\ .$$

So we have

1) $\frac{2\tau}{\gamma}$ is always an integer,

2) $q = \dim V \Leftrightarrow \frac{2\tau}{\gamma} = (\dim Gv,\ v \in V$ is a point in general position).

As a matter of fact to derive the conclusions we used only that $k[V]^G$ is Gorenstein (but not that G is a connected semisimple group). Specifically, these conclusions are also valid for those finite G with a property that $k[V]^G$ is Gorenstein (e.g. if $G \subset SL_n$, cf. [13]). It is remarkable that one can describe γ and τ for an arbitrary finite group G completely in terms of the internal properties of G itself, [13]:

(47) $$\gamma = \frac{1}{|G|}, \qquad \tau = \frac{\text{the number of pseudo-reflections in } G}{2|G|} .$$

Therefore in this case

(48) $$\frac{2\tau}{\gamma} = \text{the number of pseudo-reflections in } G.$$

If G is finite and $k[V]^G$ is Gorenstein then it follows from (46) that

$$q = \dim V + (\text{the number of pseudo-reflections in } G) .$$

If for instance $G \subset SL_n$ then $q = \dim V$ (it should be noted also conversely that if $k[V]^G$ is Gorenstein and $q = \dim V$ then $G \subset SL_n$, cf. [13]).

It seems to me to be very interesting to establish whether there exists an interpretation of the numbers γ, τ and $\frac{2\tau}{\gamma}$ completely in terms of the internal properties of G itself, that is analogous to (47), (48), when G is a connected semisimple group.

It is proved in [14] that if G is a connected semisimple and the trivial character of T is "critial" then

(47) $$q = \dim V, \text{ hence } \frac{2\tau}{\gamma} = (\dim Gv,\ v \in V \text{ is a point in general position}).$$

If for instance G is an image of SL_2 under its natural representation in the space of binary forms of degree n-1, then the trivial character is critical and (47) is fulfilled when $n \geqslant 4$, [14] (and (47) is not fulfilled when n = 2,3).

8. We shall discuss in this last section an approach to the solution of the problem of constructive invariant theory suggested in [2].

The general strategy in [2] is the same as in n.3 - it provides an effective way to calculate the constant C satisfying the property (18). (It should be incidentally noted that the reasoning on page 67 in [2] is not correct. That is to say, the ring B' depends on the point x_0, therefore the minimal equations of $\det(Y_{ij})$ over the fields of fractions of the rings B' constructed for the rings B' constructed for x_0 and $s.x_0$ are *a priori* different. Hence, the equality $Q_k((Y_{ij}).(s.x_0)) =$ $= Q_k((Y_{ij}).x_0)$ remains unsubstantiated. As a matter of fact, the assertion itself is correct but one has to change the proof using Reynolds operator $\natural$ as it was done above, n.4, formula (27)). After finding the constant C, the authors consider an ideal J in k[V] generated by all polynomials f_v, $v \in V-N$, see (18), and then notice: "the Nullstellensatz shows that some power of every invariant is in J. As we may (at least theoretically) pick a finite system of generators of J of degree $\leqslant M'$ (= C) we are then reduced to forming a system of generators of the "root" of J; Hilbert's own proof of his Nulstellensatz shows how this (again theoretically) may be done explicitly". But in reality this assertion leads to the solution *only if* $\sqrt{J}$ coincides with the ideal I in k[V] generated by all G-invariant polynomials of strictly positive degree (it is easy to understand that a set of invariants, that is a homogeneous system of generators of I, is also a system of generators of $k[V]^G$). However, in general we have only the equality

$$\sqrt{J} = \sqrt{I}.$$

Therefore $\sqrt{J} = I$ if and only if I is *the radical ideal*, i.e. $\sqrt{I} = I$. There seems to be an impression that in [2] the equality $\sqrt{I} = I$ is considered to be self-evident. However, in general this equality *does not* take place. Examples are given below, but here we say only that when $\sqrt{I} \neq I$ we can say nothing in general about the basis of I, knowing the basis of $\sqrt{I}$ (hence the approach suggested in [2] does not lead to the solution).

Now , for the examples. Firstly, it is evident that if G is finite then $\sqrt{I}$ always coincides with the ideal in $k[x_1,\ldots,x_n]$ generated by $x_1,\ldots,x_n$. Hence $\sqrt{I} \neq I$ if G is not trivial. It is more interesting (and more difficult) to construct such an example when G is a connected semisimple group. One can find in [3], pp. 159-160, an example when G is isomorphic to SO(2k+1). But it is only the classical case of homogeneous polynomial representations of SL_d under consideration in [2]. Therefore it will be more instructive to have an example valid for this case. We shall show now that if G is an image of SL_2 under its natural representation in the space of binary forms of degree 6, then $\sqrt{I} \neq I$.

Denote by R_d the unique (up to isomorphism) irreducible SL_2-module of degree d+1. It is known that the multiplicity c(d,e) of R_d in $k[V]_e$ is equal to the dimension of the space of covariants of binary form of degree e that are homogeneous of degree **6** in the coefficients and order d in the variables (i.e. c(d,e) = m(e,**6**,d) in the notations of [11]). Calculating these numbers by means of one known method or another (I used Sylvester's tables of Poincaré series for the covariants, [16]) one obtains the following table

e	1	2	3	4	5	6	8	10
d	2	4	2	4	2	4	4	4
c(d,e)	0	1	1	2	2	4	7	11

It is known that $k[V]^G$ has a minimal homogeneous system of generators $f_2, f_4, f_6, f_{10}, f_{15}$, $\deg f_i = i$, with f_2, f_4, f_6, f_{10} algebraically independent and $f_{15}^2 \in k[f_2, f_4, f_6, f_{10}]$, see [11].

Clearly $I_5 = k[V]_3 f_2 \oplus k[V]_1 f_4$. It follows from the table above that the multiplicity of R_2 in I_5 is equal to 1. We can conclude now from $c(2,5) = 2$ that there exists a submodule R_2 in $k[V]_5$ that does not lie in I_5. We also have $I_{10} = k[V]_8 f_2 + k[V]_6 f_4 + k[V]_4 f_6 + k[V]_0 f_{10}$. The polynomials f_2, f_4, f_6 clearly are irreducible, hence $k[V]_8 f_2 \cap k[V]_6 f_4 \cap k[V]_4 f_6 = 0$.
But $k[V]_8 f_2 + k[V]_6 f_4 \cong (k[V]_8 f_2 \oplus k[V]_6 f_4)/(k[V]_8 f_2 \cap k[V]_6 f_4)$ and it is easy to understand that $k[V]_8 f_2 \cap k[V]_6 f_4 = k[V]_4 f_2 f_4$. It follows from here and from the table above that the multiplicity of R_4 in I_{10} is equal to $c(4,8) + c(4,6) - c(4,4) + c(4,4) = 7+4 = 11 = c(4,10)$. Therefore, every irreducible submodule R_4 in $k[V]_{10}$ lies in fact in I_{10}. Let us now take a submodule R_2 in $k[V]_5$ that does not lie in I_5 and let f be an element of this submodule that is a highest vector (with respect to some fixed maximal torus and Borel subgroup B of SL_2). So f is a B-semi-invariant of the weight 2. But $f^2 \in k[V]_{10}$ is evidently a B-semi-invariant of the weight 4. Therefore f^2 is a highest vector of a module $R_4 \subset k[V]_{10}$. It follows from the above that $f^2 \in I_{10}$. We now see that $f \in k[V]$, $f \notin I$, but $f^2 \in I$, i.e. $\sqrt{I} \neq I$.

It will be interesting to clarify whether or not I is radical for the other irreducible representations of SL_2 (it is easy to see that I is radical for the representations of dimensions 2,3,4,5; apparently it is not the case for the dimensions $\geqslant 8$). The property for I to be radical seems to be very ingenious and to reflect important properties of G-module k[V] (for example one can prove that if $\sqrt{I} = I$ and there exists a dense orbit in N, then k[V] is a free $k[V]^G$-module). It would also be interesting to classify the connected semisimple groups for those $\sqrt{I} = I$.

The fact that $\sqrt{I} \neq I$ leads in general to a number of questions about the structure of I itself. There is as yet little known in this direction. We want to conclude by formulation a conjecture about the structure of the reduced primary decomposition of I.

Let X(T) be the character group of T (written additively). We consider X(T) as a lattice in $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let Δ be the set of the weights of T in V. Denote by V_λ the weight space in V of the weight $\lambda \in \Delta$. Consider all the maximal subsets $\Delta^{(\mu)}$ of Δ that lie in an open halfspace of $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (depending on $\Delta^{(\mu)}$) and for each $\Delta^{(\mu)}$ consider the following subspace in V

$$N_T^{(\mu)} = \bigoplus_{\lambda \in \Delta^{(\mu)}} V_\lambda .$$

Let $k[V]^T$ be the subalgebra of T-invariants in k[V], I_T the ideal in k[V] generated by all homogeneous elements of $k[V]^T$ of strictly positive degree and N_T the variety of zeros of I_T in V. It follows from the Hilbert-Mumford theorem, [9], that

$$N_T = \bigcup_{\mu} N_T^{(\mu)},$$

$$N = GN_T = \bigcup_{\mu} GN_T^{(\mu)}.$$

It may happen in general that $GN_T^{(\mu)} \subsetneqq GN_T^{(\nu)}$.

Conjectures (Vinberg).

1). *There is no imbedded primes belonging to a reduced primary decomposition of* I_T,

2). *The varieties defined by imbedded primes belonging to a reduced primary decompostion of* I *are exactly those* $GN_T^{(\mu)}$ *for which there exists such* ν *that* $GN_T^{(\mu)} \subsetneqq GN_T^{(\nu)}$.

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