

Astérisque

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Astérisque, tome 80 (1980), p. 53-56

[<http://www.numdam.org/item?id=AST_1980__80__53_0>](http://www.numdam.org/item?id=AST_1980__80__53_0)

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AN EXTENSION OF PIZZETTI'S FORMULA TO RIEMMANIAN MANIFOLDS

T.J. Willmore

This note summarizes joint work of Alfred Gray and myself. Let $M_m(r, f)$ denote the mean-value of a real-valued integrable function f over a sphere with centre m and radius r in n -dimensional euclidean space R^n . Then the formula of Pizzetti [PI.1], [PI.2], [CH, P.287] states that

$$(1.1.) \quad M_m(r, f) = \Gamma\left(\frac{1}{2}n\right) \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{(\Delta^k f)_m}{k! \Gamma\left(\frac{1}{2}n+k\right)}.$$

Formula (1.1) can be written more compactly as a Bessel function in $\sqrt{-\Delta}$ [ZA].

We have

$$(1.2) \quad M_m(r, f) = [j_{(n/2)-1}(r\sqrt{-\Delta})f](m)$$

where

$$j_\ell(z) = 2^\ell \Gamma(\ell + 1) J_\ell(z) / z^\ell$$

and J^1 is the Bessel function of the first kind of order .

In this paper we generalize (1.1) to arbitrary Riemannian manifolds. Our formula also generalizes the mean-value theorem for harmonic spaces [WI]. Complete proofs will appear elsewhere. In the Riemannian case we define $M_m(r, f)$ as the mean-value of f over a geodesic sphere with centre m and radius r in an n -dimensional Riemannian manifold M . More precisely we have

$$(1.3.) \quad M_m(r, f) = \frac{\int_{\exp_m(S^{n-1}(r))} f \omega}{V(\exp_m(S^{n-1}(r)))}$$

where $\exp_m$ is the exponential map of M at m .

The exponential map $\exp_m$ can be used to transfer formulas from M to the tangent space M_m . Let g_{ij} be the components of the metric tensor with respect to a system of normal coordinates, $(x_1, \dots, x_n)$ and put

$$\theta = \sqrt{\det(g_{ij})}, \quad \tilde{\Delta}_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Both θ and $\tilde{\Delta}_m$ are independent of the choice of normal coordinates at m .

We have

Theorem 1.

$$(1.4) \quad M_m(r, f) = \frac{\sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma\left(\frac{1}{2}n+k\right)} \tilde{\Delta}_m^k [f\theta]_m}{\sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma\left(\frac{1}{2}n+k\right)} \tilde{\Delta}_m^k [\theta]_m}$$

$$= \frac{J_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})[f\theta]_m}{J_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})[\theta]_m}$$

The operator $\tilde{\Delta}_m$ is of some interest in itself. We prove that there is a globally defined differential operator L_{2k} of degree $2k$ which coincides with the k^{th} power of $\tilde{\Delta}_m$ at m . For example we find that

$$(1.5) \quad (\tilde{\Delta}_m f)_m = (\Delta f)_m$$

$$(1.6) \quad (\tilde{\Delta}_m^2 f)_m = (\Delta^2 f + \frac{1}{3} \langle df, d\tau \rangle + \frac{2}{3} \langle \nabla^2 f, \rho \rangle)_m$$

where Δ denotes the ordinary Laplacian, τ is the scalar curvature, ρ the Ricci curvature and $\nabla^2 f$ the Hessian of f . Using (1.4), (1.5) and (1.6) we prove

Theorem 2.

$$M_m(r, f) = f(m) + A(m)r^2 + B(m)r^4 + O(r^6)$$

as $r \rightarrow 0$, where

$$A = \frac{1}{2n} \Delta f$$

$$B = \frac{1}{24n(n+2)} (3\Delta^2 f - 2 \langle \nabla^2 f, \rho \rangle - 3 \langle \nabla f, \nabla \tau \rangle + \frac{4\tau}{n} \Delta f).$$

We have also computed the coefficient of r^6 but it is too complicated to write down here.

2. A CHARACTERIZATION OF EINSTEIN MANIFOLDS. As an immediate corollary to theorem 2 we have

$$(2.1) \quad M_m(r, f) = f(m) + O(r^4) \text{ as } r \rightarrow 0$$

if and only if f is harmonic near m . We prove

Theorem 3.

Let M be an Einstein manifold and let $m \in M$. Then for small $r > 0$, every function harmonic near m has the mean-value property

$$(2.2) \quad M_m(r, f) = f(m) + O(r^6) \text{ as } r \rightarrow 0$$

Conversely, we have

Theorem 4. Let M be a analytic manifold and let $m \in M$. If for small $r > 0$, every function harmonic near m has the mean-value property

$$(2.3) \quad M_m(r, f) = f(m) + o(r^6) \text{ as } r \rightarrow 0.$$

then M is Einstein.

The analyticity is required in theorem 4 because our proof depends upon the Gauchy-Kowalewski existence theorem for elliptic operators. We thus obtain a characterization of Einstein manifolds by the mean-value property (2.2). A similar but more complicated characterization of Einstein manifolds is given in [FR].

3. A CHARACTERIZATION OF SUPER-EINSTEIN MANIFOLDS. We denote by $\dot{R}$ the symmetric tensor field given by

$$\dot{R}(x,y) = \sum_{i,j,k=1}^n R(e_i, e_j, e_k, x) R(e_i, e_j, e_k, y)$$

for $x, y \in M_m$, where $e_1, \dots, e_n$ is an orthonormal basis of M_m . We define a manifold to be super-Einstein if it is Einstein and in addition satisfies the condition

$$(3.1) \quad \dot{R}(x,y) = \frac{1}{n} ||R||^2 \langle x, y \rangle \quad \text{for } n > 4,$$

$$||R||^2 = \text{constant for } n = 4.$$

This definition is suggested in [BE, p.165]. It is easy to see that an irreducible symmetric space is super-Einstein. There exist metrics on spheres of dimension $4n + 3$ which are Einstein but not super-Einstein.

We prove

Theorem 5. Let M be a super-Einstein manifold, and let $m \in M$. Then for small $r > 0$, every function harmonic near m has the mean-value property

$$M_m(r, f) = f(m) + O(r^8) \text{ as } r \rightarrow 0.$$

The proof depends upon the explicit calculation of the coefficient of r^6 in the expansion of $M_m(r, f)$. Conversely, we have

Theorem 6. Let M be an analytic manifold, and let $m \in M$. If for small $r \rightarrow 0$, every function harmonic near m has the mean-value property

$$M_m(r, f) = f(m) + O(r^8) \text{ as } r \rightarrow 0, \text{ then } M \text{ is super-Einstein.}$$

Again we make use of the Gauchy-Kowalewski theorem to prove theorem 6. We thus obtain a characterization of super-Einstein manifolds.

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