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REALIZATIONS OF A SINGLE VOLTERRA KERNEL

par

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ABSTRACT. - The input-output map of a nonlinear analytic system can be expanded in a suitable domain as a Volterra series. The Volterra kernels,each of which defines a term in the series,can be expressed in terms of the system data. In this paper these expressions are used to realize a single term in the series, again directly in terms of the system data. This will lead in later work to a synthesis algorithm for systems with finite Volterra series.

I. - INTRODUCTION. - It has been shown in BROCKETT [1], BROCKETT and GILBERT [2], KRENER and LESIAK [7], that the input-output maps of a large class of nonlinear analytic system have convergent Volterra series expansions. In the linear and bilinear cases the Volterra kernels have well known expressions in terms of the system matrices. KRENER and LESIAK [7] have provided similar formulas for the Volterra kernels in terms of the vector fields and functions defining the system.

Generalizing to the bilinear case P. d'ALLESSANDRO et al.[5] provided an algorithm which synthesised bilinear realizations of Volterra series from the Volterra kernels. Another method was given in BROCKETT [1] for the case of finite Volterra series.

In GILBERT [3] and CROUCH [4], it was shown that a finite Volterra series has a nonlinear realization in the form of a cascade of linear systems with polynomial link maps. In CROUCH [4] it was shown that the state space of a minimal realization (in the sense of SUSSMANN [6]) of a finite Volterra series has a vector space structure, and can also be written as a cascade of linear systems. However to date no algorithm has been given which syntheses cascade realizations of finite Volterra series, in general.

The purpose of this paper is to provide a cascade realization of a single term in a Volterra series expansion of a non-linear stationary system, directly in terms of the system data. In conjunction with the previous work in CROUCH [4], this will provide the necessary structure for the synthesis algorithm above.

II. - PRELIMINARY DEFINITIONS AND RESULTS.

The following non-linear analytic system will be considered :

$$(1) \qquad \begin{cases} \dot{x} = f(x) + u\, g(x), & x(o) = x_o, \quad x \in \mathbb{R}^n \\ y = h(x) \end{cases}$$

where f and g are analytic vector fields on $\mathbb{R}^n$ (column n-vectors) and **h** is an analytic function.

THEOREM. - KRENER+LESIAK [7], BROCKETT [1], BROCKETT+ GILBERT [2].

If the equations with $u \equiv 0$, have a solution on $[0, T]$ then for all integrable u, satisfying $\int_o^T |u|\, ds < t$, and t sufficiently small, the input-output map of non-linear analytic system can be written as a uniformly convergent Volterra series on $[0, T]$:

$$(2) \qquad Y(t) = W_o(t)\,(x_o) + \sum_{i=1}^{\infty} \int_o^t \int_o^{\sigma_1} \cdots \int_o^{\sigma_{i-1}} W_i(t, \sigma_1 \ldots \sigma_i)\,(x_o)\, u(\sigma_1) \ldots u(\sigma_i)\, d\sigma_i \ldots d\sigma_1 .$$

Since the Volterra kernels W_i not only depend on the real parameters $t, \sigma_1 \ldots \sigma_i$, but also on the initial condition x_o they are viewed as real valued functions on $\mathbb{R}^n$:

$$x \longrightarrow W_i\,(t, \sigma_1 \ldots \sigma_i)\,(x) .$$

To express these kernel functions in terms of f, g and h, some convenient notation is introduced.

If a and b are analytic vector fields on $\mathbb{R}^n$, define a covariant derivative $\nabla_a b$ as the vector field :

$$x \longrightarrow (\nabla_a b)\,(x) = \nabla_{a(x)} b$$

whose i^{th} component is given by :

$$(\nabla_{a(x)} b)_i = \sum_{j=1}^{n} \frac{\partial b_i}{\partial x_j}(x)\, a_j(x) \quad .$$

When b is an analytic function $\nabla_a b$ simply represents the directional derivative of b in the direction a . An easy computation shows :

$$\nabla_a b - \nabla_b a - [a, b] = 0 \tag{3}$$

for arbitrary vector fields a and b , where [,] is the Lie bracket. The Lie derivative of a vector field b by a vector field a well be denoted by :

$$L_a b = [a, b]$$

and higher order derivatives by :

$$L_a^k b = L_a^{k-1} [a, b] \; ; \quad L_a^o b = b \quad .$$

If a is a vector field, let γ_a denote the flow of a . Thus on some maximal neighborhood of $0 \in \mathbb{R}$ depending on $x \in \mathbb{R}^n$

$$\frac{d}{dt} \gamma_a(t)(x) = a(\gamma_a(t)(x)) \; , \quad \gamma_a(0)(x) = x \quad .$$

Let $\gamma_a(t)_*$ denote the differential of the local diffeomorphism $\gamma_a(t) : \mathbb{R}^n \to \mathbb{R}^n$. For t sufficiently small the one parameter vector field $\gamma_a(-t)_* b(\gamma_a(t)(x))$ is given by the convergent series :

$$\sum_{i=o}^{\infty} t^i/_{i!} \; L_a^i \, b\,(x)$$

and sometimes denoted by $\exp t\, L_a(b)(x)$.

In the case where $a = f$ and $b = g$ set :

$$g(\sigma)(x) = \gamma_f(-\sigma)_* \; g(\gamma_f(\sigma)(x))$$

and also define inductively the n-parameter vector fields :

$$g_n(\sigma_1 \ldots \sigma_n) = \nabla_{g(\sigma_n)} g_{n-1}(\sigma_1 \ldots \sigma_{n-1}) \; , \quad g_1(\sigma_1) = g(\sigma_1) \; . \tag{4}$$

The kernel functions are now described in the following result :

THEOREM. - KRENER + LESIAK [7].

$$W_n(t,\sigma_1\ldots\sigma_n)(x) = \nabla_{g(\sigma_n)}(x)\,\nabla_{g(\sigma_{n-1})}\,\nabla_{g(\sigma_{n-2})}\cdots\nabla_{g(\sigma_1)}\,(h\circ\gamma_f(t))$$
$$W_o(t)(x) = h\circ\gamma_f(t)(x)\ .$$

COROLLARY. -

$$W_n(0,\sigma_1-t,\ldots,\sigma_n-t)\,(\gamma_f(t)(x)) = W_n(t,\sigma_1,\ldots,\sigma_n)(x)\ .$$

Proof : Let a be a vector field, b a function, and γ a local diffeomorphism, then by definition of the differential :

$$(\nabla_a\, b\circ\gamma)(x) = \nabla_{a(x)}\, b\circ\gamma = \nabla_{\gamma_* a(x)}\, b = (\nabla_{\gamma_* a}\, b)\circ\gamma(x)$$

where $(\gamma_* a)(\mathbf{x}) = \gamma_* a(\gamma^{-1}(x))$. Since

$$(\gamma_f(t)_*\, g(\sigma_i))(x) = g(\sigma_i - t)(x)$$

the corollary now follows by applying the theorem. QED.

The main aim of this paper is to find a realization of the p' th term of the Volterra series expansion in equation (2) where the system is stationary, that is $f(x_o) = 0$. By appealing to the corollary this amounts to finding a realization of the following input-output map when $x = x_o$.

(5) $$y_p(t)(x) = \int_o^t \int_o^{\sigma_1}\cdots\int_o^{\sigma_{p-1}} W'_p(\sigma_1 - t,\ldots,\sigma_p - t)(x)\,u(\sigma_1)\ldots u(\sigma_p)\,d\sigma_p\ldots d\sigma_1$$

where

(6) $$W'_p(\sigma_1\ldots\sigma_p)(x) = \nabla_{g(\sigma_p)(x)}\,\nabla_{g(\sigma_{p-1})}\cdots\nabla_{g(\sigma_1)}\,h\ .$$

In fact the realization constructed is valid for all $x\in\mathbb{R}^n$. The key observation in obtaining such a realization is the following result.

LEMMA. - Let $a, b, a_1,\ldots,a_r$ be analytic vector fields and define a differential operator on vector fields by :

$$d \rightarrow \nabla_c d = \nabla_{a_r}\nabla_{a_{r-1}}\cdots\nabla_{a_2}\nabla_{a_1} d\ .$$

Then the following identity holds.

(7) $$L_b(\nabla_c a) + (\nabla_{\nabla_c a}\, b - \nabla_c\nabla_a b) = \nabla_c(L_b a) + \sum_{i=1}^{r}\nabla_{a_r}\nabla_{a_{r-1}}\cdots\nabla_{L_b a_i}\cdots\nabla_{a_1} a\ .$$

Proof : By using equation (3) the identity reduces to :

$$\nabla_b \nabla_c a = \nabla_c \nabla_b a + \sum_{i=1}^{r} \nabla_{a_r} \nabla_{a_{r-1}} \cdots \nabla_{L_b(a_i)} \cdots \nabla_{a_1} a .$$

However this simply follows from the identity.

(8) $$\nabla_b \nabla_d a - \nabla_d \nabla_b a - \nabla_{[b,d]} a = 0$$

which is valed for any vector fields a, b and d. Q.E.D.

§3.- The first stage in obtaining a realization is to isolate the dependence of W'_p on h. Let $x \to h^{(r)}(x)$ denote the r'th derivative of h, where for each x, $h^{(r)}(x)$ is a symmetric r-linear map $(v_1, \dots, v_r) \to h^{(r)}(x)(v_1 \dots v_r)$ $\overbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}^{r} \to \mathbb{R}$. It is clear from the definition of W'_p in equation (6) that it has an expansion of the form :

$$W'_p(\sigma_1 \dots \sigma_p)(x) = h^{(1)}(x)(g_p(\sigma_1 \dots \sigma_p)(x)) + \dots + h^{(p)}(x)(g_1(\sigma_1)(x) \dots g_1(\sigma_p)(x)).$$

Denote this expansion by :

(9) $$W'_p(\sigma_1 \dots \sigma_p) = H'_p(g_1 \dots g_p)$$

where H'_p is a linear function in the components of the vectors $g_1(\sigma_1)(x), \dots,$ $g_1(\sigma_p)(x), \dots, g_p(\sigma_1 \dots \sigma_p)(x)$, with coefficients depending on x.

The terms in this expansion are grouped in the following way. For each s $1 \leqslant s \leqslant p$, consider those terms involving the s'th derivative of h only, and in which g_j terms $1 \leqslant j \leqslant q$, appear r_j times, $q = p-(s-1)$. It is easily shown that will be $p!/(r_1! \dots r_q!)\,(2!)^{r_2} \dots (q!)^{r_q}$ terms in this group specified by the integers $(q, r_1 \dots r_q)$, which must satisfy $\sum_{i=1}^{q} r_i = s$, $\sum_{i=1}^{q} i\, r_i = p$.

By introducing the control dependent vector fields

(10) $$x_i(t)(x) = \int_0^t \int_0^{\sigma_1} \dots \int^{\sigma_{i-1}} g_i(\sigma_1 - t, \dots, \sigma_i - t)(x)\, a(\sigma_1) \dots a(\sigma_i)\, d\sigma_i \dots d\sigma_1$$

it is easily verified that $y_p(t)(x)$ is a sum of terms of the form

$$1/(r_1! \dots r_q!)\, h^{(s)}(x)\, (\overbrace{x_1(t)(x) \dots x_1(t)(x)}^{r_1}\ \overbrace{x_2(t)(x) \dots x_2(t)(x)}^{r_2} \dots x_q^{r_q}(x))$$

where each term is due to the contribution of a group of terms as described above, and is therefore specified by a set of integers $(q, r_1 \dots r_q)$. This sum of terms will be denoted by setting

(11) $$y_p(t) = H_p(x_1(t) \dots x_p(t))$$

where H_p is a polynomial function in the components of the vectors $x_1(t) \dots x_p(t)$. Notice that since for each q , $\sum_{i=1}^{q} i\, r_i = p$, $\lambda^p H_p(x_1 \dots x_p) = H_p((\lambda x_1) \dots (\lambda^p x_p))$ for $0 \neq \lambda \in \mathbb{R}$. The coefficient of a term in H_p specified by $(q, r_1 \dots r_q)$ is simply related to those of H'_p through the division by $r_1! \dots r!$ of the coefficient of any term in the group specified by the same integers.

The realization of the input-output map given in equation (5) is now equivalent to realizing the vector fields $x_i(t)$ $1 \leq i \leq p$, and applying H_p as in equation (11).

§ 4. - In this section the identity appearing in the lemma is used to provide a set of non-linear differential equations for the vector fields g_i , which are then solved using a variation of parameters formula.

Recalling the definition of $g(\sigma)$ and $g_i(\sigma_1 \dots \sigma_i)$ it is easily verified that ${}^d/d\epsilon\, g(\sigma+\varepsilon) = L_f g(\sigma+\varepsilon)$ and hence

$$^d/d\epsilon\, g_{r+1}(\sigma+\varepsilon, \sigma_1+\varepsilon, \dots, \sigma_r+\varepsilon) = \sum_{i=1}^{r} \nabla_{g(\sigma_r+\varepsilon)} \cdots \nabla_{L_f g(\sigma_i+\varepsilon)} \cdots \nabla_{g(\sigma_1+\varepsilon)} g(\sigma+\varepsilon)$$
$$+ \nabla_{g(\sigma_r+\varepsilon)} \cdots \nabla_{g(\sigma_i+\varepsilon)} \cdots \nabla_{g(\sigma_1+\varepsilon)} L_f g(\sigma+\varepsilon) .$$

Setting $a = g(\sigma+\varepsilon)$, $b = f$, $a_i = g(\sigma_i+\varepsilon)$ in equation (7) gives

$$^d/d\epsilon\, g_{r+1}(\sigma+\varepsilon, \sigma_1+\varepsilon, \dots, \sigma_r+\varepsilon) = L_f\, g_{r+1}(\sigma+\varepsilon, \sigma_1+\varepsilon, \dots, \sigma_r+\varepsilon) +$$
$$+ \nabla_{g_{r+1}(\sigma+\varepsilon, \dots, \sigma_r+\varepsilon)} f - \nabla_{g(\sigma_r+\varepsilon} \nabla_{g(\sigma_i+\varepsilon)} \cdots \nabla_{g(\sigma+\varepsilon)} f .$$

The last two terms on the right hand side of this equation yield terms involving second and higher derivative of f . Letting

$$F'_r(g_1 \dots g_{r-1}) = \nabla_{g(\sigma_r)} \cdots \nabla_{g(\sigma_i)} \cdots \nabla_{g(\sigma_1)} f - \nabla_{g_r(\sigma_1 \dots \sigma_r)} f$$

the above equation can be written as

(12) $$-{}^d/d\epsilon\, g_{r+1}(\sigma+\varepsilon, \sigma_1+\varepsilon, \dots, \sigma_r+\varepsilon) = -L_f\, g_{r+1}(\sigma+\varepsilon, \dots, \sigma_r+\varepsilon) + F'_{r+1}(g_1, \dots, g_r) .$$

Notice that the resulting set of equations for g_i $1 \leq i \leq p$ can be solved inductively using the variation of parameters formula. In fact

$$-{}^{d}/d\varepsilon \ \exp\varepsilon - L_f \, g_{r+1}(\sigma+\varepsilon, \sigma_1+\varepsilon, \ldots, \sigma_r+\varepsilon) = \exp\varepsilon - L_f \, F'_{r+1}(g_1 \ldots g_r)$$

and hence by integrating with respect to t, between 0 and $-\sigma$

$$g_{r+1}(\sigma, \sigma_1 \ldots \sigma_r) = \exp -\sigma - L_f \, g_{r+1}(0, \sigma_1 - \sigma, \ldots, \sigma_r - \sigma) + \\ + \int_0^{-\sigma} \exp\varepsilon - L_f \, F'_{r+1}(g_1 \ldots g_r) \, d\varepsilon .$$

The "initial condition" can be reformulated as a sum of derivatives of g (as H'_p and F'_r) since

$$g_{r+1}(0, \sigma_1 \ldots \sigma_r) = \nabla_{g(\sigma_r)} \ldots \nabla_{g(\sigma_i)} \ldots \nabla_{g(\sigma_1)} \, g .$$

Thus letting

$$g_{r+1}(0, \sigma_1 \ldots \sigma_r) = G'_{r+1}(g_1 \ldots g_r) \tag{13}$$

the expression for g_{r+1} can now the written in the form

$$-g_{r+1}(\sigma - t, \sigma_1 - t, \ldots, \sigma_r - t) = \exp(t-\sigma) - L_f \, G'_{r+1}(g_1 \ldots g_r) + \\ + \int_\sigma^t \exp(t-s) - L_f \, F'_{r+1}(g_1 \ldots g_r) \, ds . \tag{14}$$

§ 5. - Equation (14) is now reformulated in terms of the vector fields $x_i(t)$ in order to obtain the desired set of equations.

Using the definition of $x_{r+1}(t)$ equation (14) can be written in the form

$$x_{r+1}(t) = \int_0^t \exp(t-\sigma) - L_f \left(\int_0^\sigma \int_0^{\sigma_1} \ldots \int_0^{\sigma_{r-1}} G'_{r+1}(g_1 \ldots g_r) \, u(\sigma_1) \ldots u(\sigma_r)) \, d\sigma_r \ldots d\sigma_1 \right) u(\sigma) d\sigma \\ + \int_0^t \int_\sigma^t \exp(t-s) - L_f \left(\int_0^\sigma \int_0^{\sigma_1} \ldots \int_0^{\sigma_{r-1}} F'_{r+1}(g_1 \ldots g_r) \, u(\sigma_1) \ldots u(\sigma_r) \, u(\sigma) \, d\sigma \right. \\ \left. d\sigma_r \ldots d\sigma_1 \right) u(\sigma) \, ds \, d\sigma .$$

The second term on the right hand side of the equation can be reexpressed, by interchanging the order of integration between s and σ to give

$$\int_0^t \exp(t-s) - L_f \left(\int_0^s \int_0^\sigma \ldots \int_0^{\sigma_{r-1}} F'_{r+1}(g_1 \ldots g_r) \, u(\sigma) \, u(\sigma_1) \ldots u(\sigma_r) \, d\sigma_r \ldots d\sigma_1 \, d\sigma \right) ds .$$

By making the obvious definitions of the vector valued polynomial functions G_{r+1} and F_{r+1} the following expression for $x_{r+1}(t)$ is obtained

$$x_{r+1}(t) = \int_o^t \exp(t-s) - L_f(F_{r+1}(x_1(s), \ldots, x_r(s)) + u(s)\, G_{r+1}(x_1(s)\ldots x_1(s)))\, ds \; .$$

This is easily recognised as the solution to the equation

$$\dot{x}_{r+1}(t) = -L_f\, x_{r+1}(t) + F_{r+1}(x_1(t)\ldots x_r(t)) + u(t)\, G_{r+1}(x_1(t)\ldots x_r(t)) \; .$$

The main result now follws.

THEOREM. - The input-output map of equations (5) has the following realization

$$\begin{aligned}
\dot{x}_1 &= -L_f\, x_1 && + u\, g && x_1(0) = 0 \\
\dot{x}_2 &= -L_f\, x_2 + F_2(x_1) && + u\, G_2(x_1) && x_2(0) = 0 \\
&\vdots \\
\dot{x}_p &= -L_f\, x_p + F_p(x_1\ldots x_{p-1}) && + u\, G_p(x_1\ldots x_{p-1}) && x_p(0) = 0 \\
y_p &= H_p(x_1\ldots x_p) \; .
\end{aligned}$$

F_i , G_i and H_p are (vector valued) polynomials in the components of the state vectors $x_1\ldots x_p$, satisfying the homogeneity relations $(0 \neq \lambda \in \mathbb{R})$

$$\lambda^i\, F_i(x_1\ldots x_{i-1}) = F_i(\lambda\, x_1\ldots \lambda^{i-1}\, x_{i-1}) \quad \lambda^{i-1} G_i(x_1\ldots x_{i-1}) = G_i(\lambda\, x_1\ldots \lambda^{i-1}\, x_{i-1})$$
$$\lambda^p\, H_p(x_1\ldots x_p) = H(\lambda\, x_1\ldots \lambda^p\, x_p) \; .$$

Note that F_i , G_i and H_p are related to f , g and h via the kernel functions g_i defined in equation (4) and F_i' , G_i' and H_p' defined in equations (12), (13) and (9) respectively. The solutions of the above equations are expressed directly in terms of the kernel functions g_i via equation (10).

The techniques involved here can easily be extended to multi-input, multi-output non-linear systems using a generalization of the expression for the Volterra kernels given in equation(4) (see CROUCH [4]). Moreover since all the analysis performed is of a local nature these results apply equally as well to non-linear systems defined on manifolds. It is noted however that the covariant derivative defined here satisfies the equations (3) and (8). That is the torsion and curvature tensors vanish identically

$$0 \equiv T(a, b) = \nabla_a b - \nabla_b a - [a, b]$$

$$0 \equiv R(a, b)\, c = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a, b]} c \ .$$

Both these properties are used in the analysis and so other choices of covariant derivative cannot be used.

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