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ON CANONICAL FORM FOR COMPLETELY
REACHABLE DYNAMICAL SYSTEMS

par

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INTRODUCTION.

In this paper we investigate the existence of continuous (algebraic) canonical forms for linear, time-invariant, completely reachable dynamical systems on a field K .

Roughly speaking, the situation is the following: a dynamical system σ is given by a pair (F, G) , F and G being respectively $n \times n$ and $n \times m$ matrices, up to the equivalence induced by a change of basis in state space. A canonical form is the choice of a representative element in the equivalence class of pairs (F, G) which defines σ (see [9]).

Endowing the set $SCR(m, n)$ of all completely reachable pairs (F, G) with a topological structure (if K coincides with $\mathbb{R}$ or $\mathbb{C}$) or with a geometric one (if K is an algebraically closed field) and interpreting a canonical form as a particular endomorphism c of $SCR(m, n)$ (see 1.4), one can demand that c is also continuous or algebraic. Canonical forms of this kind are useful in e.g. identification problems (see [1, 2, 10]), but, as proved in [5], there are no globally defined continuous (algebraic) canonical forms on $SCR(m, n)$ when $m > 1$.

Here, we describe (see 1.6) the equivalence between local continuous (algebraic) canonical forms and local triviality of a particular vector bundle on the

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variety $M_{m,n}$ of completely reachable systems (see [5]). This allows us to determine (see 2.4) a class of open subsets of $SCR(m,n)$, useful in systems theory, on which there exists a local continuous (algebraic) canonical form. Obviously, the property, for a family Σ of systems, to be contained in one of the previous subsets is sufficient to assure that there is one local continuous (algebraic) canonical form defined for all the elements of Σ . Moreover, we prove (see 2.6) that if Σ is finite and K has infinitely many elements the above condition is satisfied.

1. - We consider linear time-invariant completely reachable dynamical systems

$$\begin{aligned}\dot{x} &= Fx(t) + Gu(t) \quad (\text{continuous time}) \quad \text{and} \\ x(t+1) &= Fx(t) + Gu(t) \quad (\text{discrete time})\end{aligned}$$

where F, G are respectively $n \times n$ and $n \times m$ matrices with entries in a field K . I.e. the state space dimension is n and there are m inputs.

1.1. - A change of basis in state space changes the pair (F, G) as follows

$$(F, G) \longmapsto (T \cdot F \cdot T^{-1}, T \cdot G) \quad T \in GL(n, K).$$

Then the systems we are considering are represented by the orbits of the above described action of $GL(n, K)$ on the set $SCR(m, n)$ of all completely reachable pairs (F, G) .

This action may be considered from two different points of view. Namely one may assume that K coincides with $\mathbb{R}$ or $\mathbb{C}$ or that K is an algebraically closed field. So what we have is a continuous action in the first case and an algebraic action in the second (see [3] and [4]).

Our treatment is applicable to both the cases and distinction will be made only when necessary.

1.2. - PROPOSITION. ([5] 3.7 and [7]). Let $G_{n,r}$ be the Grassman variety of n -dimensional subspaces V^n of K^r , $r = (n+1)m$.

The orbit space $SCR(m, n)/GL(n, K)$, denoted $M_{m, n}$, is a quasi projective subvariety of $G_{n, r}$.

1.3. - Since $G_{n, r}$ may be considered as the orbit space of the action, given by rows by columns product, of $GL(n, K)$ on the space $A_{reg}^{n \times r}$ of all maximal rank $n \times r$ matrices, the situation of 1.2 may be described by the following commutative diagram

$$\begin{array}{ccc} SCR(m, n) & \xrightarrow{R} & A_{reg}^{n \times r} \\ p_1 \downarrow & & \downarrow p_2 \\ M_{m, n} & \xrightarrow{\text{inclusion}} & G_{n, r} \end{array}$$

where R is the continuous (algebraic) one-one morphism defined by

$$R(F, G) = (G \quad FG \quad F^2G \dots F^nG)$$

and where p_1, p_2 are the projection onto the orbit spaces.

1.4. - DEFINITION. Let $L \subset A_{reg}^{n \times r}$ be a $GL(n, K)$ -invariant subspace. A continuous (algebraic) canonical form on L is a continuous (algebraic) morphism

$$c : L \longrightarrow L \text{ such that}$$

- i) $c(S) = c(S')$ iff $S, S' \in L$ are $GL(n, K)$ -equivalent ;
- ii) S and $c(S)$ are $GL(n, K)$ -equivalent for any $S \in L$.

1.5. - Let $L^* = R^{-1}(L) \subset SCR(m, n)$. Since $SCR(m, n)$ is isomorphic to its R -image, a canonical form c on L defines a morphism (see [8] 2.2.)

$$c^* : L^* \longrightarrow L^*$$

which verifies i), ii) of 1.4 modified in the obvious way.

Then c^* is a continuous (algebraic) canonical form for the completely reachable systems which belong to L^* .

The question whether a canonical form exists on a given subset of $SCR(m, n)$ is motivated by e. g. identification of systems theory. To investigate this problem we first need the following.

1.6. - PROPOSITION. Let $\gamma = (E, p, G_{n,r})$ be the n -dimensional (algebraic) canonical vector bundle over $G_{n,r}$ (see [6] 2.2.5) and let $L \subset A_{reg}^{n \times r}$. Then there exists a continuous (algebraic) canonical form c on L iff the restricted bundle $\gamma|_{p_2(L)}$ is trivial.

Proof. The existence of a continuous (algebraic) canonical form c on L is equivalent to the existence of a continuous (algebraic) morphism

$$\tilde{c} : p_2(L) \longrightarrow L$$

such that $p_2 \cdot \tilde{c} = \text{identity}$.

Now, $\gamma|_{p_2(L)}$ is trivial iff there is an isomorphism

$$\alpha : p_2(L) \times K^n \longrightarrow E|_{p_2(L)} \quad \text{given by}$$

$$\alpha(x, v) = (x, v \cdot S_x) \quad \text{where } S_x \in A_{reg}^{n \times r} \quad \text{and}$$

$$\tilde{c} : x \longrightarrow S_x$$

is a continuous (algebraic) morphism between $p_2(L)$ and L such that $p_2 \cdot \tilde{c} = \text{identity}$.

1.7. - The bundle γ is non trivial and also $\gamma|_{M_{m,n}}$ is non trivial if the considered systems have more than one input (i. e. $m > 1$) (see [5] 6.2.).

Our purpose is then to describe a proper subset of $G_{n,r}$, useful in systems theory, on which γ is trivial.

2. - We consider the classical embedding of $G_{n,r}$ into $\mathbb{P}^N$, $N = \binom{r}{n} - 1$, obtained denoting by $x_0, \dots, x_N$ the grassmann coordinates, lexicographically ordered, of V^n .

Let λ denote a linear homogeneous form in the coordinates $x_0, \dots, x_N$ or, equivalently, a hyperplane of $\mathbb{P}^N$. Both the intersection subvariety $\lambda \cdot G_{n,r}$ and the

corresponding set of V^n in K^r will be denoted by $[\lambda]$.

2.1. DEFINITIONS (see [11] 2) For any linear homogeneous form λ , $[\lambda]$ is called linear complex.

A linear complex $[\lambda]$ is called special if it represent the set of all V^n of K^r which meet a fixed V^{r-n} , axis of the complex, in a proper subspace.

2.2. - PROPOSITION. A linear complex $[\lambda]$ is a special linear complex with axis a fixed V^{r-n} iff the coefficients of λ are the grassmann cohomordinates, antilexicographically ordered, of the V^{r-n} .

Proof. Expand by Laplace rule, respect to the first n rows, the determinant of the $r \times r$ matrix $\begin{pmatrix} S \\ S_0 \end{pmatrix}$ where $S \in A_{\text{reg}}^{n \times r}$ and S_0 is an $(r-n) \times r$ matrix whose rows span the fixed V^{r-n} .

Given a linear complex $[\lambda]$ we denote by W_λ the open subvariety of $G_{n,r}$ of the points $x \in G_{n,r}$ such that $x \notin [\lambda]$.

2.3. -REMARK - i) Consider for any $i = 0, \dots, N$ the linear homogeneous form x_i . We have that $[x_i]$ is a special linear complex and, in particular, the axis of $[x_0]$ is the $V^{r-n} \langle e_{n+1}, \dots, e_r \rangle$.

ii) The bundle $\gamma|_{W_{x_i}}$ is trivial for any $i = 0, \dots, N$ (see [6] 3.1.4).

A continuous (algebraic) canonical form on $L_i = p_2^{-1}(W_{x_i})$ is given by

$$L_i \ni S \longmapsto (S_i)^{-1} \cdot S$$

where $\det(S_i)$ is the i -th cohomordinate of $p_2(S)$.

2.4. -PROPOSITION Let $[\lambda]$ be a special linear complex, then $\gamma|_{W_\lambda}$ is trivial.

Proof. Let $\rho: K^r \longrightarrow K^r$ be a change of basis such that the ρ -image of the $V^{r-n} \langle e_{n+1}, \dots, e_r \rangle$, axis of $[x_0]$, is the V^{r-n} axis of $[\lambda]$.

ρ induces a continuous (algebraic) automorphisme ρ^* of γ and, since

$$\rho^*(\gamma|_{W_{x_0}}) = \gamma|_{W_\lambda} \text{ (see [8] 4.5), by 2.3 ii) } \gamma|_{W_\lambda} \text{ is trivial.}$$

2.5.-COROLLARY. Let $L \subset A_{\text{reg}}^{n \times r}$. A sufficient condition for the existence of a continuous (algebraic) canonical form on L is that there exists a special linear complex $[\lambda]$ with $p_2(L) \subset W_\lambda$.

2.6.-PROPOSITION Let the field K have infinitely many elements and let $L = \{S_1, \dots, S_n\} \times GL(n, K) \subset A_{\text{reg}}^{n \times r}$. Then there exists a special linear complex $[\lambda]$ such that $p_2(L) \subset W_\lambda$.

Proof. Let

$$A_i = \{S \in A_{\text{reg}}^{(r-n) \times r} \text{ such that } \det \begin{pmatrix} S \\ S_i \end{pmatrix} \neq 0, i = 1, \dots, n\}$$

$\bigcap_{i=1}^n A_i$ is non empty (see [8] 4.4), let X be a point in $\bigcap_{i=1}^n A_i$. We have that

$$\det \begin{pmatrix} T \cdot S_i \\ X \end{pmatrix} \neq 0 \text{ for } T \in GL(n, k), i = 1, \dots, n \text{ and so the conclusion follows from 2.2.}$$

2.7. - EXAMPLES Let $[\lambda]$ be a special linear complex and let $L_\lambda = p_2^{-1}(W_\lambda)$. By 2.3 ii) and 2.4 a continuous (algebraic) canonical forme on L_λ is the following : let ρ be as in 2.4 and let Y denote the associated $r \times r$ non singular matrix, then

$$L_\lambda \ni S \longrightarrow ((S \cdot Y^{-1})_0)^{-1} \cdot S.$$

In [8] 5 there are examples of families of systems, verifying the condition of 2.5, on wich the " classical " canonical form of 2.3 ii) are not defined .

In the same paper a canonical form of the above kind for these families is described .

Here we show that the condition of 2.5 is not necessary for the existence of continuous canonical form on connected subsets of $SCR(m, n)$ (see also [5] 7.2.).

At this aim assume $K = \mathbb{C}, n = 4, m = 3$ and denote grassmannn cohordinates by

$$x_{i_1 i_2 i_3 i_4}.$$

Let $V_1 = \{ x \in M_{3,4} \text{ such that}$

$$\left[\begin{array}{l} x_{i_1 i_2 i_3 i_4} = 0 \quad \text{if } \{i_1, i_2, i_3, i_4\} \not\subset \{1, 2, 3\} \\ x_{1235} + x_{1236} = 0 \\ x_{1235} x_{1239} = (x_{1234})^2 \end{array} \right. \quad \left. \vphantom{\begin{array}{l} x_{i_1 i_2 i_3 i_4} = 0 \\ x_{1235} + x_{1236} = 0 \\ x_{1235} x_{1239} = (x_{1234})^2 \end{array}} \right\}$$

$V_2 = \{ x \in M_{3,4} \text{ such that}$

$$\left[\begin{array}{l} x_{i_1 i_2 i_3 i_4} = 0 \quad \text{if } \{i_1, i_2, i_3, i_4\} \not\subset \{1, 2, 3\} \\ x_{1234} x_{1236} = (x_{1235})^2 \\ x_{1235} x_{1237} = (x_{1236})^2 \end{array} \right. \quad \left. \vphantom{\begin{array}{l} x_{i_1 i_2 i_3 i_4} = 0 \\ x_{1234} x_{1236} = (x_{1235})^2 \\ x_{1235} x_{1237} = (x_{1236})^2 \end{array}} \right\}$$

Then $p_1^{-1}(V_1) = L_1 = \{ (F_t, C) \times GL(n, K) \subset SCR(3, 4) \text{ with}$

$$F_t = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ t & 1 & 1 & t^2 \end{bmatrix} \quad t \in \mathbb{C}, \quad G = \begin{bmatrix} I_3 \\ 0 \ 0 \ 0 \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ t & 1 & 1 & t^2 \end{bmatrix}} \right\}$$

$p_1^{-1}(V_2) = L_2 = \{ (F'_s, G) \times GL(n, K) \subset SCR(3, 4) \text{ with}$

$$F'_s = \begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ 1 & s & s^2 & s^3 \end{bmatrix} \quad s \in \mathbb{C}, \quad G = \begin{bmatrix} I_3 \\ 0 \ 0 \ 0 \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ 1 & s & s^2 & s^3 \end{bmatrix}} \right\}$$

$L_1 \cup L_2$ is connected since $L_1 \cap L_2 = (F_1, G) \times GL(n, K) = (F'_1, G) \times GL(n, K)$.

The maps

$$c_1 : L_1 \longrightarrow L_1 \quad \text{and} \quad c_2 : L_2 \longrightarrow L_2$$

defined respectively as

$$c_1((F_t, G), T) = (F_t, G) \quad \text{and} \quad c_2((F'_s, G), T) = (F'_s, G)$$

are both continuous canonical forms on L_1 and on L_2 .

Since c_1 and c_2 coincide on $L_1 \cap L_2$, the map $c : L_1 \cup L_2 \longrightarrow L_1 \cup L_2$ defined as

$$c(A, B) = \begin{cases} c_1(A, B) & \text{if } (A, B) \in L_1 \\ c_2(A, B) & \text{if } (A, B) \in L_2 \end{cases}$$

is a continuous canonical form on $L_1 \cup L_2$.

Moreover if λ is a linear homogeneous form and a_{1234} denotes the coefficient of x_{1234} in λ , then if $a_{1234} = 0, \lambda(p_1(F'_0, G)) = 0$ and, if $a_{1234} \neq 0$, there exists $\bar{t} \in \mathbb{C}$ such that $\lambda(p_1(F_{\bar{t}}, G)) = 0$.

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