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R. M. HIRSCHORN

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INVERSES FOR NONLINEAR CONTROL SYSTEMS

by

R.M. HIRSCHORN

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ABSTRACT .- A nonlinear control system is invertible if the associated input-output map is injective for nonlinear systems of the form $\dot{x} = A(x) + \sum_{k=1}^m u_k B_k(x)$ $y = C(x)$ which evolve on a real analytic manifold we obtain sufficient conditions for invertibility and construct systems which act as inverse systems. In the case of single-input systems our conditions are necessary and sufficient for invertibility. For invertible systems we construct nonlinear systems which act as left-inverses for the original systems.

1.- INTRODUCTION. Consider the system

$$\begin{aligned}\dot{x}(t) &= A(x(t)) + \sum_{i=1}^m u_i B_i(x(t)) \quad ; \quad x(0) = x_0 \in M \\ y(t) &= C(x(t))\end{aligned}$$

where M is a connected real analytic manifold, $A, B_i \in V(M)$, the real vector space of real analytic vector fields on M , $C : M \rightarrow \mathbb{R}^m$ is a real analytic mapping, and $u = (u_1, \dots, u_m)$ is a real analytic control function mapping $[0, \infty)$ into $\mathbb{R}^m$. Let $x(t, u, x_0)$ denote the solution of the above differential equation and set $y(t, u, x_0) = C(x(t, u, x_0))$. The system (*) is said to be invertible at x_0 if distinct controls $u \neq \hat{u}$ result in distinct outputs $y(\cdot, u, x_0) \neq y(\cdot, \hat{u}, x_0)$ and strongly invertible if there exists an open dense submanifold M_0 of M such that for all $x_0 \in M_0$, the system is invertible at x_0 . There is a considerable amount of literal dealing with invertibility for linear control system (cf. [1], [2], [3], [4]) and some partial results are known for more general classes of systems (cf. [5], [6], [7]). The purpose of this paper is to indicate a way in which a standard linear system argument (see [3]) can be generalized to study the invertibility of certain nonlinear systems.

2. - NONLINEAR INVERTIBILITY AND INVERSE SYSTEMS.

A standard linear test for invertibility involves creating a sequence of systems by differentiating the output map (see [3]). Following this approach we let $y(t)$ denote the output $y(t, u, x_0)$ for the system (*). Differentiating y with respect to t we find that

$$\dot{y}^{(1)}(t) = dC_{x(t)}(A(x(t)) + \sum_{i=1}^m u_i(t) B_i(x(t))) = AC(x) + \sum_{i=1}^m u_i B_i C$$

where $\forall x \in V(M)$ and $f: M \rightarrow \mathbb{R}^l$, $X F(x) = df_x X(x)$ (cf. [8]). Thus we can write $\dot{y}^{(1)}(t) = AC(x(t)) + D(x(t))u$ where $D(x) = [B_1 C(x) \ B_2 C(x) \ \dots \ B_m C(x)]$ is a $m \times m$ matrix for each $x \in M$. Let $\Gamma_1 = \max_{x \in M} \{\text{rank } D(x)\}$. We assume that the components of C have been reordered so that the submatrix $D_{11}(x)$ of $D(x)$ consisting of the first Γ_1 rows of $D(x)$ has rank Γ_1 for some $x \in M$. Set $M_1 = \{x \in M \mid \text{rank } D_{11}(x) = \Gamma_1\}$. It follows from the real analyticity of the entries of $D(x)$ that M_1 is an open dense submanifold of M . Now let

$$E_o(x) = \left[\begin{array}{c|c} I_{\Gamma_1 \times \Gamma_1} & O \\ \hline F_o(x) & I_{(m-\Gamma_1) \times (m-\Gamma_1)} \end{array} \right]$$

be an $m \times m$ elementary matrix whose entries are real analytic functions on M_1 and with the property that

$$E_o(x) D(x) = \begin{bmatrix} D_{11}(x) \\ O \end{bmatrix}.$$

This results in a new system

System (1) :

$$\dot{x} = A(x) + \sum_{i=1}^m u_i B_i(x); x \in M_1$$

$$Z_1 = C_1(x) + D_1(x) u$$

where $C(x) = E_o(x) A C(x)$, $D_1(x) = E_o(x) D(x)$, and by construction $D_1(x)$ has rank Γ_1 on M_1 .

Définition. - We call Γ_1 the invertibility index of system (1). The above procedure can be repeated to produce a sequence of nonlinear systems. Suppose that

$$\dot{x} = A(x) + \sum_{i=1}^m u_i B_i(x) ; x \in M_k$$

$$Z_k = C_k(x) + D_k(x)u$$

is the k -th system and has invertibility index Γ_k , and state space M_k , an open dense submanifold of M .

$$Z_k = \begin{bmatrix} \bar{Z}_k \\ \hat{Z}_k \end{bmatrix} = \begin{bmatrix} \bar{C}_k(x) \\ \hat{C}_k(x) \end{bmatrix} + \begin{bmatrix} D_{k_1}(x) & \\ & 0 \end{bmatrix} u$$

and differentiating $\hat{Z}_k$ with respect to t we have

$$\hat{Z}_k^{(1)}(t) = A\hat{C}_k(x) + \sum_{i=1}^m u_i B_i \hat{C}_k(x) = A\hat{C}_k(x) + D_{k_2}(x)u$$

where D_{k_2} is the matrix with columns $B_i \hat{C}_k$. Set $\hat{D}_k = \begin{bmatrix} D_{k_1} \\ D_{k_2} \end{bmatrix}$ and let

$\Gamma_{k+1} = \max_{x \in M_k} \{\text{rank } \hat{D}_k(x)\}$. For simplicity we will assume that components of $C(x)$

have been reordered so that the submatrix $\hat{D}_{k_1}$ of $\hat{D}_k$ consisting of the first Γ_{k+1} rows of $\hat{D}_k$ has rank Γ_{k+1} for some $x \in M_k$. As above we set $M_{k+1} = \{x \in M_k \mid \text{rank } \hat{D}_{k_1}(x) = \Gamma_{k+1}\}$ and note that M_{k+1} is an open dense submanifold of M_k and hence of M . Finally, let

$$E_k(x) = \begin{bmatrix} I_{\Gamma_{k+1} \times \Gamma_{k+1}} & 0 \\ \hline F_k(x) & I_{(m-\Gamma_{k+1}) \times (m-\Gamma_{k+1})} \end{bmatrix}$$

be an elementary matrix whose entries are real analytic functions on M_{k+1} and such that

$$E_k(x) \hat{D}_k(x) = \begin{bmatrix} \hat{D}_{k_1}(x) \\ 0 \end{bmatrix} \quad \text{for all } x \in M_{k+1}$$

This lets us define

$$\begin{aligned} \text{System (k+1) : } \quad \dot{x} &= A(x) + \sum_{i=1}^m u_i B_i(x) ; x \in M_{k+1} \\ Z_{k+1} &= C_{k+1}(x) + D_{k+1}(x) u \\ \text{with invertibility index } \Gamma_{k+1}, D_{k+1} &= E_k \hat{D}_k, \text{ and } C_{k+1} = E_k \begin{bmatrix} \bar{C}_k \\ A \hat{C}_k \end{bmatrix}. \end{aligned}$$

Given system (*) we have constructed a sequence of systems, a sequence of indices $0 \leq \Gamma_1 \leq \Gamma_2 \leq \dots$ and a sequence of matrix valued functions $E_0(x), E_1(x), \dots$. We let α denote the least positive integer k such that $\Gamma_k = m$ or $\alpha = \infty$ if $\Gamma_k < m$ for all $k > 0$ and call α the relative order of the system (*). It is easy to verify that α is well defined (independent of the choice of $E_0(x), E_1(x), \dots$) and we will show that α is related to the highest order derivative of y used to reconstruct the input from a knowledge of $y(t, u, x_0)$. The following theorems relate the above constructions to the invertibility of the system (*) :

Theorem 1.- If $\alpha < \infty$ then the system (α) constructed above is invertible at x_0 for all $x_0 \in M_\alpha$. In particular the system (α) is strongly invertible.

Theorem 2.- Consider the system (*) with relative order α . Then if $\alpha = 1$ or if $\alpha > 1$ and for $i \in \{1, 2, \dots, m\}$

$$B_i A^j E_k(.) = 0 \quad \text{on } M$$

for $0 \leq k \leq \alpha - 2$ and $0 \leq j \leq \alpha - 2 - k$ the system (*) is invertible at $x_0 \forall x_0 \in M_\alpha$ and in particular is strongly invertible.

Corollary 1.- For single-input systems ($m=1$) the condition $\alpha < \infty$ is necessary and sufficient for strong invertibility.

Corollary 2.- Suppose that the system (*) satisfies the hypotheses of theorem 2. Then there exists a matrix function $H_\alpha(x)$ defined on M_α such that $\forall x_0 \in M_\alpha$,

$$Z_{\alpha}(t) = H_{\alpha}(x(t)) \begin{bmatrix} y^{(1)}(t) \\ y^{(2)}(t) \\ \vdots \\ y^{(\alpha)}(t) \end{bmatrix} = H_{\alpha}(x(t)) Y_{\alpha}(t)$$

and the system

$$\begin{aligned} \hat{x} &= \hat{A}(\hat{x}) + \hat{B}(\hat{x}) \hat{u} \quad ; \quad \hat{x} \in M_{\alpha} \\ (**) \quad \hat{y} &= \hat{C}(\hat{x}) + \hat{D}(\hat{x}) \hat{u} \end{aligned}$$

where

$$\begin{aligned} \hat{A} &= A - [B_1 \ B_2 \ \dots \ B_m] D_{\alpha}^{-1} C_{\alpha} \\ \hat{B} &= [B_1 \ B_2 \ \dots \ B_m] D_{\alpha}^{-1} H^{\alpha} \\ \hat{C} &= -D_{\alpha}^{-1} C_{\alpha} \quad \text{and} \quad \hat{D} = D_{\alpha}^{-1} H_{\alpha} \end{aligned}$$

acts as a left-inverse for the system (*). In particular, if $\hat{u}(t) = (y^{(1)}(t), \dots, y^{(\alpha)}(t))$ and $\hat{x}(0) = x_o$ then $\hat{y}(\cdot, \hat{u}, x_o) = u(t)$.

Corollary 3.- For multivariable time-invariant linear systems $\alpha < \infty$ is a necessary and sufficient condition for strong invertibility and $M_{\alpha} = M = \mathbb{R}^n$.

We remark that the left-inverse system described in Corollary 2 provides a "practical" way to recover $u(t)$ given $y(t, u, x_o)$.

Proof (Theorem 1) : Since $\Gamma_{\alpha} = m$ and $D_{\alpha}(x)$ is by construction a $m \times m$ matrix valued function on M_{α} of rank Γ_{α} , we know D_{α}^{-1} exists on M_{α} , and the inverse system (**) from Corollary 2 is well defined if we replace $H_{\alpha}(x)$ by an $m \times m$ identity matrix. Now we set $\hat{u}(t) = Z_{\alpha}(t, u, x_o)$, and this results in the evolution of the state vector $\hat{x}(t, \hat{u}, x_o)$. A straightforward computation shows that $x(t) = x(t, u, x_o)$ satisfies (**) when $\hat{u} = Z_{\alpha}$, and thus $\hat{y}(t) = \hat{c}(\hat{x}(t)) + \hat{D}(\hat{x}(t)) \hat{u} = -D_{\alpha}^{-1}(x) C_{\alpha}(x) + D_{\alpha}^{-1}(x) \hat{u}$. Since $\hat{u} = Z_{\alpha} = C_{\alpha}(x) + D_{\alpha}(x) u$, we have $\hat{y}(\cdot, Z_{\alpha}, x_o) = u(\cdot)$. Since u can be recovered from $y(\cdot, u, x_o)$ the system (α) is invertible at x_o .

Proof (Theorem 2): The proof used to establish theorem can be repeated here if we can show that $Z_{\alpha}(t) = H_{\alpha}(x(t)) Y_{\alpha}(t)$ for some $m \times m_{\alpha}$ matrix valued function $H_{\alpha}(x)$ on M_{α} . By assumption

$$\begin{aligned}
 B_i E_o(.) &= B_i A E_o(1) = \dots = B_i A^{\alpha-2} E_o(i) = 0 \\
 (***) \quad B_i E_1(.) &= \dots = B_i A^{\alpha-3} E_1(.) = 0 \\
 &\vdots \\
 B_i E_{\alpha-2}(.) &= 0
 \end{aligned}$$

and by construction

$$Z_1(t) = E_o(x(t)) y^{(1)}(t) = \begin{bmatrix} E_o^1 y^{(1)}(t) \\ E_o^2(x(t)) y^{(1)}(t) \end{bmatrix}$$

where E_o^1 is the submatrix of $E_o(x)$ consisting of the first Γ_1 rows and $E_o^2(x)$ is the matrix formed from the last $m - \Gamma_1$ rows. Following the construction of the systems (1), (2), ..., (α), we see that

$$Z_2 = \begin{bmatrix} E_1^1 \\ E_{12}(x) \end{bmatrix} \begin{bmatrix} E_o^1 y^{(1)} \\ (E_o^2(x) y^{(1)})^{(1)} \end{bmatrix}. \text{ Now}$$

$$\begin{aligned}
 \frac{d}{dt} E_o^2(x(t)) y^{(1)} &= E_o^2(x) y^{(2)} + \{A E_o^2(x) + \sum_{i=1}^m u_i B_i E_o^2(x)\} \\
 &= E_o^2(x) y^{(2)} + A E_o^2(x) y^{(1)}
 \end{aligned}$$

from (***) , and thus

$$Z_2 = \begin{bmatrix} E_1^1 \\ E_1^2(x) \end{bmatrix} \begin{bmatrix} E_o^1 y^{(1)} \\ E_o^2(x) y^{(2)} + A E_o^2(x) y^{(1)} \end{bmatrix} = H_2(x) \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}$$

One can continue this procedure to generate $H_{\alpha}(x)$. To complete the proof one now repeats the steps outlined in the proof of theorem 1 .

Proof (Corollary 1) : See reference [5].

Proof (Corollary 2) : This Corollary is proved in the course of proving Theorem 2.

Proof (Corollary 3) : See [3].

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REFERENCES

- [1] R.W. BROCKETT and M.D. MESAROVIC.- The reproducibility of multivariable systems. J. Math. Anal. Appl.11 (1965), p.548-563.
- [2] M.K. SAIN and J.L. MASSEY.- Invertibility of linear time-invariant dynamical systems. IEEE Trans.Aut. Control, AC-14 (1969), p.141-
- [3] L.M. SILVERMAN.- Inversion of multivariable linear systems. IEEE Trans. Aut. Control, AC-14 (1969), p.270-276.
- [4] A.S. WILLSKY.- On the invertibility of linear systems. IEEE Trans. on Aut. Control, AC-19 (1974), p.272-274.
- [5] R.M. HIRSCHORN.- Invertibility of nonlinear control systems. To appear in SIAM J. on Control and Optimization.
- [6] R.M. HIRSCHORN.- Invertibility of control systems on Lie groups. SIAM J. on Control and Optimization.
- [7] R.M. HIRSCHORN.- Invertibility of multivariable nonlinear control systems, submitted to IEEE Trans. on Aut. Control.
- [8] F. WARNER.- Foundations of Differentiable Manifold and Lie groups. Scott, Foresman and & , Glenview Ill., 1970.

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R.M. HIRSCHORN
Queen's University
KINGSTON, ONTARIO (Canada)