

Astérisque

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Astérisque, tome 75-76 (1980), p. 125-131

http://www.numdam.org/item?id=AST_1980__75-76__125_0

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LIE ALGEBRAIC METHODS FOR THE CONTROL OF INFINITE DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS

by

Henry HERMES*

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INTRODUCTION. -

Let $\mathcal{A}$, $\mathcal{B}$ be densely defined operators acting within a Banach space B of $\mathbb{R}^n$ valued functions. We consider systems of the form

$$(1) \quad dv_t/dt = \mathcal{A}(v_t) + u(t) \mathcal{B}(v_t), \quad v_0 = \varphi \in B$$

where the control u is Lebesgue measurable with values $|u(t)| \leq 1$, and unique solutions of (1) are assumed to exist for small $t > 0$. An observation of (1) is a continuous linear map $g : B \rightarrow \mathbb{R}^k$.

Let v_t denote the reference solution, at time t , corresponding to control $u \equiv 0$, and $A(t, \varphi) \subset B$ be the set of points attainable at time t by all solutions of (1). We shall study two questions. The first, that of local controllability, is to derive computable conditions to determine when $g(v_t) \in \text{interior } g(A(t, \varphi))$ for small $t > 0$. This includes the study of $\mathbb{R}^n$ controllability, say for delay equations, but not function space controllability, e.g. see [1], [4]. The second question is that of finite dimensional realizations. Specifically, if $t \mapsto g(v_t(u))$ is a solution $t \mapsto v_t(u)$ of (1), we say g admits a strong differential realization on $\mathbb{R}^k$. We shall give an example of an observation of a linear controlled parabolic equation which admits a strong (bilinear) differential realization on $\mathbb{R}^2$. This realization has "singular arcs" which may be studied by high order methods using Lie Theory. This illustrates the difficulties which may occur in "internally

* This research was supported by NSF grant MCS 76-04419-A01.

controlled" systems of partial differential equations, i.e. systems where the control is not a forcing function or boundary value.

I. - LIE PRODUCTS OF OPERATORS.

Throughout, the symbol s will be used as the independent variable for a function $v \in B$. The infinitesimal generators we will consider shall be assumed to have the form

$$(2) \quad (Cv)(s) = f^s((T^s v)(s))$$

where for each s , T^s is a closed linear operator acting within B (and these may change with the value s) while $f^s: \mathbb{R}^n \rightarrow \mathbb{R}^l$ is real analytic. We assume, throughout, that our operators generate (at least locally) strongly continuous one parameter semi-groups, i.e., that $dv_t/dt = C(v_t)$, $v_0 = \varphi$ has a unique solution for small $t > 0$.

For example, consider the delay equation on $\mathbb{R}^n$; $dx(t)/dt = W(x(t-1))$, $x(s) = \varphi(s)$, $-1 \leq s \leq 0$ where $\varphi \in C[-1, 0]$ and $W: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is real analytic. Let $\zeta(t, \varphi)$ denote a solution at time t and define $v_t(\varphi)(s) = \zeta(t+s, \varphi)$, $-1 \leq s \leq 0$. The map $t \rightarrow v_t(\varphi) \in C[-1, 0]$ is a strongly continuous semi-group with infinitesimal generator

$$(Cv_t)(s) = \begin{cases} d/ds v_t(s) = d/dt v_t(s), & -1 \leq s < 0 \\ W((Sv_t)(0)) = W(v_t(-1)), & s = 0 \end{cases}$$

where S is the unit shift operator (i.e. $(Sv)(t) = v(t-1)$). Here, referring to (2), $f^s(\lambda) = \lambda$ and $T^s = d/ds$ if $-1 \leq s < 0$ while $f^s(\lambda) = W(\lambda)$ and $T^s = S$ if $s = 0$. In the Banach space $B = C[-1, 0]$, $v_t(\varphi)$ satisfies the equation $d/dt v_t(\varphi) = C(v_t)$, $v_0 = \varphi$. Let C be an operator of the form (2) and f' denote the derivative of f . We define $((DC(v)(s))w)(s) = f'^s((T^s v)(s))(T^s w)(s)$ and the Lie Product of operators $C, \mathcal{G}$ of the form (2) as $([C, \mathcal{G}](v))(s) = ((DC(v)(s))(\mathcal{G}v))(s) - ((D\mathcal{G}(v)(s))(Cv))(s)$. As expected, if $C, \mathcal{G}$ are linear operators then $[C, \mathcal{G}]$ is just the commutator $C\mathcal{G} - \mathcal{G}C$. Furthermore, the above concepts easily generalize to operators of the form $(Cv)(s) = \sum_{\nu=1}^m f_{\nu,1}^s((T_{\nu,1}^s v)(s)) \dots f_{\nu,k}^s((T_{\nu,k}^s v)(s))$; the details and examples of computations can be found in [4], [5]. We introduce the notation $(ad C, \mathcal{G}) = [C, \mathcal{G}]$ and inductively $(ad^{k+1} C, \mathcal{G}) = [C, (ad^k C, \mathcal{G})]$.

Now consider the equation (1) with $\mathcal{A}, \mathcal{B}$ operators of the form (2). Let $\eta_t(\varphi)$ denote the solution, at time t , of $dv_t/dt = \mathcal{A}(v_t)$, $v_0 = \varphi$, and $D\eta_t(\varphi)$ the differential of the map $\varphi \rightarrow \eta_t(\varphi)$. In [4] we showed the

DECOMPOSITION THEOREM. - Assume the maps $t \rightarrow \eta_t(\varphi)(s)$ and $t \rightarrow D\eta_t(\varphi)(s)$ are real analytic for all s and that $D\eta_t(\varphi) \rightarrow \text{id}$ in the strong operator topology as $t \rightarrow 0$. Then a sufficient condition that the composition $\eta_t(\psi_t(\varphi, u))$ be a solution of equation (1) is that $\psi_t(\varphi, u)$ satisfy

$$(3) \quad dv_t/dt = u(t) \sum_{\nu=0}^{\infty} (-t)^{\nu}/\nu! (\text{ad}^{\nu} \mathcal{A}, \mathcal{B})(v_t), \quad v_0 = \varphi.$$

Such decomposition theorems provide the key to the applications of Lie theory to differential equations and control systems, [6], [7].

Let $g : B \rightarrow \mathbb{R}^k$ be continuous and linear, C an operator of the form (2), $\xi_t(\varphi)$ the solution at time t of $dv_t/dt = C(v_t)$, $v_0 = \varphi$ and define $g_*(C_{\varphi}) = \lim_{t \rightarrow 0} d/dt g(\xi_t(\varphi))$. Analogous to the case of control systems on manifolds, we associate with (1) the set of operators

$$(4) \quad \mathcal{J}^1 = \{(\text{ad}^{\nu} \mathcal{A}, \mathcal{B}) : \nu = 0, 1, \dots\}$$

and let $\mathcal{J}^1(\varphi)$ denote the elements of $\mathcal{J}^1$ evaluated at φ .

THEOREM (Local Controllability). - The observation $g : B \rightarrow \mathbb{R}^k$ of system (1) is locally controllable along the reference solution $\eta_t(\varphi)$ corresponding to $u \equiv 0$ (i.e. $g(\eta_t(\varphi)) \in \text{int } g(A(t, \varphi))$ for small $t > 0$) if $\dim. \text{span } g_*(\mathcal{J}^1(\varphi)) = k$.

Proof : Form $\hat{B} = B \times \mathbb{R}^k$; let $w(t) = g(v_t) \in \mathbb{R}^k$ and $\hat{a} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} v_t \\ g_*(\mathcal{A} v_t) \end{bmatrix}$, $\hat{\mathcal{B}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \begin{bmatrix} \mathcal{B} v_t \\ g_*(\mathcal{B} v_t) \end{bmatrix}$. The "augmented system" on $\hat{B}$ is

$$(i) \quad \frac{\partial}{\partial t} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \hat{a} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} + u(t) \hat{\mathcal{B}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix}, \quad \begin{bmatrix} v_0 \\ w(0) \end{bmatrix} = \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix}.$$

Define $\mathcal{J}^1 \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix} = \{ (\text{ad}^\nu \hat{a}, \hat{a}) \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix} : \nu = 0, 1, \dots \}$ and $\pi : \hat{B} \rightarrow \mathbb{R}^k$ be projections to $\mathbb{R}^k$, specifically for $\psi \in B$, $y \in \mathbb{R}^k$, $\pi \begin{bmatrix} \psi \\ y \end{bmatrix} = y$. Let $\hat{A}(t, \varphi) \subset \hat{B}$ denote the elements attainable at time t by solutions of (i) corresponding to all admissible controls. The reference solution of (i) corresponding to $u \equiv 0$ is $\begin{bmatrix} \eta_t \\ g(\eta_t) \end{bmatrix} \in \hat{A}(t, \varphi)$. Clearly if $v_t(u)$ denotes a solution of (1) at time t corresponding to control choice u , then $\begin{bmatrix} v_t(u) \\ g(v_t(u)) \end{bmatrix}$ is a solution of (i). It follows that

$$(ii) \quad \pi(\hat{A}(t, \varphi)) = g(A(t, \varphi)).$$

Since $\text{span } \mathcal{J}^1 \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix}$ is the "first order local set of directions" in which one can proceed via solutions of (i), from the inverse function theorem (see [8], chap. I. § 5, in particular corollaries 1, 1s) one has

$$(iii) \quad g(\eta_t) = \pi \begin{bmatrix} \eta_t \\ g(\eta_t) \end{bmatrix} \in \text{int } \pi(\hat{A}(t, \varphi)) = \text{int } g(A(t, \varphi))$$

for small $t > 0$ if $\dim \text{span } \pi(\mathcal{J}^1 \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix}) = k$. To complete the proof one need only show that $\pi(\mathcal{J}^1 \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix}) = g_*(\mathcal{J}^1(\varphi))$ which is a straightforward calculation using induction. ■

As an application, we consider a tension controlled vibrating string. Let ρ denote density, and $s = s(t)$ tension. The one dimensional equation of the vibrating string is $\rho \partial^2 w / \partial t^2 = \partial / \partial x (s(t) \partial w / \partial x)$, where w measures deflection from the rest position. Choose u_0 as a nominal value of $s(t) / \rho$, $u_0 > 0$ such that $u_0 - u > 0$ and the control $u(t) = s(t) / \rho - u_0$, with $|u(t)| \leq u$. Let $\partial w / \partial x = v^1$, $\partial w / \partial t = v^2$ to obtain the first order system

$$(i) \quad \begin{aligned} \partial v^1 / \partial t &= \partial v^2 / \partial x, & v^1(x) &= v^1(x), & 0 \leq x \leq \ell \\ \partial v^2 / \partial t &= u_0 \partial v^1 / \partial x + u(t) \partial v^1 / \partial x, & v^2(x) &= v^2(x), & 0 \leq x \leq \ell \end{aligned}$$

where we assume the string is clamped at both ends so $v^2(0) = v^2(\ell) = 0$. Take $C_0[0, \ell]$ to be those functions in $C[0, \ell]$ which vanish at 0 and ℓ and $B = C[0, \ell] \times C_0[0, \ell]$. The boundary data $w(t, 0) = w(t, \ell) = 0$ is implicit in B

if the constants chosen in recovering w from v^1, v^2 are properly chosen.

This is assumed. With $\mathcal{A} = \begin{bmatrix} 0 & \partial/\partial x \\ u_0 \partial/\partial x & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & 0 \\ \partial/\partial x & 0 \end{bmatrix}$,

$$(ii) \quad \mathcal{J}^1(\varphi) = \{(\text{ad}^v \mathcal{A}, \mathcal{B})\varphi : v = 0, 1, \dots\} = \left\{ \begin{bmatrix} 0 \\ \partial \varphi^1 / \partial x \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \partial^2 \varphi^1 / \partial x^2 \\ -\partial^2 \varphi^2 / \partial x^2 \end{bmatrix}, \begin{bmatrix} -2\partial^3 \varphi^2 / \partial x^3 \\ 2u_0 \partial^3 \varphi^1 / \partial x^3 \end{bmatrix}, \begin{bmatrix} 2^2 u_0 \partial^4 \varphi^1 / \partial x^4 \\ -2^2 u_0 \partial^4 \varphi^2 / \partial x^4 \end{bmatrix}, \begin{bmatrix} -2^3 u_0 \partial^5 \varphi^2 / \partial x^5 \\ 2^3 u_0 \partial^5 \varphi^1 / \partial x^5 \end{bmatrix}, \dots \right\}.$$

One may now readily check specific initial data and observations for local controllability. For example, if $\varphi^1(x) = (\pi/\ell) \cos(\pi x/\ell)$, $\varphi^2(x) = 0$ the solution with $u(t) \equiv 0$ (i.e. the reference solution) is $w(t, x) = \cos(\sqrt{u_0} \pi t/\ell) \sin(\pi x/\ell)$, i.e. at each time t , the position of the string is a scalar multiple of $\sin(\pi x/\ell)$. Take as observation $g_1(v_1) = v_t^1(\ell/2)$, $g_2(v_t) = v_t^2(\ell/2)$, i.e. respectively the angle of deflection of the string at $\ell/2$ and the velocity of the point at $\ell/2$. Computing shows $\text{rank } g_*(\mathcal{J}^1(\varphi)) = 1$, not 2, hence the theorem does not imply local controllability of this observation. Physically this is expected since the initial data was chosen so that $g_1(v_t)$, the angle of deflection at $\ell/2$, would be zero for all t . On the other hand, the velocity of the point at $\ell/2$, i.e. $g_2(v_t)$, can be locally controlled via tension.

One may show that for any integer k there exists initial data φ^1, φ^2 and an $\mathbb{R}^k$ valued observation g which is locally controllable along the reference via tension.

II. - FINITE DIMENSIONAL DIFFERENTIAL REALIZATIONS.

Consider a special case of equation (1) i.e.

$$(5) \quad dv_t/dt = Cv_t + u(t)b, \quad v_0 = \varphi \in B$$

where C is linear and $b \in B$. By differentiating $g(v_t(u))$ with respect to t it follows that if there exists a mapping $C_* : \mathbb{R}^k$ such that

$$(6) \quad g_* C v_t = C_* g(v_t)$$

then a strong differential realization of g exists. If C has $\ker g$ as an invariant subspace, equation (6) may be used to define C_* . This rather severe restriction leads one to "change the order of events". Given an equation of the form (8), suppose C has an invariant subspace, S , of co-dim k . Then any observation g having S as kernel does admit a strong differential realization. A specific example with entails a slightly more general evolutionary equation than (5) is

Example 2.1. - Consider the parabolic partial differential equation

$$(7) \quad \partial v_t / \partial t = \partial^2 v_t / \partial x^2 + u(t) (\partial v_t / \partial x + \sin x) \\ v_0(x) = 2 \cos x + 2.$$

Let S^1 be the one sphere parametrized by $-\pi \leq x \leq \pi$ and consider equation (7) on $S^1 \times [0, \infty]$. Let $B = \{w \in L_2[-\pi, \pi] : w(-\pi) = w(\pi) = 0\}$, $a v_t = \partial^2 v_t / \partial x^2$, $B v_t = \partial v_t / \partial x$ and $b \in B$ be given by $b(x) = \sin x$. For the observation we choose $g = (g_1, g_2)$ with $g_1(v_t) = \frac{1}{\pi} \int_{-\pi}^{\pi} v_t(x) \sin x dx$, $g_2(v_t) = \frac{1}{\pi} \int_{-\pi}^{\pi} v_t(x) \cos x dx$. Then $S = \text{span}\{1, \sin x, \cos x : x = 2, 3, \dots\} = \ker g$ while $\text{codim } S = 2$ and $a, B : S \rightarrow S$. To compute the strong differential realization, let $v_t(x) = \sum_{j=1}^{\infty} \sigma_j(t) \sin jx + \sum_{j=0}^{\infty} \gamma_j(t) \cos jx$ and form the equation $d/dt g(v_t) = g_* a v_t + u(t) g_* B v_t + u(t) g_* b$. Letting $\sigma_1(t) = y_1(t)$, $\gamma_1(t) = y_2(t)$ the differential realization on $\mathbb{R}^2$ is $(\dot{y} = dy/dt)$

$$(8) \quad \begin{aligned} \dot{y}_1 &= -y_1 + u(t) (1 - y_2), \quad y_1(0) = 0 \\ \dot{y}_2 &= -y_2 + u(t) y_1, \quad y_2(0) = 2. \end{aligned}$$

If we consider the problem, for (7), of finding that measurable control u with $|u(t)| \leq 1$ such that $g(v_t) = (0, 1)$ in minimum time t , this leads to the problem of reaching $(0, 1)$ in minimum time by a solution of (8). The method of [9, §22] may be used to show that the optimal solution of this latter "bilinear problem" is $y_1(t) \equiv 0$, $y_2(t) = 2e^{-t}$, a singular arc obtained with control $u(t) \equiv 0$.

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