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A RATIO LIMIT THEOREM

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1.- Introduction.-

Let G be a (topological) group, μ a probability measure on G and denote by μ^n the n -th convolution power of μ . We are interested in results of the type

$$\lim_{n \rightarrow \infty} \frac{\mu^n(f)}{\mu^n(g)} \text{ exists} = \frac{\sigma(f)}{\sigma(g)}$$

for certain functions f, g and a certain measure σ on G . In the case of abelian groups very complete results have been obtained in [7], in the case of discrete groups a general theorem is proved in [1] (for μ symmetric). There are only a few other results of this type for nondiscrete and nonabelian groups ([8]). We will prove a ratio limit theorem for certain positive-definite and for certain symmetric probability measures on (certain) amenable groups; theorem 2 contains as a special case an extension of the result in [1]. It will also be shown that these results can only hold in unimodular, amenable groups.

2.- Preliminaries.-

Let G be always a locally compact group.

For a function f on G we write

$${}_x f(y) = f(xy), \quad f_x(y) = f(yx), \quad \tilde{f}(x) = f(x^{-1}) \quad .$$

By K we denote the linear space of real-valued continuous functions on G vanishing outside compact sets, by K^+ the positive elements in K .

For a measure μ on G and a complex-valued function f on G we write

$$\mu(f) = \int f(x) d\mu(x) \quad .$$

All properties of convolutions of functions and measures we use can be found in [5]

The convolution powers of a probability measure μ on G will be denoted by μ^n .

If G is a unimodular group the Haar measure will be denoted by λ and its differential by dx ; we have then for $y \in G$:

$$\lambda(f) = \int f(x) dx = \int f(xy) dx = \int f(yx) dx = \int \tilde{f}(x) dx \quad .$$

A measure μ on the group G is called positive definite if

$$\mu(f \circ \tilde{f}) \geq 0 \quad \text{for all } f \in K \quad (\text{or } f \in L^2(G))$$

(see [3] for properties of positive-definite measures and functions).

If μ is a positive-definite probability measure on G then all the convolution powers μ^n are positive-definite probability measures on G .

Lemma 1 : if μ is a positive-definite probability measure on the unimodular group G , $x \in G$ and $g \in L^2(G)$ then

$$|\mu^n(g_x \overset{\sim}{*} g)| \leq \mu^n(g \overset{\sim}{*} g) \quad (n = 1, 2, \dots) .$$

Proof : Since $g_x \overset{\sim}{*} g_x = g \overset{\sim}{*} g$ for every $x \in G$ we have by the Cauchy-Schwarz inequality

$$|\mu^n(g_x \overset{\sim}{*} g)| \leq (\mu^n(g \overset{\sim}{*} g))^{1/2} (\mu^n(g_x \overset{\sim}{*} g_x))^{1/2} = \mu^n(g \overset{\sim}{*} g) .$$

Lemma 2 : If μ is a positive-definite probability measure on G then

$$\lim_{n \rightarrow \infty} \frac{\mu^{n+1}(g \overset{\sim}{*} g)}{\mu^n(g \overset{\sim}{*} g)} = \|\mu\|_2 \quad \text{for all } g \in K^+, g \neq 0$$

($\|\mu\|_2$ is the norm of the operator $T : L^2(G) \rightarrow L^2(G)$ given by $Tf = \mu * f$ for $f \in L^2(G)$)

Proof : Since by assumption T is a positive-definite operator on the Hilbertspace $L^2(G)$ there exists a unique positive-definite operator S on $L^2(G)$ such that $S^2 = T$ ([5] , C.35). Therefore we have for any positive integer n and $g \in K^+ (<, > \text{ denotes the inner product in } L^2(G)) :$

$$\begin{aligned} \mu^n(g \overset{\sim}{*} g) &= \langle T^n g, g \rangle = \langle S^{2n} g, g \rangle = \langle S^{n-1} g, S^{n+1} g \rangle \\ &\leq \langle S^{n-1} g, S^{n-1} g \rangle^{1/2} \langle S^{n+1} g, S^{n+1} g \rangle^{1/2} \\ &= \langle T^{n-1} g, g \rangle^{1/2} \langle T^{n+1} g, S^{n+1} g \rangle^{1/2} \\ &= (\mu^{n-1}(g \overset{\sim}{*} g) \mu^{n+1}(g \overset{\sim}{*} g))^{1/2} . \end{aligned}$$

In [2] it is shown that under the hypotheses of lemma 2 we have

$$\|\mu\|_2 = \lim_{n \rightarrow \infty} (\mu^n(g \overset{\sim}{*} g))^{1/n} \quad \text{for all } g \in K^+, g \neq 0$$

and that the sequence $(\mu^n(g \overset{\sim}{*} g))^{1/n}$ is increasing. Therefore $\|\mu\|_2 > 0$ and to every $g \in K^+, g \neq 0$ there exists an integer $n_0(g)$ such that

$$\mu^n(g \overset{\sim}{*} g) > 0 \quad \text{for all } n \geq n_0(g) .$$

From the calculation above we get then

$$\frac{\mu^n(g \overset{\sim}{*} g)}{\mu^{n-1}(g \overset{\sim}{*} g)} \leq \frac{\mu^{n+1}(g \overset{\sim}{*} g)}{\mu^n(g \overset{\sim}{*} g)}$$

for n sufficiently large ; so the sequence $\frac{\mu^{n+1}(g \overset{\sim}{*} g)}{\mu^n(g \overset{\sim}{*} g)}$ is increasing, therefore its limit exists and is equal to $\|\mu\|_2$.

3.- Positive-definite probability measures.

If μ is a probability measure on G then we write

$$\mu' = \sum_{n=1}^{\infty} 2^{-n} \mu^n$$

which is again a probability measure on G .

A function f on G is called central if $f(xy) = f(yx)$ for all $x, y \in G$.

For definition and properties of amenable groups see [4].

We will prove here the following

Theorem 1 : Assume that

- a) G is a locally compact, amenable, unimodular group with Haar-measure λ
- b) μ is a positive-definite probability measure on G such that λ is absolutely continuous with respect to μ' (i.e. $\mu'(A) = 0 \Rightarrow \lambda(A) = 0$)
- c) there exists a nonzero function $g \in K^+$ such that $g * \mu = \mu * g$.

Then for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ we have

$$\lim_{n \rightarrow \infty} \frac{\mu^n(g * f_1 * \tilde{g})}{\mu^n(g * f_2 * \tilde{g})} = \frac{\int g * f_1 * \tilde{g}(x) dx}{\int g * f_2 * \tilde{g}(x) dx} = \frac{\int f_1(x) dx}{\int f_2(x) dx}.$$

Proof : By lemma 1 we have

$$\tilde{g} * \mu^n * g(x) = \mu^n(\tilde{g} * g)_x \leq \mu^n(\tilde{g} * g) = \tilde{g} * \mu^n * g(e).$$

Since G is amenable $\|\mu\|_2 = 1$ ([4], Th. 3.2.2.). Now (using b))

$$\begin{aligned} 0 &< \int \left(1 - \frac{\tilde{g} * \mu^n * g(x)}{\tilde{g} * \mu^n * g(e)}\right) d\mu'(x) = \int \sum_{s=1}^{\infty} 2^{-s} \left(1 - \frac{\tilde{g} * \mu^n * g(x)}{\tilde{g} * \mu^n * g(e)}\right) d\mu^s(x) = \\ &= \sum_{s=1}^L 2^{-s} \int \left(1 - \frac{\tilde{g} * \mu^n * g(x)}{\tilde{g} * \mu^n * g(e)}\right) d\mu^s(x) + \int \sum_{s=L+1}^{\infty} 2^{-s} \left(1 - \frac{\tilde{g} * \mu^n * g(x)}{\tilde{g} * \mu^n * g(e)}\right) d\mu^s(x) \\ &\leq \sum_{s=1}^L 2^{-s} \left(1 - \frac{\mu^{n+s}(\tilde{g} * g)}{\mu^n(\tilde{g} * g)}\right) + 2^{-L} \leq \sum_{s=1}^L 2^{-s} \left(1 - \frac{\mu^{n+L}(\tilde{g} * g)}{\mu^n(\tilde{g} * g)}\right) + 2^{-L} \\ &\leq \left(1 - \frac{\mu^{n+L}(\tilde{g} * g)}{\mu^n(\tilde{g} * g)}\right) + 2^{-L} \rightarrow 2^{-L} \text{ for } n \rightarrow \infty \text{ (lemma 2)} \end{aligned}$$

which implies that

$$\int \left(1 - \frac{\tilde{g} * \mu^n * g(x)}{\tilde{g} * \mu^n * g(e)}\right) d\mu'(x) \rightarrow 0 \text{ for } n \rightarrow \infty.$$

Therefore if (n') is any subsequence of the positive integers there exists a subsequence $(n'') \subset (n')$ such that ([5], (11.27))

$$\lim_{n'' \rightarrow \infty} \frac{\tilde{g} * \mu^{n''} * g(x)}{\tilde{g} * \mu^{n''} * g(e)} = 1 \quad \mu' \text{ almost everywhere}$$

and so by assumption b) λ almost everywhere. Lebesgue dominated convergence theorem then implies

$$\lim_{n'' \rightarrow \infty} \frac{\tilde{g} * \mu^{n''} * g(x)}{\tilde{g} * \mu^{n''} * g(e)} f(x) dx = \int f(x) dx = \lambda(f)$$

for all $f \in L^1(G)$. Since $\tilde{g} * \mu^n * g$ is positive-definite

$$\int \tilde{g} * \mu^n * g(x) f(x) dx = \mu^n(g * f * \tilde{g})$$

and we obtain : every subsequence (n') of the positive integers contains a sub-subsequence $(n'') \subset (n')$ such that

$$\lim_{n''} \frac{\mu^{n''}(g * f_1 * \tilde{g})}{\mu^{n''}(g * f_2 * \tilde{g})} = \frac{\lambda(f_1)}{\lambda(f_2)} ;$$

since the limit is independent of (n') , (n'') this implies that

$$\lim_n \frac{\mu^n(g * f_1 * \tilde{g})}{\mu^n(g * f_2 * \tilde{g})} \text{ exists} = \frac{\lambda(f_1)}{\lambda(f_2)} ;$$

The function g is by assumption positive and so we get from the invariance of the Haar measure

$$\int g * f * \tilde{g}(x) dx = \|g\|_1^2 \int f(x) dx$$

which proves theorem 1 .

4.- Symmetric probability measures.

A probability measure μ is called symmetric if $\mu(f) = \mu(\tilde{f})$.

If μ is symmetric then $\mu^{2n}(n=1, 2, \dots)$ is positive-definite. We will write

$$\mu'' = \sum_{n=1}^{\infty} 2^{-n} \mu^{2n} .$$

Theorem 2 : Assume that

- a) G is a locally compact, unimodular, amenable group with Haar measure λ
- b) μ is a symmetric probability measure on G such that λ is absolutely continuous with respect to μ''
- c) there exists a nonzero central function $g \in K^+$.

Then for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ we have

$$\lim_{n \rightarrow \infty} \frac{\mu^n(g * f_1 * \tilde{g})}{\mu^n(g * f_2 * \tilde{g})} = \frac{\int g * f_1 * \tilde{g}(x) dx}{\int g * f_2 * \tilde{g}(x) dx} = \frac{\int f_1(x) dx}{\int f_2(x) dx} .$$

Proof : Since μ^2 is positive-definite we get by theorem 1

$$\lim_{n \rightarrow \infty} \frac{\mu^{2n}(g * f_1 * \tilde{g})}{\mu^{2n}(g * f_2 * \tilde{g})} = \frac{\int f_1(x) dx}{\int f_2(x) dx}$$

for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$. We have to show that this relation also holds if $2n$ is replaced by $2n+1$. Now

$$\mu^{2n+1}(g * f * \tilde{g}) = \int \mu^{2n}(g * f * \tilde{g})(x) d\mu(x) = \int \mu^{2n}(g * f * \tilde{g})(x) d\mu(x) .$$

Therefore we get for $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ by theorem 1 and Lebesgue's dominated convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mu^{2n+1}(g * f_1 * \tilde{g})}{\mu^{2n}(g * f_2 * \tilde{g})} &= \lim_n \int \frac{\mu^{2n}(g * x f_1 * \tilde{g})}{\mu^{2n}(g * f_2 * \tilde{g})} d\mu(x) = \\ &= \int \frac{f_1(x) dy}{f_2(y) dy} d\mu(x) = \frac{\int f_1(y) dy}{\int f_2(y) dy} . \end{aligned}$$

But this implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mu^{2n+1}(g * f_1 * \tilde{g})}{\mu^{2n+1}(g * f_2 * \tilde{g})} &= \frac{\lim_{n \rightarrow \infty} \frac{\mu^{2n+1}(g * f_1 * \tilde{g})}{\mu^{2n}(g * f_2 * \tilde{g})}}{\lim_{n \rightarrow \infty} \frac{\mu^{2n+1}(g * f_2 * \tilde{g})}{\mu^{2n}(g * f_2 * \tilde{g})}} = \frac{\int f_1(x) dx}{\int f_2(x) dx} \end{aligned}$$

and theorem 2 follows.

5.- Some remarks.

Remark 1: We will write $\text{Supp } \mu$ for the support of μ . Theorem 2 specialized to discrete groups gives the ratio limit theorem in [1] namely:

If G is a discrete amenable group, μ a symmetric probability measure on G whose support generates G ($G = \bigcup_{n=1}^{\infty} \text{Supp } \mu^n$) and if there exists an integer k such that

$$\mu^{2k+1}(e) > 0 \text{ then}$$

$$\lim_{n \rightarrow \infty} \frac{\mu^n(f_1)}{\mu^n(f_2)} = \frac{\sum_{x \in G} f_1(x)}{\sum_{x \in G} f_2(x)}$$

for all $f_1, f_2 \in L^1(G)$ such that $\sum_{x \in G} f_2(x) \neq 0$

Proof: We can take $g = \delta_e$ as the nonzero central function in K^+ .

Now since $e \in \text{Supp } \mu^{2k+1}$ we have for $n = 1, 2, \dots$

$$\text{Supp } \mu^{2n+1} \subset (\text{Supp } \mu^{2n+1}) (\text{Supp } \mu^{2k+1}) = \text{Supp } \mu^{2n+k+1}$$

and therefore

$$G = \bigcup_{n=1}^{\infty} \text{Supp } \mu^n \subset \bigcup_{n=1}^{\infty} \text{Supp } \mu^{2n} \subset G .$$

Since G is discrete the Haar measure λ is the counting measure and the last relation means that λ is absolutely continuous with respect to μ^n . Therefore theorem 2 implies the result.

Remark 2: The results of theorem 1 and theorem 2 can only hold for unimodular groups, namely:

If μ is a symmetric probability measure on the locally compact group G with left Haar measure λ and if there exists a function g on G such that

$$\lim_{n \rightarrow \infty} \frac{\mu^n(g*f*g)}{\mu^n(g*g)} = \lambda(f) \quad \text{for all } f \in K,$$

then G is unimodular.

Proof : Since μ is symmetric

$$\mu^n(g*f*\tilde{g}) = \mu^n((g*f*\tilde{g})^\sim) = \mu^n(g*\tilde{f}*g)$$

and so we get $\lambda(f) = \lambda(\tilde{f})$, i.e. λ is inversion invariant.

Therefore G is unimodular ([5], (15.16)).

Remark 3 : Theorem 1 and theorem 2 can only hold for amenable groups namely :

If μ is a symmetric probability measure on the locally compact unimodular group G such that for some $g \geq 0$

$$\lim_{n \rightarrow \infty} \frac{\mu^n(g*f*g)}{\mu^n(g*g)} = \lambda(f)$$

for all f which are characteristic functions of compact sets, then G is amenable.

Proof : Since

$$\mu^n(g*f*\tilde{g}) = \int \tilde{g} * \mu^n * g(t) f(t) dt$$

the assumption is equivalent to

$$\lambda(f) - \frac{\mu^n(g*f*\tilde{g})}{\mu^n(g*\tilde{g})} = \int (1 - \frac{\tilde{g} * \mu^n * g(t)}{\mu^n(g*\tilde{g})}) f(t) dt \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

for all f which are characteristic functions of compact sets. Now write

$$s_n(x) = \frac{\mu^n * g(x)}{(\mu^{2n}(g*\tilde{g}))^{1/2}}.$$

Then

$$\tilde{s}_n * s_n(x) = \frac{\tilde{g} * \mu^{2n} * g(x)}{\mu^{2n}(g*\tilde{g})}$$

and

$$\begin{aligned} \|s_n\|_2^2 &= \frac{1}{\mu^{2n}(g*\tilde{g})} \int \mu^n * g(x) \mu^n * g(x) dx = \\ &= \frac{1}{\mu^{2n}(g*\tilde{g})} \int \int \int g(y^{-1}x) \tilde{g}(x^{-1}z) dx \, d\mu^n(y) \, d\mu^n(z) \\ &= \frac{1}{\mu^{2n}(g*\tilde{g})} \int \int \tilde{g} * g(y^{-1}z) d\mu^n(y^{-1}) d\mu^n(z) \\ &= \frac{\mu^n * \mu^n(g*\tilde{g})}{\mu^{2n}(g*\tilde{g})} = 1. \end{aligned}$$

Since $\mu^{2n}(n=1, 2, \dots)$ is a positive-definite probability measure on G we have by lemma 1

$$0 \leq \tilde{s}_n * s_n(x) \leq 1 \quad \text{for all } x \in G.$$

Now let K be a compact set (of positive Haar-measure), let $\varepsilon > 0$, $\delta > 0$ be given and let f -characteristic function of K , then there exists an integer n_0 such that

$$\int_K (1 - \tilde{s}_n * s_n(x)) dx < \varepsilon \delta \quad \text{for all } n \geq n_0.$$

Therefore if $K_n(\varepsilon) = \{x \in K \mid 1 - \tilde{s}_n * s_n(x) > \varepsilon\}$ the Haar-measure of $K_n(\varepsilon)$

$$\lambda(K_n(\varepsilon)) < \delta.$$

This implies that to every compact set K , every $\varepsilon > 0$, $\delta > 0$ there exists an integer n_0 and there exist nonnegative functions $s_n \in L^2(G)$ and subsets $K_n(\varepsilon) \subset K$ with the properties :

$$|1 - \tilde{s}_n * s_n(x)| < \varepsilon \quad \text{for all } n \geq n_0, x \in K \setminus K_n(\varepsilon), \lambda(K_n(\varepsilon)) < \delta$$

This means that (in the terminology of [6]) the sequence $\tilde{s}_n * s_n$ converges almost uniformly to 1 for $n \rightarrow \infty$. Therefore prop. 0.1 and prop. 6.1 (see the proof there) of [6] imply that G is amenable.

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Résumé :

Dans cet article on démontre les résultats suivants :

1) Soit G un groupe localement compact, moyennable, de unimodulaire de mesure de Haar λ , μ une mesure de probabilité sur G telle que λ soit absolument continue par rapport à $\mu' = \sum_{n=1}^{\infty} 2^{-n} \mu^n$, supposons qu'il existe une fonction non

nulle $g \in K^+$ (ensemble des éléments positifs de K où K est l'espace des fonctions réelles continues tendant vers zéro en dehors des compacts) telle que $g * \mu = \mu * g$. Alors pour tout $f_1, f_2 \in L^1(G)$ telle que $\lambda(f_2) \neq 0$ on a :

$$\lim_{n \rightarrow +\infty} \frac{\mu^n(g * f_1 * \tilde{g})}{\mu^n(g * f_2 * \tilde{g})} = \frac{\int g * f_1 * \tilde{g}(x) dx}{\int g * f_2 * \tilde{g}(x) dx} = \frac{\int f_1(x) dx}{\int f_2(x) dx} \quad (I)$$

2) Soit G un groupe localement unimodulaire, moyennable de mesure de Haar λ , μ une mesure de probabilité symétrique sur G telle que λ soit absolument continue par rapport à $\mu'' = \sum_{n=1}^{\infty} 2^{-n} \mu^{2n}$, supposons qu'il existe une fonction centrale

non nulle $g \in K^+$

Alors pour tout $f_1, f_2 \in L^1(G)$ telle que $\lambda(f_2) \neq 0$ on a (I) .

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