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by

J. W. Jenkins

Recall that a locally compact group G is said to be amenable if there is an element $p \in L^\infty(G)^*$ such that i) $p \geqslant 0$, ii) $< p, \delta_g * \psi > = < p, \psi >$ for all $g \in G$ and $\psi \in L^\infty(G)$ (here δ_g is the unit mass at g and $*$ denotes convolution), and iii) $< p, \chi_G > = 1$, where χ_A denotes the characteristic function of a subset $A \subseteq G$. In this note we consider the situation in which the "normalization condition" iii) is replaced by iii)' $< p, \theta > = 1$, where θ is a given non zero, non negative element of $L^\infty(G)$. The first results on this problem were obtain by Rosenblatt [6].

We begin with the following observation : if G is not compact and K is a compact subset of G then there is an infinite sequence of elements $\{g_i\} \subseteq G$ such that for any positive integer N,

$$\chi_G \geqslant \sum_{i=1}^{N} \delta_{g_i} * \chi_K \quad .$$

If p is an element of $L^\infty(G)^*$ that satisfies i), ii), iii)' for $\theta = \chi_K$, then for all positive integers N,

$$< p, \chi_G > \geqslant \sum_{i=1}^{N} < p, \delta_{g_i} * \chi_K > = N \quad .$$

This contradiction shows that, in general, given a θ, the desired functional can exist only on a proper subspace of $L^\infty(G)$. Hence we begin by considering the smallest translation invariant subspace of $L^\infty(G)$ containing θ.

Definition 1.- A locally compact group is said to be type A if for any non zero, non negative $\theta \in L^\infty(G)$, there is a linear functional p

defined on

$$S_\theta = \{\Sigma\, \alpha(g)\, \delta_g * \theta \mid \operatorname{supp}\alpha \text{ finite}\}$$

such that i) $p \geqslant 0$, ii) $< p, \delta_g * \psi > \; = \; < p, \psi >$ for $g \in G$, $\psi \in S_\theta$, iii) $< p, \theta > \; = 1$.

In order to state our first theorem we need to recall the notion of polynomial growth. Fix a left Haar measure on G and denote the measure of a subset A of G by $|A|$. For any positive integer n , let $A^n = \{a_1 a_2 \ldots a_n \mid a_i \in A\}$. G is said to have polynomial growth if for every compact neighborhood of e in G , U , there is a polynomial p such that $|U^n| \leqslant p(n)$ for $n = 1,2,3,\ldots$. The following theorem was first proved in [6] for discrete groups and in [3] for arbitrary groups.

<u>Theorem 2</u>.- A locally compact group with polynomial growth is type A .

Proof. We define a linear functional p on S_θ by setting $< p, \Sigma\, \alpha(g)\, \delta_g * \theta > \; = \Sigma\, \alpha(g)$. It is obvious that p satisfies conditions ii) and iii) of definition 1. It only remains to show that p is positive, i.e. if $\Sigma\, \alpha(g)\, \delta_g * \theta \geqslant 0$ then $\Sigma\, \alpha(g) \geqslant 0$. For this purpose, let $U = U^{-1}$ be a compact neighborhood of e in G containing the support of α and such that $< \chi_U, \theta > \; > 0$. Set

$$A_n = \int_{U^n} \theta(s)\, ds \quad ,$$

and note that $0 < A_n \leqslant \|\theta\|_\infty \, |U^n|$. Hence $\underline{\lim}(A_n)^{1/n} = 1$. Thus there is a subsequence $\{n_k\}$ such that $A_{n_k-1} / A_{n_k+1} \to 1$. Now if $g \in \operatorname{supp}\alpha$ then $U^{n-1} \subseteq gU^n \subseteq U^{n+1}$, and hence, for such g ,

$$A_{n-1} \leqslant \int_{gU^n} \theta(s)\, ds \leqslant A_{n+1} \quad .$$

Recalling that $\Sigma\, \alpha(g)\, \delta_g * \theta \geqslant 0$, we have

$$0 \leqslant \int_{U^{n_k}} \Sigma\, \alpha(g)\, \delta_g * \theta(s)\, ds = \Sigma\, \alpha(g) \int_{gU^{n_k}} \theta(s)\, ds \quad .$$

Dividing the inequality by A_{n_k} and taking the limit on k we have, since $\operatorname{supp}\alpha$ is finite, $\Sigma\, \alpha(g) \geqslant 0$.

There is a particial converse to theorem 2.

Theorem 3.- Suppose that G is a finitely generated, discrete, solvable group or that G is a connected group, and that G is type A . Then G has polynomial growth.

Proof. Suppose that G a finitely generated, discrete, solvable group or that G is a connected group, and that G does not have polynomial growth. Then by [4] and [6] , there exist elements $a,b \in G$ and a compact neighborhood U of e in G such that if $[a,b]$ denotes the semi group generated by a and b , we have i) $a[a,b] \cap b[a,b] = \emptyset$ and ii) $sU \cap tU = \emptyset$ if $s,t \in [a,b]$, $s \neq t$. Let $V = [a,b]\, U$. Then

$$\begin{aligned} \operatorname{supp} \delta_a * \chi_V \cap \operatorname{supp} \delta_b * \chi_V &\subseteq aV \cap bV \\ &= (a[a,b] \cap b[a,b]\,)\, U = \emptyset \quad . \end{aligned}$$

Thus $\chi_V \geqslant \delta_a * \chi_V + \delta_b * \chi_V$. Hence there can not be a functional p on S_{χ_V} satisfying the conditions of definition 1.

S_θ is the smallest subspace of $L^\infty(G)$ on which we might hope to find a translation invariant positive functional normalized with respect to θ . By contrast, the largest subspace is

$$L_\theta = \{\psi \in L^\infty(G) \mid |\psi| \leqslant \mu * \theta \ , \ \mu \in M(G)\} \quad ,$$

where $M(G)$ denotes the algebra of all bounded, regular Borel measures on G . Further, instead of asking only for invariance with respect to convolution by point masses, one could require that $< p, \mu * \psi > = \mu(G) < p, \psi >$ for all $\psi \in L_\theta$ and all $\mu \in M(G)$. We will construct such functionals p , inductively, obtaining at each stage a functional invariant with respect to a larger subalgebra of $M(G)$. This construction also requires continuity of the functionals with respect to the following norm, $\|\cdot\|_\theta$, on L_θ.

$$\|\psi\|_\theta = \inf \{ \|\mu\| \mid |\psi| \leqslant \mu * \theta , \quad \mu \in M(G) \} .$$

Definition 4.- Let A be a subalgebra of $M(G)$. G is said to have A-invariance if for all $0 \neq \theta \in L^\infty(G)$ for which $0 \leqslant \theta \leqslant f * \theta$ for some $f \in L^1(G)$, there is a $p \in L_\theta^*$ such that

i) $p \geqslant 0$,

ii) $< p, \mu * \psi > = \mu(G) < p, \psi >$, $\mu \in A$, $\psi \in L_\theta$,

iii) $< p, \mu * \psi > = \int < p, \delta_X * \psi > d\mu(X)$, $\mu \in M(G)$, $\psi \in L_\theta$,

iv) $< p, \theta > = 1$.

The following theorem and proof were inspired by Ludwig [5].

Theorem 5.- Suppose G is a connected group with polynomial growth or that G is a compact extension of a nilpotent group. Then G has $M(G)$-invariance.

The proof of this theorem requires the following.

Lemma 6.- Suppose G has polynomial growth and that G contains normal subgroups H and K with $K \subseteq H$. Suppose further that each element of H/K is contained in a compact neighborhood that is invariant under the inner-automorphisms from G/K. Then, if G has $M(K)$-invariance it has $M(H)$-invariance.

Proof. Let θ be a non zero, non negative element of $L^\infty(G)$ and let $p \in L_\theta^*$ satisfying i) - iv) of definition 4 with respect to $M(K)$.

Let $\dot{U}$ be a compact neighborhood of e in H/K that is invariant under inner-automorphism from G/K. Given $\varepsilon > 0$, we define $f_{\varepsilon, \dot{U}}$ on H/K by $f_{\varepsilon, \dot{U}}(\dot{x}) = (1 + \varepsilon)^{-1}$ if $\dot{x} \in \dot{U}$, $f_{\varepsilon, \dot{U}}(\dot{x}) = (1 + \varepsilon)^{-n}$ if $\dot{x} \in \dot{U}^n - \dot{U}^{n-1}$, $n \geqslant 2$, and $f_{\varepsilon, \dot{U}}(\dot{x}) = 0$ if $\dot{x} \notin < \dot{U} >$, the closed (and open) subgroup of H/K generated by $\dot{U}$. Since H/K has polynomial growth

$$\| f_{\varepsilon, \dot{U}} \|_1 = (1+\varepsilon)^{-1} |\dot{U}| + \sum_{n=2}^{\infty} (1+\varepsilon)^{-n} |\dot{U}^n - \dot{U}^{n-1}| < \infty \quad .$$

Furthermore, for $\dot{y} \in \dot{U}$ and $\dot{x} \in < \dot{U} >$

$$|f_{\varepsilon, \dot{U}}(\dot{x}\,\dot{y}) - f_{\varepsilon, \dot{U}}(\dot{x})| \leqslant \varepsilon\, f_{\varepsilon, \dot{U}}(\dot{x}) \quad .$$

Since $\theta \leqslant f_o * \theta$ for some $f_o \in L^1(G)$,

$$1 = < p, \theta > \leqslant < p, f_o * \theta > = \int_G f_o(x) < p, \delta_x * \theta > \; dx \quad .$$

Thus, $< p, \delta_x * \theta > > 0$ on some open subset of G . Hence, for some $a \in G$, $< p, \delta_{ax} * \theta > > 0$ for x is some open subset of $< \dot{U} >$.

Since $< p, \delta_k * \psi > = < p, \psi >$ for all $k \in K$, we can define $< p, \delta_{\dot{x}} * \psi >$ for $\dot{x} \in H/K$ by setting $< p, \delta_{\dot{x}} * \psi > = < p, \delta_x * \psi >$. We define $p'_{\varepsilon, \dot{U}}$ on L_θ by

$$< p'_{\varepsilon, \dot{U}} , \psi > = \int_{H/K} f_{\varepsilon, \dot{U}}(\dot{x}) < p, \delta_{\dot{a}\dot{x}} * \psi > \; d\dot{x} \quad .$$

Fix $\dot{b} \in H/K$ so that

$$(1-\varepsilon)^{-1} < p'_{\varepsilon, \dot{U}} , \delta_{\dot{b}} * \theta > \geqslant \sup_{\dot{y}} \; < p'_{\varepsilon, \dot{U}} , \delta_{\dot{y}} * \theta > := \alpha \quad ,$$

and define $p_{\varepsilon, \dot{U}}$ on L_θ by setting

$$< p_{\varepsilon, \dot{U}} , \psi > = \alpha^{-1} < p'_{\varepsilon, \dot{U}} , \delta_{\dot{b}} * \psi > \quad ,$$

for all $\psi \in L_\theta$.

We first show that $\| p_{\varepsilon, \dot{U}} \|_\theta \leqslant 1$. For this, suppose that $\psi \in L_\theta$ and that $|\psi| \leqslant \mu * \theta$. Then

$$|< p_{\varepsilon,\dot{U}}, \psi >| \leqslant < p_{\varepsilon,\dot{U}}, |\psi| > = \alpha^{-1} < p'_{\varepsilon,\dot{U}}, \delta_b * |\psi| >$$

$$= \alpha^{-1} \int_{H/K} f_{\varepsilon,\dot{U}}(\dot{x}) < p, \delta_{axb} * |\psi| > d\dot{x}$$

$$\leqslant \alpha^{-1} \int_{H/K} f_{\varepsilon,\dot{U}}(\dot{x}) < p, \delta_{axb} * \mu * \theta > d\dot{x}$$

$$= \alpha^{-1} \int_G \int_{H/K} f_{\varepsilon,\dot{U}}(\dot{x}) < p, \delta_{axbz} * \theta > d\dot{x}\, d\mu(z)$$

$$= \alpha^{-1} \int_G < p'_{\varepsilon,\dot{U}}, \delta_{bz} * \theta > d\mu(z)$$

$$\leqslant \int_G d\mu(z) \leqslant \|\mu\| .$$

Also note that if $\dot{y} \in \dot{U}$, $\psi \in L_\theta$, then

$$|< p_{\varepsilon,\dot{U}}, \delta_y * \psi > - < p_{\varepsilon,\dot{U}}, \psi >|$$

$$\leqslant \alpha^{-1} \left|\int f_{\varepsilon,\dot{U}}(\dot{x}) < p, \delta_{axby} * \psi - \delta_{axb} * \psi > d\dot{x}\right|$$

$$= \alpha^{-1} \int |f_{\varepsilon,\dot{U}}(\dot{x}\dot{b}\dot{y}^{-1} \dot{b}^{-1}) - f_{\varepsilon,\dot{U}}(\dot{x})| < p , \delta_{axb} * \psi > d\dot{x}$$

$$\leqslant \varepsilon\, \alpha^{-1} \int f_{\varepsilon,\dot{U}}(x) < p, \delta_{axb} * \psi > d\dot{x}$$

$$= \varepsilon < p_{\varepsilon,\dot{U}}, \psi > .$$

Choose a sequence of compact neighborhoods of e in H/K, $\dot{U}_n$, that are invariant under the inner-automorphisms from G/K and such that $\bigcup_{n=1}^{\infty} \dot{U}_n = H/K$. For each positive integer n, let $p_n = p_{1/n,\dot{U}_n}$. By ω^*-compactness of the unit ball in L_θ^*, the sequence $\{p_n\}$ has a cluster point p_∞. Clearly $p_\infty \geqslant 0$ and, since

$$1 \geqslant < p_n , \theta > = < p_{1/n,\dot{U}_n}, \theta >$$

$$= \alpha^{-1} < p'_{1/n,\dot{U}_n}, \delta_{a_n} * \theta >$$

$$\geqslant 1 - 1/n ,$$

$< p_\infty, \theta > = 1$. It is also clear that for $k \in K$ and $\psi \in L_\theta$, $< p_\infty, \delta_k * \psi > = < p_\infty, \psi >$ and hence that $< p_\infty, \delta_{\dot{y}} * \psi > = < p_\infty, \delta_y * \psi > = < p_\infty, \psi >$ for $\dot{y} \in H/K$.

If $\psi \in L_\theta$ and $\mu \in M(G)$, then

$$< p_\infty, \mu * \psi > = \lim < p_{1/n_\alpha}, \dot{U}_{n_\alpha}, \mu * \psi >$$
$$= \lim \int_G < p_{1/n_\alpha}, \dot{U}_{n_\alpha}, \delta_x * \psi > d\mu(x) \quad .$$

Since

$$|< p_{1/n}, \dot{U}_n, \delta_x * \psi >| \leqslant ||\psi||_\theta ,$$

the dominated convergence theorem gives

$$< p_\infty, \mu * \psi > = \int_G < p_\infty, \delta_x * \psi > d\mu(x) \quad .$$

Finally, note that if $\mu \in M(H)$, $\psi \in L_\theta$,

$$< p_\infty, \mu * \psi > = \int_H < p_\infty, \delta_x * \psi > d\mu(x)$$
$$= \mu(G) < p_\infty, \psi > \quad .$$

Therefore p_∞ satisfies i) - iv) of definition 4.

Proof of theorem 5. If N is a normal, nilpotent subgroup of G such that G/N is compact and if $N = N_1 \supset \ldots \supset N_k = \{e\}$ is the lower center series for N , then for each $i = 1,\ldots, k-1$, N_i/N_{i+1} has large compact neighborhoods invariant under the inner-automorphisms form G/N_{i+1} .
It G is a connected Lie group with polynomial growth then the eigenvalues of Ads are of modulus 1 for all $s \in G$ (see [4] or [1] , where the following fact was first pointed out). Let S be the solu-radical of G , LS its Lie algebra and $LS_{\mathbb{C}}$, the complexification of LS . By Lie's theorem, there is an ordered basis for $LS_{\mathbb{C}}$, $\{X_1 ,\ldots, X_m\}$ so that the matrix

representation for Ads with respect to this basis is upper triangular for all $s \in S$. Let V_j be the subspace spanned by $\{X_1, \ldots, X_j\}$. Then the action of Ads on V_j/V_{j-1} is multiplication by $\alpha_j(s)$ where $|\alpha_j(s)| = 1$, i.e. Ads acts by rotation on each space V_j/V_{j-1} . Thus we can find subspaces $\{0\} = W_o \subset W_1 \subset \ldots \subset W_n$ of LS , each invariant under Ads with $\dim(V_j / V_{j-1}) \leqslant 2$, and with Ads acting by rotation on each space W_j / W_{j-1} , $j = 1, 2, \ldots, n$. Thus W_j is an ideal in LS and if S_j is the correspondinc closed normal subgroup of S , each element of S_j / S_{j-1} has large compact neighborhoods invariant under the inner-automorphisms from S / S_{j-1} , and also G/S , since, G/S is compact.

If G is a connected group with polynomial growth, then there is a compact, normal subgroup K of G such that G/K is a Lie group with polynomial growth. The above argument applied to G/K produced the desired series of subgroups.

In order to show that polynomial growth is not sufficient to imply M(G)-invariance, as is the case with being type A , we prove the following.

Theorem 7.- Let A be a subalgebra of M(G) that contains the point masses and is closed with respect to involution. If G has A-invariance then for each $f \in A \cap L^1(G)$, $-1 \notin sp(f * f^*)$.

Proof. Assume there is an $f \in A \cap L^1(G)$ such that $-1 \in sp(f * f^*)$. Then for all $g \in L^1(G)$, $\|g * f * f^* + g + f * f^*\| \geqslant 1$, for other wise,

$$[(g-1)(-f * f^* - 1)]^{-1} = \{1 + [(g-1)(- f * f^* - 1) - 1]\}^{-1}$$

exists in $\mathbb{C}\, 1 \oplus L^1(G)$, contradicting the assumption that $-1 \in sp(f * f^*)$. Hence, there is a $\varphi \in L^\infty(G)$ such that $< g * f * f^* + g , \varphi > = 0$ for all $g \in L^1(G)$ and $< f * f^* , \varphi > = 1$. Thus $f * f^* * \varphi = - \varphi$. Let $\theta = \|\varphi\|^2$. Note that for $h \in L^1(G)$

$$|h * \varphi|^2(s) = |\int h(t)\, \varphi(t^{-1}s)\, dt|^2$$
$$\leq \left[\int |h(t)|^{1/2}\, |h(t)|^{1/2}\, |\varphi(t^{-1}s)|\, dt\right]^2$$
$$\leq \int |h(t)|\, dt \int |h(t)|\, |\varphi(t^{-1}s)|^2\, dt$$
$$= ||h||_1\, |h| * |\varphi|^2 .$$

Hence, $\theta = |\varphi|^2 = |f * f^* * \varphi|^2 \leq ||f * f^*||\, |f * f^*| * \theta$.
Thus, there is a functional p on h_θ satisfying i) - iv) of definition 4 with respect to A . Now, if $g,h \in L^1(G)$,

$$(g * \varphi)\, (\overline{h * \varphi}) \leq \frac{1}{2}\, \{|g * \varphi|^2 + |h * \varphi|^2\} \in L_\theta .$$

We define a bilinear form B on $A \cap L^1(G)$ by

$$B(g,h) = < p, (g * \varphi)\, (\overline{h * \varphi}) > .$$

Clearly $B(g,g) \geq 0$ for $g \in A \cap L^1(G)$ and $B(f * f^*, f * f^*) = < p, |\varphi|^2 > = 1$. Also, for $g \in A \cap L^1(G)$,

$$B(g,g) = < p, |g * \varphi|^2 > \leq ||g||_1 < p , |g| * \theta > = ||g||_1^2 .$$

Thus B is bounded, and if $g,h \in A \cap L^1(G)$, $s \in G$,

$$B(\delta_s * g, \delta_s * h) = < p, \delta_s * [(g * \varphi)(\overline{h * \varphi})] > = B(g,h) .$$

Hence, for $g,h,k \in A \cap L^1(G)$

$$B(g * h, k) = B(h, g^* * k) .$$

Recalling that $f * f^* * \varphi = - \varphi$, we have

$$0 \leq B(f^*, f^*) = - B(f^* * f * f^*, f^*) = - B(f * f^*, f * f^*) = - 1 .$$

This contradiction implies that $-1 \notin sp(f * f^*)$.

A Banach *-algebra A is said to be symmetric if $-1 \notin sp(aa^*)$ for all $a \in A$. Theorems 5 et 7 show that $L^1(G)$ is symmetric if G is a connected group with polynomial growth or if G is a compact extension of a nilpotent group. This was first proved by Ludwig [5].

Hulanicki [2] , considered the following group : let D be a countable direct sum of $\mathbb{Z}_2$, and let

$$\underline{D} = \prod_{d \in D} (D)_d = \{(X_d)_{d \in D} \mid X_d \in D\}$$

be the direct product of D copies of D . Define $t : D \to \mathrm{Aut}(\underline{D})$ by $t(c)\ (X_d)_{d \in D} = (X_{d+c})_{d \in D}$. Then $D \ltimes_t \underline{D}$ is a solvable group and for each $s \in D \ltimes_t \underline{D}$, $s^4 = e$. Hulanicki showed that $\ell^1(D \ltimes_t \underline{D})$ is not symmetric, hence $D \ltimes_t \underline{D}$ is a solvable group with polynomial growth that does not have $M(D \ltimes_t \underline{D})$-invariance. Hence even for solvable groups being type A is a weaker condition that having $M(G)$-invariance.

Let $F(G)$ denote the subalgebra of $M(G)$ consisting of all measures with finite support.

Using the Krein extension theorem, one can easily see that G is type A if and only if G has $F(G)$-invariance. Hence polynomial growth implies $F(G)$-invariance. Using the same techniques as in the proof of theorem 2, on can show that polynomial growth implies $K(G)$-invariance where $K(G)$ denotes the measures with compact support, or even invariance with respect to algebras of measures that vanish sufficiently rapidly at infinitly. (In Hulanicki's example, for instance, one selects a sequence of subsets $U_n \subset U_{n+1}$ such that $\cup U_n = D \ltimes_t \underline{D}$ and requires the measure μ to satisfies

$$\mathrm{Lim}_n\ |< u_n >|\ |\mu|\ \ (D \ltimes_t \underline{D} \setminus < u_n >) = 0\ )\ .$$

It is not known how large a subalgebra will give invariance for arbitrary groups with polynomial growth.

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