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EXPOSÉ 14

FIBRATIONS OVER S^1 WITH PSEUDO-ANOSOV MONODROMY

by David FRIED

We will develop Thurston's description of the collection of fibrations of a closed three manifold over S^1. We will then show that the suspended flows of pseudo-Anosov diffeomorphisms are canonical representatives of their nonsingular homotopy class, thus extending Thurston's theorem for surface homeomorphisms to a class of three dimensional flows. Our proof uses Thurston's work on fibrations and surface homeomorphisms and our criterion for cross-sections to flows with Markov partitions. We thank Dennis Sullivan for introducing Thurston's results to us. We are also grateful to Albert Fathi, François Laudenbach and Michael Shub for their helpful suggestions.

A smooth fibration $f : X \to S^1$ of a manifold over the circle determines a nonsingular (i.e. never zero) closed 1-form $f^*(d\theta)$ with integral periods. Conversely if ω is a nonsingular closed 1-form and X is closed, then the map $f(x) = \int_{x_0}^{x} \omega$ from X to $\mathbb{R}$/periods (ω) will be a fibration over S^1 provided the periods of ω have rational ratios. For since $\pi_1 X$ is finitely generated, the periods of ω will be a cyclic subgroup of $\mathbb{R}$ (not trivial since X is compact and f open) and we have $\mathbb{R}$/periods $(\omega) \cong S^1$. By constructing a smooth flow ψ on X with $\omega(\frac{d\psi}{dt}) = 1$, we see that f is a fibration. The relation of nonsingular closed 1-form to fibrations over S^1 is very strong indeed, as the following theorem (which gives strong topological constraints on the existence of nonsingular closed 1-forms) indicates.

Theorem 1 [15]. For a compact manifold X, the collection $\mathcal{C}$ of nonsingular classes, that is the cohomology classes of nonsingular closed 1-forms on X, is an open cone in $H^1(X;\mathbb{R}) - \{0\}$. The cone $\mathcal{C}$ is nonempty if and only if X fibers over S^1.

<u>Proof</u>. The openness of $\mathcal{C}$ follows easily from de Rham's Theorem. If $\eta_1, \ldots, \eta_d$ are closed 1-forms that span $H^1(X ; \mathbb{R})$ and if ω_0 is a closed 1-form, then the forms $\omega_a = \omega_0 + \sum_{i=1}^{d} a_i \eta_i$, $|a_i| < \epsilon$, represent a neighborhood of $[\omega_0]$ in $H^1(X ; \mathbb{R})$. If ω_0 is nonsingular and ϵ sufficiently small, then the ω_a are nonsingular. The forms $\lambda \omega_a$, with $\lambda > 0$ represent all positive multiples of $[\omega_a]$, so $\mathcal{C}$ is an open cone.

Choosing a so that the periods of ω_a are rationally related, we see that X fibers over S^1. We already noted that $0 \notin \mathcal{C}$. Q.E.D.

In dimension 3, Stallings characterized the elements of $\mathcal{C} \cap H^1(X ; \mathbb{Z}) \subset$ $\subset H^1(X ; \mathbb{R})$. We note that if X is closed, connected and oriented and does fiber over S^1 with fibers of positive genus, then X will be covered by Euclidean space $\mathbb{R}^3$. Thus X will be <u>irreducible</u>, that is every sphere S^2 embedded in X must bound a ball (this follows from Alexander's theorem showing $\mathbb{R}^3$ is irreducible). <u>We assume henceforward</u> that M is a closed, connected oriented and irreducible 3-dimensional manifold.

<u>Theorem</u> 2 [12]. If $u \in H^1(M ; \mathbb{Z}) - \{0\}$, then there is a fibration $f : M \to S^1$ with $[f^*(d\theta)] = u$, if and only if $\ker (u : \pi_1 M \to \mathbb{Z})$ is finitely generated.

We observe that the forward implication holds even for finite complexes since the homotopy exact sequence identifies the kernel as the fundamental group of the fiber.

Theorem 2 reduces the geometric problem of fibering M to an algebraic problem, with only two practical complications. First, whenever $\dim H^1(M ; \mathbb{R}) > 1$, there are infinitely many u to check. Secondly, it is difficult to decide if $\ker u$ is finitely generated. An infinite presentation may be readily constructed by the Reidemeister-Schreier process ; this yields an effective procedure for deciding if the abelianization of $\ker u$ is finitely generated (we work out an example of this at the end of the chapter).

Thurston's theorem (theorem 5 below) helps to minimize the first problem and make Stallings criterion more practical. It will be seen that one need only examine finitely many u, provided one can compute a certain natural seminorm on $H^1(M ; \mathbb{R})$.

As $H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R})$ is a lattice of maximal rank, the seminorm will be determined by its values on $H^1(M;\mathbb{Z})$. Each $u \in H^1(M;\mathbb{Z})$ is geometrically represented by framed surfaces under the Pontrjagin construction [6]. A framed (that is, normally oriented) surface S represents u whenever there is a smooth map $f : M \to S^1$

with regular value x so that $S = f^{-1}(x)$ and $u = [f^*(d\theta)]$. By irreducibility of M, any framed sphere in M represents the 0 class so S may be taken <u>sphereless</u> (that is, all components of S have Euler characteristic ≤ 0).

<u>Definition</u>. $\|u\| = \min \{-\chi(S) \mid S$ is a sphereless framed surface representing $u\}$.

It is important to observe that a sphereless framed surface S in M, with $\|u\| = -\chi(S)$, must be <u>incompressible</u> (that is, for each component $S_i \subset S$, $\pi_1(S_i) \to \pi_1 M$ is injective). For (see Kneser's lemma [11]), one could otherwise attach a 2-handle to S_i so as to lower $-\chi(S)$ without introducing spherical components.

The justification for the notation $\|u\|$ is the following result.

<u>Theorem</u> 3 [13]. $\|u\|$ is a seminorm on $H^1(M;\mathbb{Z})$.

This follows from standard 3-manifold techniques. The triangle inequality follows from the incompressibility of minimal representatives and some cut and paste arguments. The homogeneity follows by the covering homotopy theorem for the cover $z^n : S^1 \to S^1$.

One instance where $\|u\|$ is easily computed is when u is represented by the fiber K of a fibration $f: M \to S^1$. We have :

<u>Proposition</u> 1 [13]. If $K \to M \xrightarrow{f} S^1$ is a fibration, then $\|[f^*(d\theta)]\| = -\chi(K)$.

<u>Proof</u>. By homogeneity we may suppose that $u = [f^*(d\theta)]$ is indivisible, that is $u(\pi_1 M) = \pi_1 S^1$. This implies that K is connected and that $K \times \mathbb{R}$ is the infinite cyclic cover of M determined by u. If K is a torus we are done, so assume $-\chi(K) > 0$. Any sphereless framed surface S representing u lifts to $K \times \mathbb{R}$, since for any component $S_0 \subset S$ we have $\pi_1 S_0 \subset \ker u = \pi_1 K$. If $-\chi(S) = \|u\|$, then S is incompressible and $\pi_1 S_0 \to \pi_1(K \times \mathbb{R}) = \pi_1 K$ is injective. Since subgroups of $\pi_1 K$ of infinite index are free, we see that S_0 is a finite cover of K, hence $\|u\| = -\chi(S) \geq -\chi(S_0) \geq -\chi(K)$, as desired. Q.E.D.

In fact, we see that any sphereless framed surface S representing u with minimal $-\chi(S)$ is homotopic to the fiber K.

The behaviour of $\|\ \|$ is decisively determined by the fact that integral classes have integral seminorms. We will show :

<u>Theorem</u> 4 [13]. A seminorm $\|\ \| : \mathbb{Z}^n \to \mathbb{Z}$ extends uniquely to a seminorm

$\| \; \| : \mathbb{R}^n \to [0,\infty)$. A seminorm on $\mathbb{R}^n$ takes integer values on $\mathbb{Z}^n \Leftrightarrow \|x\| = \max_{\ell \in F} |\ell(x)|$, where $F \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is finite.

This enables us to state Thurston's description of the cone $\mathcal{C}$ of nonsingular classes, $\mathcal{C} \subset H^1(M;\mathbb{R}) - \{0\}$.

We will consistently use certain natural isomorphisms of the homology and cohomology groups of M . By the Universal Coefficient Theorem, $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}\,(H_1(M;\mathbb{Z})\;;\;\mathbb{Z})$ and $H_1(M;\mathbb{Z})/\text{torsion} \cong \mathrm{Hom}\,(H^1(M;\mathbb{Z})\;;\;\mathbb{Z})$. With real coefficients, $H^i(M;\mathbb{R})$ and $H_i(M;\mathbb{R})$ are dual vector spaces for any i . By Poincaré Duality, we may identify $H^2(M;\mathbb{Z})$ with $H_1(M;\mathbb{Z})$. Thus we regard the Euler class χ_F of a plane bundle F on M , which is usually taken to be in $H^2(M;\mathbb{Z})$, as an element of $H_1(M;\mathbb{Z})$ and thus as a linear functional on $H^1(M;\mathbb{R})$.

<u>Theorem</u> 5 [13] . $\mathcal{C}$ is the union of (finitely many) convex open cones $\mathrm{int}\,(T_i)$ where T_i is a maximal region on which $\| \; \|$ is linear. The region T_i containing a given nonsingular 1-form ω is $T_i = \{u \in H^1(M;\mathbb{R}) \mid \|u\| = -\chi_F(u)\}$ where χ_F is the Euler class of the plane bundle $F = \ker \omega$.

<u>Note</u>. When $\| \; \|$ is a norm, we may say that $\mathcal{C}$ is all vectors $v \neq 0$ such that $\frac{v}{\|v\|}$ belongs to certain "nonsingular faces" of the polyhedral unit ball. Incidentally, we have that $\| \; \|$ is a norm $\Leftrightarrow$ all $T^2 \subset M$ separate M $\Leftrightarrow$ all incompressible $T^2 \subset M$ separate M .

We give our own analytic proof of theorem 4.

<u>Proof of theorem</u> 4. Clearly $\| \; \|$ extends by homogeneity to a seminorm $\| \; \|$ on $\mathbb{Q}^n$. This function is Lipschitz, hence has a unique continuous extension to a function $\mathbb{R}^n \to [0,\infty[$. The triangle inequality and homogeneity follow by continuity.

By convexity, all one-sided directional derivatives of $N(x) = \|x\|$ exist. Suppose $\tau = (0, \frac{1}{q}p)$, $q \in \mathbb{Z}^+$, $p = (p_2,\ldots,p_n) \in \mathbb{Z}^{n-1}$ is a rational point. For integral m , we compute

$$\begin{aligned}\frac{\partial_+ N}{\partial x_1}(\tau) &= \lim_{m\to\infty} \frac{N(\tau + 1/qm\, e_1) - N(\tau)}{1/qm} \\ &= \lim_{m\to\infty} \left(N(1, mp_2,\ldots,mp_n) - N(0,mp_2,\ldots,mp_n)\right) \\ &\in \mathbb{Z} \quad ,\end{aligned}$$

since $\mathbb{Z}$ is closed.

By induction on n, we assume that $N(0,\bar{x})$, $\bar{x} \in \mathbb{R}^{n-1}$, is given by the supremum of finitely many functionals $\ell(\bar{x}) = a_2x_2 + \ldots + a_nx_n$, $a_2,\ldots,a_n \in \mathbb{Z}$, $\bar{x} = (x_2,\ldots,x_n)$. By convexity, any supporting line L to graph $(N) \subset \mathbb{R}^n \times \mathbb{R}$ lies in a supporting hyperplane H (<u>supporting</u> means intersects the graph without passing above it). We choose $\bar{x}$ a rational point for which $N \mid 0 \times \mathbb{R}^{n-1}$ is locally given by ℓ and choose L to pass through $(0,\bar{x},N(0,\bar{x})) \in \mathbb{R}^n \times \mathbb{R}$ in the direction $(1,0,\frac{\partial_+ N}{\partial x}(0,\bar{x}))$. Then we see that H is uniquely determined as the graph of $(\frac{\partial_+ N}{\partial x_1}(0,\bar{x}))\, x_1 + a_2x_2 + \ldots + a_nx_n$. So for a dense set of $\bar{x}$, the graph of N has a supporting functional at $(0,\bar{x})$ with integral coefficients.

Reasoning for each integrally defined hyperplane as we have for $\{x_1 = 0\}$, we find integral supporting functionals $\ell(x) = a_1x_1 + \ldots + a_nx_n$, $a_i \in \mathbb{Z}$, to the graph of N exist at a dense set in $\mathbb{R}^n$. Since N is Lipschitz, there is a bound $|a_i| \leq K$, $i = 1,\ldots,n$. Thus the supporting functionals form a finite set F, so $S(x) = \sup_{\ell \in F} |\ell(x)|$ is clearly a seminorm. But $S(x) \leq N(x)$ and equality holds on a dense set, implying that $S(x) = N(x)$ by continuity. Q.E.D.

Before giving the proof of theorem 5, let us observe one elementary consequence of theorem 4. Since $\|\ \|$ is natural, any diffeomorphism $h: M \to M$ induces an isometry h^* of $H^1(M;\mathbb{R})$. If $\|\ \|$ is a norm, then the finite set of vertices of the unit ball spans $H^1(M;\mathbb{R})$ and is permuted by h^*.

<u>Corollary</u>. If all incompressible $T^2 \subset M$ separate M, then the image of $\mathrm{Diff}(M)$ in $GL(H^1(M;\mathbb{R}))$ is finite.

<u>Proof of theorem</u> 5. Suppose ω, ω' are nonsingular closed 1-forms that are C^0 close. Then the oriented plane fields $F = \ker \omega$, $F' = \ker \omega'$ are homotopic and so determine the same Euler class $\chi_{F'} = \chi_F \in H_1(M;\mathbb{R})$.

If $[\omega']$ is rational, let $q[\omega'] = \beta' \in H^1(M;\mathbb{Z})$, where $0 < q \in Q$ and β' is indivisible. Then if K' is the (connected) fiber of the fibration associated to $q\omega'$, we have $\chi(K') = \chi_{F'}(K') = \chi_F(K')$. Using this and proposition 1, we find $\|[\omega']\| = \frac{1}{q}(-\chi(K')) = -\frac{1}{q}\chi_F(K') = -\chi_F[\omega']$. Thus for all rational classes $[\omega']$ near $[\omega]$, $\|\ \|$ is given by the linear functional $-\chi_F$. This shows that $\|\ \|$ agrees with $-\chi_F$ on a neighborhood of any nonsingular class $[\omega]$, as desired.

It only remains to show that every $\alpha \in \operatorname{int} T$ is a nonsingular class, where $T = \{\alpha \in H^1(M;\mathbb{R}) \mid \|\alpha\| = -\chi_F(\alpha)\}$ is the largest region containing $[\omega]$ on which $\|\ \|$ is linear.

For this, we need a result of Thurston's thesis [14] concerning the isotopy of an incompressible surface $S \subset M$ when M is foliated without "dead end components". In fact, this result is only explicitly stated for tori, and one must see [8] for a published account of this case. Restricting our attention to the foliation $\mathfrak{F}$ defined by ω ($\mathfrak{F}$ is tangent to $\ker \omega = F$), we may state this result as follows : any incompressible, oriented and connected surface $S_0 \subset M$ with $-\chi(S_0) > 0$ may be isotoped so as to either lie in a leaf of $\mathfrak{F}$ or so as to have only saddle tangencies with $\mathfrak{F}$. (We call a tangency point s of S_0 with $\mathfrak{F}$ a <u>saddle</u> if for some open ball B around s, the map $\int_s^x \omega : B \cap S_0 \to \mathbb{R}$ has a nondegenerate critical point at s which is not a local extremum.)

Suppose $\alpha \in T \cap H^1(M;\mathbb{Z})$ is not a multiple of $[\omega]$. Represent α by a framed sphereless surface with $-\chi(S) = \|\alpha\|$. As S is incompressible, each component of S may be isotoped (independently) to a surface S_i which either lies in a leaf of $\mathfrak{F}$ or has only saddle tangencies with $\mathfrak{F}$. If some S_i lies in a leaf L of $\mathfrak{F}$, then (as in proposition 1) $\pi_1 S_i$ would be of finite index in $\pi_1 L = \ker [\omega]$. Since $\pi_1 S_i \subset \ker \alpha$, we would find that α is a multiple of $[\omega]$. Thus each S_i has only saddle tangencies with $\mathfrak{F}$.

<u>Lemma</u>. For each i, the normal orientations of S_i and $\mathfrak{F}$ agree at all tangencies.

<u>Proof of lemma</u>. We compute $\|\alpha\|$ in two ways. First, $\|\alpha\| = -\chi(S) = \sum_i -\chi(S_i)$. Choosing some Riemannian metric on M, we may use the vector field V_i on S_i dual to $\omega | S_i$ to compute $-\chi(S_i)$. V_i will have only nondegenerate zeroes of index -1, since all tangencies are saddles. The Hopf Index theorem [6] gives $-\chi(S_i) = n_i$, where n_i is the number of tangencies of S_i with $\mathfrak{F}$. Thus $\|\alpha\| = \sum n_i$.

On the other hand, we know that $\alpha \in T$ implies $\|\alpha\| = -\chi_F(\alpha)$. The natural normal orientations of F and S gives us preferred orientations on F and S_i, for each i. Each oriented plane bundle $F | S_i$ has an Euler class $\chi_F(S_i)[S_i]$ where $[S_i] \in H^2(S_i;\mathbb{Z})$ is the orientation class. We compute $\chi_F(S_i)$ as the self-intersection number of the zero section of $F | S_i$. For this purpose, look at the field W_i of vectors on S_i tangent to $\mathfrak{F}$, which are the projection onto F of the unit normal vectors of S_i. Regarding W_i as a perturbation of the zero section of $F | S_i$, we compute the self-intersection number using the local orientations of F and S_i. When these orientations agree, one counts the singularity as -1 (just as in the tangent bundle case already considered) but when the orientations disagree one counts $+1$. Thus $-\chi_F(S_i) = n_i^+ - n_i^-$, where n_i^+ is the number of tangencies at which the orientations agree and n_i^- is the number of tangencies at which the orientations disagree. Thus

$\|\alpha\| = \Sigma n_i^+ - \Sigma n_i^-$.

Since $n_i = n_i^+ + n_i^-$, we have $\Sigma n_i^+ + \Sigma n_i^- = \|\alpha\| = \Sigma n_i^+ - \Sigma n_i^-$, whence all the nonnegative integers n_i^- must be zero. This proves the lemma.

Because of the lemma, we may define a framing N_i of S_i with $\omega(N_i) > 0$ everywhere. This framing may be extended to a product neighborhood structure on $U_i \supset S_i$, where $h : S_i \times [-1,1] \to U_i$ is a diffeomorphism, $h_*(\frac{\partial}{\partial t}) = N_i$ on $S_i = S_i \times 0$ and $\omega(h_*(\frac{\partial}{\partial t})) > 0$. Let $B : [-1,1] \to [0,\infty]$ be a smooth function vanishing on $|x| > \frac{1}{2}$ with $\int_{-1}^{+1} B = +1$. Letting $\eta_i = (\pi_2 h^{-1})^* B\,dt$ we find that, for all $s > 0$, $(\omega + s\eta_i)(h_* \frac{\partial}{\partial t}) > 0$ on U. But since $\omega + s\eta_i = \omega$ away from U, we see that the closed 1-form $\omega + s\eta_i$ is nonsingular.

The portion of theorem 5 already proven gives $[\omega + s\eta_i] \in \operatorname{int} T$. Thus, $[\eta_i] = \lim_{s\to\infty} \frac{[\omega + s\eta_i]}{s} \in T \cap H^1(M;\mathbb{Z})$, for all i. So replacing $[\omega]$ by $[\omega] + s_1[\eta_1] + \ldots + s_{i-1}[\eta_{i-1}]$, we see inductively that $[\omega] + s_1[\eta_1] + \ldots + s_i[\eta_i]$ is nonsingular for all $s_1, \ldots, s_i \geq 0$. In particular, for all $s \geq 0$, $[\omega] + s\alpha = [\omega] + s\,\Sigma[\eta_i]$ is nonsingular.

We just showed that if $\beta = [\omega] \in \operatorname{int} T$ is a nonsingular class, then $\beta + s\alpha$ is nonsingular for all $\alpha \in T \cap H^1(M;\mathbb{Z})$ and $s \geq 0$. Now consider an arbitrary $\gamma \in \operatorname{int} T$, $\gamma \neq \beta$. By convexity we may find $v_1, \ldots, v_d \in \operatorname{int} T$, $d = \dim H^1(M;\mathbb{R})$, so that γ is in the interior of the d-simplex spanned by β, $v_1, \ldots, v_d$. We may choose $v_1 \ldots v_d$ rational, say $v_j = \frac{1}{N}\alpha_j$, some $N \in \mathbb{Z}^+$, $\alpha_j \in \operatorname{int} T \cap H^1(M;\mathbb{Z})$. We have $\gamma = t_0\beta + \sum_{j=1}^{d} t_j\alpha_j$, with all $t_j > 0$. By induction on k, we see that each $\beta + \sum_{j=1}^{k} (t_j/t_0)\alpha_j$ is nonsingular. Setting $k = d$ and multiplying by $t_0 > 0$, we see that γ is nonsingular as well. Thus if one point $\beta \in \operatorname{int} T$ is nonsingular all $\gamma \in \operatorname{int} T$ are nonsingular. Q.E.D.

We will sharpen Thurston's theorem 5 in the case when M is <u>atoroidal</u> (contains no incompressible imbedded tori) and $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$. We show (theorem 7) that a nonsingular face T (i.e. one containing a nonsingular class) of the unit $\|\ \|$-ball determines a canonical flow $\varphi_t : M \to M$ such that $\operatorname{int} T$ consists precisely of all $[\omega]$ where ω is a closed one form with $\omega(\frac{d\varphi}{dt}) > 0$. We must begin by relating the atoroidal condition to Thurston's classification of surface homeomorphisms.

We suppose $f : M \to S^1$ is a fibration. Then flows ψ_t for which $\frac{d}{dt} f(\psi_t m) > 0$ (we will only consider flows having a continuous time derivative) determine an isotopy class of surface homeomorphisms. For any $k \in K = f^{-1}(1)$, we consider the smallest time $T(k) > 0$ for which $\psi_{T(k)}(k) \in K$. This map $T(k) : K \to (0,\infty)$ is smooth (since the flow lines of ψ are transverse to K) and the <u>return map</u> $R(k) = \psi_{T(k)}(k)$ is a homeomorphism. By varying ψ, we obtain an isotopy class of homeomorphisms of the fiber K as return maps ; this <u>isotopy</u> class will be called the <u>monodromy</u> of f and denoted $m(f)$.

We remark that the monodromy of f is determined algebraically by the cohomology class $\beta = f^*[d\theta] \in H^1(M;\mathbb{Z})$, or equivalently by the map $f_* : \pi_1 M \to \pi_1 S^1$. First assume that β is indivisible. From the exact homotopy sequence $1 \to \pi_1 K \to$ $\to \pi_1 M \xrightarrow{f_*} \pi_1 S^1 \to 1$, we see that $\pi_1 M$ is the semidirect product $\pi_1 K \ltimes_\alpha \mathbb{Z}$, where α is the outer automorphism of $\pi_1 K$ determined by the monodromy of f. Thus $\pi_1 K$ $(= \ker f_*)$ and α are determined by f_* alone. Clearly the topological type of K is determined by $\pi_1 K$; but Nielsen also showed that isotopy classes in $\mathrm{Diff}(K)$ correspond 1 – 1 to outer automorphisms of $\pi_1 K$. In general, $\beta = n\beta'$ is a positive integer multiple of an indivisible class β', and n is determined by $\mathrm{coker}\, f_* = \mathbb{Z}/n\mathbb{Z}$. We see that the fiber of f consists of n copies of K (where $\pi_1 K = \ker f_*$) which are permuted cyclically by the monodromy. The nth power of the monodromy preserves K and acts on $\pi_1 K$ by α (the outer automorphism of $\ker f_*$). Thus we may unambiguously speak of the monodromy of a nonsingular class $\beta \in H^1(M;\mathbb{Z})$.

We say that the monodromy $m(f)$ of a fibration $f : M \to S^1$ is <u>pseudo-Anosov</u> if the isotopy class has a pseudo-Anosov representative R. This representative is then uniquely determined within <u>strict conjugacy</u>, that is for any two pseudo-Anosov representatives $R_0, R_1 \in m(f)$ there will be a homeomorphism g isotopic to the identity for which $R_0 g = g R_1$.

<u>Proposition</u> 2. Suppose that $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$. Given a fibration $f : M \to S^1$, M is atoroidal precisely when the monodromy $m(f)$ is pseudo-Anosov and the fibers of f are not composed of tori.

<u>Proof</u>. Suppose M contains an incompressible torus S and let $\mathcal{F}$ be the foliation of M by the fibers of f. Again using the result of Thurston's thesis discussed in the proof of theorem 5 [8, 14], we may isotope S to either lie in a leaf of $\mathcal{F}$ or to be transverse to $\mathcal{F}$ (since $\chi(S) = 0$, the presence of saddle tangencies would force there to be tangencies of other types). If S does lie in a leaf, then the fibers of f are composed of tori parallel to S. If the torus S is transverse to $\mathcal{F}$, then one may define a

flow ψ on M that preserves S and satisfies $\frac{d}{dt}(f \circ \psi_t) = 1$. Thus the return map $\psi_1 : K \to K$, $K = f^{-1}(1)$, preserves the family of curves $S \cap K$. Since S is incompressible, each of those curves is homotopically nontrivial in K. If the monodromy of f were pseudo-Anosov, these curves would grow exponentially in length under iteration by ψ_1. So we see that when $m(f)$ is pseudo-Anosov and the fibers of f are not unions of tori, then M must be atoroidal.

Conversely, when the fibers of f are unions of tori, these tori are essential. So we assume the components of the fibers have higher genus and that the monodromy is not pseudo-Anosov (hence reducible or periodic) and look for an incompressible torus. If $m(f)$ is reducible, we may construct ψ with $\frac{d}{dt}(f \circ \psi_t) = 1$ for which ψ_1 cyclically permutes a family of homotopically nontrivial closed curves $C \subset K$. Then $\{\psi_t C\}$ is an incompressible torus. If $m(f)$ has period n, after Nielsen (see exposé 11), we may choose ψ with $\frac{d}{dt}(f \circ \psi_t) = 1$ for which ψ_n = identity. Thus M is Seifert fibred. One may easily compute that $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g+1}$, where g is the genus of the topological surfaces which is the orbit space of ψ [7]. As we assumed $H^1(M;\mathbb{Z}) \neq \mathbb{Z}$, we must have a homologically nontrivialcurve in this orbit space which corresponds to an incompressible torus in M. Q.E.D.

We may consider flows transverse to a fibration over S^1 from three viewpoints. The first is to begin with the fibration and produce transverse flows and an isotopy class of return maps. The second is to begin with a homeomorphism $R : K \to K$ and produce a fibration over S^1 with fiber K and a transverse flow φ with return map R. This is the well-known mapping torus construction, for which one sets $X = K \times [0,1]/(k,1) = (R(k),0)$, $f : X \to ([0,1]/0=1) = S^1$ the natural fibration and defines ψ to be the flow along the curves $k \times [0,1]$ with unit speed. Clearly $\psi_1 | K \times 0 = R$ is the return map of ψ, as desired. This flow ψ is called the <u>suspension</u> of R. The third viewpoint is to begin with a flow ψ on X and to seek a fibration f over S^1 to which ψ is transverse - a fiber K is called a <u>cross-section</u> to ψ. Note that K and ψ determine the return map R and an isotopy class of fibrations f.

In general, one has little hope of finding cross-sections, since many manifolds don't fiber over S^1 at all. But there is a classification of the fibrations transverse to ψ which is especially concrete in the case of interest to us now.

Suppose that some cross-section K to a flow φ has a return map $R : K \to K$ admitting a Markov partition $\mathfrak{m} = \{S_1, \ldots, S_m\}$ (see exposé 10 - the case we need is when R is pseudo-Anosov). There is a directed graph with vertices $S_1, \ldots, S_m$ and arrows $S_i \to S_j$ for each i and j for which $R(S_i)$ meets $\mathrm{int}(S_j)$. A <u>loop</u> ℓ

for $\mathfrak{m}$ is a cyclic sequence of arrows $S_{i_1} \to S_{i_2} \to \dots \to S_{i_k} \to S_{i_1}$. Each loop ℓ determines a periodic orbit for R and thus a periodic orbit $\gamma(\ell)$ for φ . If all of $i_1,\dots,i_k$ are distinct, we call ℓ minimal. There are only finitely many minimal loops ℓ .

We now discuss the classification and existence of cross-sections to flows. Given a flow ψ on a compact manifold X there is a nonempty compact set of homology directions $D_\psi \subset H_1(X ; \mathbb{R})/\mathbb{R}^+$, where the quotient space is topologized as the disjoint union of the origin and unit sphere. A homology direction for ψ is an accumulation point of the classes determined by long, nearly closed trajectories of ψ . We note that when K is a cross-section to ψ , K is normally oriented by ψ and so determines a dual class $u \in H^1(X ; \mathbb{Z})$. Let $C_{\mathbb{Z}}(\psi) = \{u \in H^1(X ; \mathbb{Z}) \mid u$ is dual to some cross-section K to $\psi\}$.

Theorem 6 [1,2] . $C_{\mathbb{Z}}(\psi) = \{u \mid u(D_\psi) > 0\}$. If φ , as above, has a cross-section K and the return map R admits a Markov partition $\mathfrak{m}$, then $C_{\mathbb{Z}}(\varphi) = \{u \mid u(\gamma(\ell)) > 0$ for all minimal loops ℓ for $\mathfrak{m}\}$.

Thus, $C_{\mathbb{Z}}(\psi)$ consists of all lattice points in a (possibly empty) open convex cone $C_{\mathbb{R}}(\psi) = \{u \mid u(D_\psi) > 0\} \subset H^1(X ; \mathbb{R}) - \{0\}$. It follows easily from theorem 6 that $C_{\mathbb{R}}(\psi) = \{[\omega] \mid \omega$ is a closed 1-form with $\omega(\frac{d\psi}{dt}) > 0\}$.

Returning to our discussion of three-manifolds, we call a flow φ on M pseudo-Anosov if it admits some cross-section for which the return map is pseudo-Anosov. We now describe the cross-sections to pseudo-Anosov flows, and show they are uniquely determined by their homotopy class among nonsingular flows on M .

Theorem 7. Suppose M fibers over S^1 . Then each flow ψ on M that admits a cross-section determines a nonsingular face $T(\psi)$ for the norm $\|\ \|$ on $H^1(M ; \mathbb{R})$. Here $T(\psi) = \{\|u\| = -\chi_{\psi^\perp}(u)\}$ and $\psi^\perp$ denotes the normal plane bundle to the vector field $\frac{d\psi}{dt}$. One has $C_{\mathbb{R}}(\psi) \subset \operatorname{int} T(\psi)$.

For any pseudo-Anosov flow φ on M , $C_{\mathbb{R}}(\varphi) = \operatorname{int} T(\varphi)$.

The face $T(\varphi)$ (or the class $\chi_{\varphi^\perp}$) determines the pseudo-Anosov flow φ up to strict conjugacy. Thus any nonsingular face T on an atoroidal M with $H^1(M ; \mathbb{Z}) \neq \mathbb{Z}$ determines a strict conjugacy class of pseudo-Anosov flows.

Proof. For $u \in C_{\mathbb{Z}}(\psi)$, there is a cross-section K to ψ dual to u . We have $\|u\| = -\chi(K)$, by proposition 1 . Since the restriction $\psi^\perp \mid K$ is the tangent bundle of K , we have $-\chi(K) = -\chi_{\psi^\perp}(u)$. Thus $-\chi_{\psi^\perp}$ is a linear functional on $H^1(M;\mathbb{R})$ that agrees with $\|\ \|$ on $C_{\mathbb{Z}}(\psi)$ and the first paragraph of theorem 7 is shown.

We now observe

<u>Lemma</u>. Any cross-section K to a pseudo-Anosov flow φ on M will have pseudo-Anosov return map R_K .

<u>Proof</u>. By definition there is some cross-section L to φ with pseudo-Anosov return map R_L , but K and L will generally not be homeomorphic (one calls return maps to distinct cross-sections to the same flow <u>flow-equivalent</u>). In any case, any structure on L invariant under R_L is carried over to structure on K invariant under R_K under the system of local homeomorphisms between K and L determined by φ . This shows that R_K preserves a pair of transverse foliations $\mathfrak{F}^u_K$ and $\mathfrak{F}^s_K$ with the same local singularity structure as a pseudo-Anosov diffeomorphism.

We now show that the closure $\overline{P}$ of any prong P of $\mathfrak{F}^u_K$ or $\mathfrak{F}^s_K$ is the component K_0 of K which contains P . By passing to a cyclic cover $M_n \to M$ determined by the composite homomorphism $\pi_1 M \to (\pi_1 M/\pi_1 K_0) \cong \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ and restricting to the cross-section $K_0 \subset M_n$ we may assume that K is connected and that R_K leaves P invariant (choose n so that P is invariant under $R^n_{K_0}$) . Consider the closed R_L invariant subset $\{\varphi_t \overline{P}\} \cap L = I$. Since I contains the closure of a prong for the pseudo-Anosov diffeomorphism R_L , we know that I is dense in some component $L_0 \subset L$. As L_0 is a cross-section to φ , we find that $\{\varphi_t \overline{P}\} = M$. As $\overline{P}$ is R_K invariant, we find $\overline{P} = K$ as desired.

Similarly we can check that the foliations $\mathfrak{F}^u_K$ and $\mathfrak{F}^s_K$ have no closed leaves.

It follows by the Poincaré-Bendixson theorem that each leaf closure contains a singularity, and thus a prong. So we find that all leaves of $\mathfrak{F}^s_K$ and $\mathfrak{F}^u_K$ are dense in their component of K .

We may see from this density of leaves and the fact that the local stretching and shrinking properties of R_K are the same as those of R_L that the Markov partition construction of exposé 10 works for R_K . (It is easiest to construct birectangles for R_K by "analytic continuation", from immersed birectangles in L. This makes sense because K and L have the same universal cover.) As in the Anosov case [9] , the Parry measures for the one-sided subshifts of finite type associated to $\mathfrak{R}$ push forward to give transverse measures on $\mathfrak{F}^u_K$ and $\mathfrak{F}^s_K$ that transform under R_K by factors λ_K^{-1} and λ_K , for some $\lambda_K > 1$. As leaves are dense, these measures have positive values on any transverse interval but vanish on points. Thus R_K is pseudo-Anosov.

Q.E.D.

Now suppose φ^1 and φ^2 are pseudo-Anosov flows on M for which $C_{\mathbb{R}}(\varphi^1)$

intersects $C_{\mathbb{R}}(\varphi^2)$. Then we may choose $u \in C_{\mathbb{R}}(\varphi^1) \cap C_{\mathbb{R}}(\varphi^2) \cap H^1(M;\mathbb{Z})$ and find fibrations $f_i : M \to S^1$ with $\frac{d}{dt}(f_i \circ \varphi^i_t) > 0$ and $u = [f_i^*(d\theta)]$, $i = 1,2$.

As discussed earlier, u determines $m(f_i)$. This gives a homeomorphism $h : M \to M$ such that $f_1 \circ h = f_2$ where h acts on $\pi_1 M$ by the identity. Thus h is isotopic to the identity [16]. Hence, by this preliminary isotopy, we assume $f_1 = f_2 = f$ and denote the fiber by K.

Each φ^i determines a return map $R_i : K \to K$. By the lemma above, these R_i are pseudo-Anosov. Since the maps R_i are in the same isotopy class $h(f)$, they are strictly conjugate by the uniqueness of pseudo-Anosov diffeomorphisms (exposé 12).

Now suppose that $gR_1 = R_2 g$, with g isotopic to the identity. Then the map $C_0 : M \to M$ defined by $C_0(\varphi^1_s k) = (\varphi^2_s gk)$, $k \in K$, $0 \leq s \leq 1$, is a homeomorphism conjugating flows φ^1 and φ^2 and $f \circ C_0 = f$. As $C_0 | K = g$ is isotopic to the identity, C_0 may be isotoped to C_1 where $f \circ C_t = f$, for $t \in [0,1]$ and C_1 fixes K. Since Diff K is simply connected [4], we may isotop C_1 to the identity C_2 (through C_t satisfying $f \circ C_t = f$, $t \in [1,2]$).

We have shown so far that if φ^i are pseudo-Anosov flows, $i = 1,2$, then either $C_{\mathbb{Z}}(\varphi^1)$ equals $C_{\mathbb{Z}}(\varphi^2)$ or is disjoint from it, since conjugating a flow by a conjugacy isotopic to the identity doesn't affect $C_{\mathbb{Z}}$. It follows easily that the open cones $C_{\mathbb{R}}(\varphi^1)$ and $C_{\mathbb{R}}(\varphi^2)$ are either disjoint or equal.

Now suppose that φ is pseudo-Anosov but $C_{\mathbb{R}}(\varphi)$ is a proper subcone of $\operatorname{int} T(\varphi)$. By theorem 6, $C_{\mathbb{R}}(\varphi)$ is defined by linear inequalities with integer coefficients, and so there is an integral class $u \in \operatorname{int} T \cap \partial C_{\mathbb{R}}(\varphi)$. Then u is nonsingular (theorem 5), the fibration corresponding to u has pseudo-Anosov monodromy (proposition 2) and one obtains an Anosov flow ψ with $u \in C_{\mathbb{R}}(\psi)$. This shows that $C_{\mathbb{R}}(\psi)$ and $C_{\mathbb{R}}(\varphi)$ are neither disjoint nor equal, contradicting the previous paragraph.

Thus we see that pseudo-Anosov flows satisfy $C(\varphi) = \operatorname{int} T(\varphi)$. Q.E.D.

Theorem 7 shows that pseudo-Anosov maps satisfy an interesting extremal property within their isotopy class. Suppose $h_0 : K \to K$ has suspension flow $\psi^0_t : M \to M$, where we take K connected and dual to the indivisible class $u \in H^1(M;\mathbb{Z})$. Given an isotopy h_t starting at h_0, we may deform ψ^0 through flows ψ^t with cross-section K and return map h_t. We regard $u^{-1}(1)$ as a subset of $H_1(M;\mathbb{R})/\mathbb{R}^+$ and note that we always have $D_{\psi^t} \subset u^{-1}(1)$. By the Wang exact sequence :

$$H_1(K;\mathbb{R}) \xrightarrow{h_{0*}-Id} H_1(K;\mathbb{R}) \to H_1(M;\mathbb{R}) \xrightarrow{u} \mathbb{R} \to 0 ,$$

we may identify $u^{-1}(1)$ with $u^{-1}(0) = \operatorname{coker}(h_{0*} - Id)$ by some fixed splitting of u. Whenever $h_s = h_t$, the simple connectivity of Diff K [4] implies that $D_{\psi^s} = D_{\psi^t}$.

Thus we may unambiguously associate a set of homology directions $D_h \subset \operatorname{coker}(h_{0*}-\mathrm{Id})$ to homeomorphisms h isotopic to h_0. Now assume that h_0 is pseudo-Anosov. By theorem 7, we have $\mathcal{C}_{\mathbb{R}}(\psi^s) \subset \operatorname{int} T(\psi^s) = \operatorname{int} T(\psi^0) = \mathcal{C}_{\mathbb{R}}(\psi^0)$. Thus we find, using theorem 6, that the convex hull of D_{h_s} (which may be identified with the asymptotic cycles of ψ^s in this situation [3,10]) always contains the convex polygon determined at $s = 0$. Thus we may say that pseudo-Anosov diffeomorphisms have the fewest generalized rotation numbers in their isotopy class.

We may analyze the topological entropy of the return-maps R_K of the various cross-sections K to a pseudo-Anosov flow $\mathcal{C}$. We parametrize these cross-sections K by their dual classes $u \in H^1(M;\mathbb{Z})$ and define $h : \mathcal{C}_{\mathbb{Z}}(\varphi) \to (0,\infty)$ by $h([K]) = h(R_K)$, the topological entropy of R_K. We showed in [3] that $1/h$ extends uniquely to a homogeneous, downwards convex function $1/h : \overline{\mathcal{C}_{\mathbb{R}}(\varphi)} \to [0,\infty]$ that vanishes exactly on $\partial\mathcal{C}_{\mathbb{R}}(\varphi)$. Thus $h(u)$ may be defined for all $u \in H^1(M;\mathbb{R})$ in a natural way. The smallest value of h on $\operatorname{int} T \cap \{\|u\| = 1\}$ defines an interesting measure of the complexity of φ (or equivalently, by theorem 7, of the face $T = T(\varphi)$). The integral points at which h is largest give the "simplest" cross-section to the flow φ (see [3]).

If one is given a pseudo-Anosov diffeomorphism $h : K \to K$ and a Markov partition $\mathfrak{m}$ for h, theorems 6 and 7 give an effective description of the nonsingular face T determined by the suspended flow $\varphi_t : M \to M$ of h, in terms of the orbits corresponding to minimal loops. As the computation of minimal loops in a large graph is difficult, we observe that there is a more algebraic way of using $\mathfrak{m}$ to obtain a system of inequalities defining T. (We refer the reader to [3] for details, where we used this method to construct a rational zeta function for axiom A and pseudo-Anosov flows.) For sufficiently fine $\mathfrak{m}$, we may associate to $\mathfrak{m}$ a matrix A with entries in $H_1(M;\mathbb{Z})/\text{torsion} \doteq H$. The expression $\det(I - A)$, regarded as an element in the group ring of the free abelian group H, may be uniquely written as $1 + \Sigma\, a_i\, g_i$, $g_i \in H - \{0\}$, $a_i \in \mathbb{Z} - \{0\}$, g_i distinct. Then T is defined by the inequalities $u(g_i) > 0$.

To illustrate Thurston's theory, it is convenient to work on a bounded M^3. The norm considered above can be extended to such M by omitting spheres and discs before computing the negative Euler characteristic. One should restrict to the case where ∂M is incompressible, and then theorems 2 and 5 and proposition 1 extend [5,13].

We let K be the quadruply connected planar region and h the indicated composite of the two elementary braids (figure 1) which fixes the outer boundary

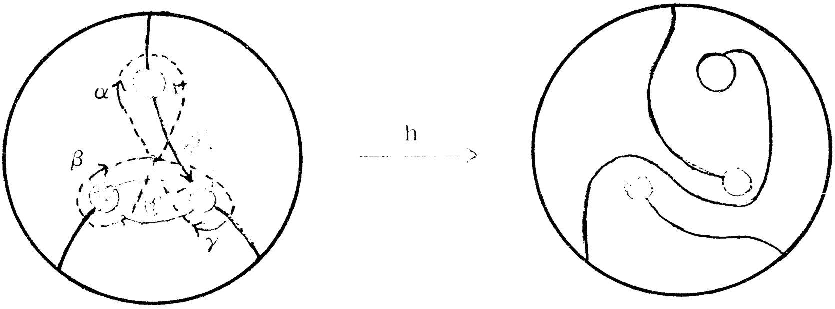

Figure 1

component. We will let M be the mapping torus of h and compute $\|\ \|$. Rather than finding a pseudo-Anosov map isotopic to h, which would only help compute one face, we will instead compute $\ker(u : \pi_1 M \to \mathbb{Z})$ for several indivisible $u \in H^1(M;\mathbb{Z})$. When this kernel is finitely generated, theorem 2 shows u is nonsingular and proposition 1 enables us to compute $\|u\|$. From a small collection of values of $\|\ \|$, theorem 5 allows us to deduce all the others, indicating the existence of nonsingular classes that would be hard to detect using only theorem 2.

We first compute $\pi_1 M = \pi_1 K \ltimes \mathbb{Z}$. Writing $\pi_1 K$ as the free group on the loops α, β and γ shown in the diagram, we find :

$$\pi_1 M = \langle \alpha,\beta,\gamma,t \mid t^{-1}\alpha t = \gamma\ ,\quad t^{-1}\beta t = \gamma^{-1}\alpha\gamma\ ,$$

$$t^{-1}\gamma t = (\gamma^{-1}\alpha\gamma)\beta(\gamma^{-1}\alpha\gamma)^{-1} \rangle$$

$$= \langle \alpha,\beta,\gamma,t \mid t^{-1}\alpha t = \gamma\ ,\ t^{-1}\beta t = \gamma^{-1}\alpha\gamma\ ,\ \gamma\beta t = \beta t\beta \rangle$$

$$= \langle \gamma,t \mid (t\gamma^{-1}t\gamma t^{-1}\gamma)^2 = \gamma(t\gamma^{-1}t\gamma t^{-1}\gamma)t \rangle \quad .$$

Abelianizing gives $H_1(M;\mathbb{Z}) = \mathbb{Z}\gamma \oplus \mathbb{Z}t$. Suppose $u \in H^1(M;\mathbb{Z})$ is indivisible, so that $a = u(\gamma)$ and $b = u(t)$ are relatively prime. The Reidemeister-Schreier process gives a presentation for $\ker(u : \pi_1 M \to \mathbb{Z})$ (essentially by computing the fundamental group of the infinite cyclic cover corresponding to u) which is very ungainly for large a . When $a = 1$, one finds the relatively simple expression :

$$\ker u = \langle t_i \mid t_i t_{i+b-1} t_{i+b}^{-1} t_{i+b+1} t_{i+2b} t_{i+2b+1}^{-1} = t_{i+1} t_{i+b} t_{i+b+1}^{-1} t_{i+b+2} \rangle .$$

For $b > 1$, this relation expresses t_i in terms of $t_{i+1}, \ldots, t_{i+2b+1}$ and expresses t_{i+2b+1} in terms of $t_i, \ldots, t_{i+2b}$. Thus $\ker u$ is free on $t_1, \ldots, t_{2b+1}$. Similarly if $b < -1$, then $\ker u$ is free on $t_1, \ldots, t_{1-2b}$ and if $b = 0$, then $\ker u$ is free on t_1, t_2, t_3 . If $b = \pm 1$, however, one may abelianize and obtain $(\ker u)^{ab} = \mathbb{Z}[t, t^{-1}]/ 2t^3 - 3t^2 + 3t - 2$ which maps onto the collection of all 2^n th roots of unity, and so $\ker u$ is not finitely generated.

By theorem 2 (Stallings), there is a fibration for $u = (1,b)$ when $b \neq \pm 1$, with fiber K_u satisfying $\pi_1(K_u) = \ker u$. By proposition 1 , $\|u\| = -\chi(K_u)$, which is clearly $-1 + \operatorname{rank}(H_1 K_u) = \begin{cases} |2b| , & b > 1 , \ b \in \mathbb{Z} \\ 2 , & b = 0 . \end{cases}$

We will see that these values determine $\| \ \|$ completely. Using the dual basis to (γ, t) , we know that :

$$\|(1,b)\| = \begin{cases} |2b| , & b > 1 , \ b \in \mathbb{Z} \\ 2 , & b = 0 \end{cases} .$$

But $\|(1,b)\|$ is a convex function f of b by theorem 3 and it takes integer values at integer points. By convexity, $f(1)$ must be 2 or 3 . Were $f(1) = 3$, convexity would force $f(x) = \begin{cases} 2 + x & \text{for } 0 \leq x \leq 2 \\ 2x & \text{for } x \geq 2 \end{cases}$ and then $(1,2)$ would not lie in an open face of the unit ball, contradicting theorem 5 . Thus one must have $f(1) = 2$, and likewise $f(-1) = 2$. By convexity, we find $f(x) = \max(|2x|, 2)$. Homogenizing shows $\|(a,b)\| = \max(|2a|, |2b|)$, i.e. $\|u\| = \max(|u(2\gamma)|, |u(2t)|)$.

By theorem 5, $u \in H^1(M; \mathbb{R})$ is nonsingular $\Leftrightarrow$ $|u(\gamma)| \neq |u(t)|$.

This example embeds in a larger one, constructed with the mapping torus M_0 of the transformation h^3 (M_0 is a triple cyclic cover of M) . $H_1(M_0, \mathbb{Z})$ is free abelian on α, β, γ, t , so there is a norm on $H^1(M_0; \mathbb{R})$ whose restriction to $H^1(M; \mathbb{R}) \cong \{ u \in H^1(M_0; \mathbb{R}) \mid u(\alpha) = u(\beta) = u(\gamma) \}$ is $3\| \ \|$. We leave its computation as an exercise.

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