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RAMIFICATION IN P-ADIC LIE EXTENSIONS

by

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Let $\mathcal{O}$ be a complete discrete valuation ring, with residue field k algebraically closed of characteristic $p > 0$. Let K be the field of fractions, $\overline{K}_s$ the separable closure of K, $\overline{K}$ the algebraic closure of K, and $\mathcal{G} = \mathrm{Aut}_K(\overline{K}) = \mathrm{Gal}(\overline{K}_s/K)$.

If G is a p-divisible group over $\mathcal{O}$, its general fibre determines a continuous Galois representation:

$$\rho : \mathcal{G} \longrightarrow \prod_{\lambda} GL(d_\lambda, D_\lambda)$$

where the D_λ are division algebras with center $\mathbb{Q}_p$. When K has characteristic zero this representation is well-known; it is given by the Galois action on the Tate module $T(G)$ [10]. When K has characteristic p, I will show how to define ρ as a Galois action on a generalized Tate module and will calculate its determinant.

In both cases the image of ρ is a closed subgroup of $\prod_{\lambda} GL(d_\lambda, D_\lambda)$ and inherits the structure of a p-adic Lie group. It carries two filtrations: an arithmetic filtration by the upper ramification subgroups of $\mathcal{G}$, and an analytic filtration by the p-saturated subgroups of Lie theory. When $\mathrm{char}(K) = 0$, Sen has shown that these two filtrations are related in a striking manner [7]; unfortunately, his results hold for <u>any</u> p-adic Galois representation and have nothing to do with the group G. When $\mathrm{char}(K) = p$ the ramification behavior of an <u>arbitrary</u> p-adic Galois representation can be quite random [11], but it seems that there <u>is</u> an interesting relation between the two filtrations when the representation comes from a p-divisible group over $\mathcal{O}$. Such a relation would reflect a favorable arithmetic property of ρ in the equicharacteristic case, much as $T(G)$ enjoys a Hodge-Tate decomposition in the case of mixed characteristic [10].

In this paper I will present evidence for such a filtration relation when G has dimension one. In this case the ramification calculations can be made quite explicitly, and one can appeal to the theory of formal A-modules when G has additional endomorphisms. It is a pleasure to express my appreciation to Jon Lubin and John Tate, who taught me this subject and offered many helpful suggestions.

§1. <u>Review of ramification theory</u> [8]

Let K be a local field, with algebraically closed residue field k of characteristic $p > 0$. Let v_K be the valuation on $\overline{K}$ with value group $\mathbb{Z}$ on K^*.

If E is a finite separable extension of K, we may filter the set

$$\Gamma = \Gamma_{E/K} = \mathrm{Hom}_K(E, \overline{K})$$

as follows. Since E is totally ramified over K, it is generated by any uniformizing parameter β. Let $e = [E:K]$ and define for $x \geq 0$ the subset

$$\Gamma_x = \{\sigma \in \Gamma : e v_K(\beta^\sigma - \beta) \geq x + 1\}$$

For large enough x, Γ_x consists only of the identity homomorphism; furthermore this filtration is independent of the choice of β.

We call x a <u>break</u> in the filtration if $\Gamma_x \neq \Gamma_{x+\varepsilon}$ for all $\varepsilon > 0$. When E is a Galois extension of K, the set Γ may be identified with the Galois group and the filtration we have defined coincides with the lower ramification filtration of $\mathrm{Gal}(E/K)$. In this case the breaks all occur at integers; in the general case the breaks may be rational, as $(\beta^{\sigma}-\beta)$ may ramify over E.

If $x = 0$ is the only break in the filtration of Γ then E/K is tamely ramified (hence cyclic). We shall henceforth assume there are further breaks. Define the Herbrand transition function:

$$\phi_{E/K}(x) = \frac{1}{e}\int_0^x \mathrm{Card}(\Gamma_t)dt \tag{1.1}$$

This is monotone increasing and piecewise linear. Let $\psi(x)$ be the inverse function on the interval $[0,\infty)$ and define the upper filtration of Γ by setting $\Gamma^y = \Gamma_{\psi(y)}$ for $y \geq 0$. The upper breaks are the values of y such that $\Gamma^{y+\varepsilon} \neq \Gamma^y$ for all $\varepsilon > 0$.

The lower numbering passes well to a subgroup, and the upper numbering to a quotient. To be precise: let L be a finite Galois extension of K containing E. Let $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L/E)$, so $\Gamma \simeq G/H$. Then

$$H_x = H \cap G_x \qquad \text{for all } x \geq 0\,. \tag{1.2}$$

$$\Gamma^y = G^y H/H \qquad \text{for all } y \geq 0\,. \tag{1.3}$$

$$\phi_{L/K} = \phi_{E/K} \circ \phi_{L/E} \tag{1.4}$$

Using (1.3) we may define an upper filtration on the Galois group of an <u>infinite</u> Galois extension L/K by setting:

$$\mathrm{Gal}(L/K)^y = \{\sigma \in \mathrm{Gal}(L/K) : \text{ for all subfields } E \text{ of finite degree over } K,\ \sigma \in \Gamma^y_{E/K}\,\mathrm{Gal}(L/E)\}\,.$$

We say y is a break in this filtration if it occurs as a break in some finite quotient. Then every non-negative rational number occurs as a break in

$\mathrm{Gal}(\overline{K}_s/K)$; on the other hand, when $\mathrm{Gal}(L/K)$ is a p-adic Lie group, the breaks form a discrete subset of the reals [7], [11]. If L is the maximal abelian extension of K, the breaks occur exactly at the non-negative integers.

We now show how to calculate the upper breaks in $\Gamma_{E/K}$ when E is given as the root field of a separable Eisenstein polynomial. By (1.3) these breaks will also occur in the filtration of the Galois group of the normal closure of E.

Lemma 1.5 (Tate)

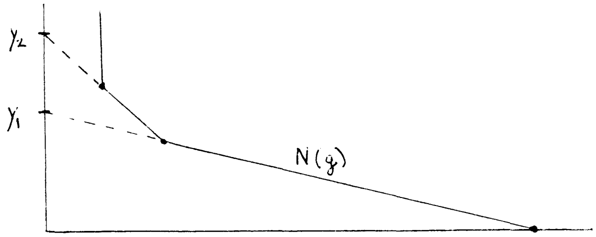
Assume $E = K(\beta)$, *where* β *satisfies the separable equation*:
$f(x) = x^e + a_{e-1}x^{e-1}+\ldots+a_0$ *with* $a_i \in K$, $v_K(a_i) \geq 1$, *and* $v_K(a_0) = 1$.

Let $g(x)$ *be the polynomial*:

$$g(x) = (\tfrac{1}{\beta})^e f(\beta x+\beta) = x^e + b_{e-1}x^{e-1}+\ldots+b_1 x$$

and let $N(g)$ *be its Newton polygon: the convex hull of the points* $(i,v_K(b_i))$ *in the plane*.

Then the upper breaks in the filtration of $\Gamma_{E/K}$ *occur at the y-intercepts of the non-trivial sides of* $N(g)$.



Proof. The roots of $g(x)$ are the values $a_\sigma = (\beta^\sigma/\beta) - 1$, where σ runs through $\mathrm{Hom}_K(E,\overline{K})$. Thus the distinct rational numbers in the set $S = \{ev_K(a_\sigma) : \sigma \neq 1\}$ give the lower breaks of Γ .

On the other hand, the numbers $-v_K(a_\sigma)$ are precisely the slopes of $N(g)$. Since the non-trivial sides of the polygon satisfy linear equations of the form

$$y + \lambda x = \phi_{E/K}(e\cdot\lambda)$$

we see that the y-intercepts give the upper breaks.

Corollary 1.6

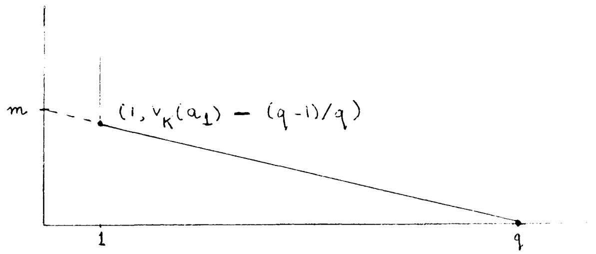
Suppose $\mathrm{char}(K) = p$ and $E = K(\beta)$ is a separable extension of degree $q = p^f$, where β satisfies $f(x)$ as in 1.5.

1) If $v_K(a_i) \geq v_K(a_1)$ for all $i \geq 1$ then $a_1 \neq 0$ and the upper and lower filtrations of $\Gamma_{E/K}$ have a unique break at the point

$$m = (qv_K(a_1)/q-1) - 1 .$$

2) If E/K is Galois then $q - 1$ divides $v_K(a_1)$ and $\mathrm{Gal}(E/K) \simeq \mathbb{F}_q^+$.

Proof. 1) The coefficient a_1 is non-zero as $f(x)$ is assumed separable. If we graph the Newton polygon of $g(x)$ as in (1.5) we find it has but one slope:



The y-intercept is at $(qv_K(a_1)/q-1) - 1$, which is the only upper break. By (1.1) it is also the only lower break.

2) If E/K is Galois the lower break must be integral. As there is only one break point and this point is positive, $\mathrm{Gal}(E/K)$ is an elementary abelian p-group [8].

§2. P-divisible groups and Galois representations

Let K be a field, and G a p-divisible group over K of height h. If $p \neq \mathrm{char}(K)$ then G is étale and is completely determined by its Tate module:

(2.1) $$T(G) = \mathrm{Hom}_{\overline{K}}(\mathbb{Q}_p/\mathbb{Z}_p, G) .$$

This module is free of rank h over $\mathbb{Z}_p = \mathrm{End}_K(\mathbb{Q}_p/\mathbb{Z}_p)$ and admits a left action of $\mathcal{G} = \mathrm{Aut}_K(\overline{K})$ which is continuous and $\mathbb{Z}_p$-linear:

(2.2) $$\rho : \mathcal{G} \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T(G)) \simeq GL(h, \mathbb{Z}_p) .$$

The functor $G \longmapsto T(G)$ from étale groups to Galois modules is fully faithful [9], [10].

When $p = \mathrm{char}(K)$ the situation is more complicated as G need not be étale. The Tate module, as defined in (2.1), can only give information on the maximal étale quotient of G. To construct a more sensitive functor into the category of p-adic Galois modules, we need a larger supply of initial objects (like $\mathbb{Q}_p/\mathbb{Z}_p$).

These objects are furnished by Dieudonné theory. For any reduced rational number $\lambda = r/s$ in the interval $[0,1]$ there is a canonical p-divisible group G_λ defined over $\mathbb{F}_p$ of dimension r and height s. The group G_λ is specified by its Dieudonné module:

$$\mathbb{D}(G_\lambda) = \mathbb{Z}_p[F,V]/(F^{s-r}=V^r, FV=VF=p) .$$

All endomorphisms of G_λ are defined over $\mathbb{F}_{p^s}$, and

$$\mathrm{End}_{\mathbb{F}_{p^s}}(G_\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq D_\lambda ,$$

where D_λ is the central division algebra over $\mathbb{Q}_p$ with invariant $\lambda \pmod{\mathbb{Z}}$. The central assertion of the classical theory is that the category of p-divisible groups up to isogeny over $\overline{K}$ is semi-simple and that the groups G_λ represent the distinct simple objects [1]. If G is any group over K we therefore have

$$G \sim \prod_\lambda G_\lambda^{d_\lambda} \qquad \text{over } \overline{K} ,$$

where the d_λ are integers, almost all zero, determined by G. We can generalize the construction (2.1) by defining

$$V^\lambda(G) = \mathrm{Hom}_{\overline{K}}(G_\lambda, G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{2.3}$$

Then $V^\lambda(G)$ is a right module over D_λ of dimension d_λ, or a left module for the dual algebra D_λ°. It admits a continuous left action of $\mathcal{G}$; when K contains the field $\mathbb{F}_{p^s}$ this action is D_λ°-linear:

$$\rho^\lambda : \mathcal{G} \longrightarrow \mathrm{Aut}_{D_\lambda^\circ}(V^\lambda(G)) \simeq GL(d_\lambda, D_\lambda) .$$

If K contains the algebraic closure of the prime field we can thus define the representation $\rho = \bigoplus_\lambda \rho^\lambda$ on the generalized Tate module $V(G) = \bigoplus_\lambda V^\lambda(G)$.

Now suppose $\mathcal{O}$ is a complete discrete valuation ring, as in the introduction, with quotient field K and residue field k (algebraically closed of characteristic $p > 0$). Let G be a p-divisible group <u>defined over $\mathcal{O}$</u>. The special fibre G_k and the general fibre G_K are groups over a field; therefore

(2.5)
$$G_k \sim \Pi\, G_\lambda^{c_\lambda}$$
$$G_K \sim \Pi\, G_\lambda^{d_\lambda} \qquad \text{over } \overline{K} ,$$

where we accept the convention that $G_{0/1} = \mathbb{Q}_p/\mathbb{Z}_p$ and $d_{0/1} = h$ if $\mathrm{char}(K) = 0$. Consider the Galois representation arising from the general fibre:

(2.6)
$$\rho : \mathcal{G} \longrightarrow \prod_\lambda GL(d_\lambda, D_\lambda) .$$

How can we distinguish this from an arbitrary p-adic Galois representation?

First, we can compose ρ with the homomorphism

$$\det = \prod_\lambda \mathrm{Nm}_\lambda : \prod_\lambda GL(d_\lambda, D_\lambda) \longrightarrow \mathbb{Q}_p^*$$

where Nm_λ is the reduced norm in the algebra $\mathrm{Mat}(d_\lambda, D_\lambda)$ over $\mathbb{Q}_p$. We obtain a p-adic character $\varepsilon = \det(\rho)$ of $\mathcal{G}$.

<u>Theorem 2.7</u>

<u>If</u> $\mathrm{char}(K) = p$ <u>then</u> $\varepsilon = 1$ <u>in</u> $\mathrm{Hom}(\mathcal{G}, \mathbb{Q}_p^*)$.

<u>Proof</u>. The group G gives rise to an F-crystal $E(G)$ over the perfect closure of $\mathcal{O}$ [3]. The special fibre of this crystal is isogenous to the direct sum $\bigoplus E_\lambda^{c_\lambda}$, where $E_{r/s} = \mathbb{Z}_p[F]/(F^s = p^r)$. Over $\overline{K}$ the general fibre is isogenous

to $\oplus E_\lambda^{d_\lambda}$; over $K^{\mathrm{perf.}}$ it is isogenous to this crystal, twisted by the representation ρ .

The category of F-crystals has an exterior power operation which commutes with fibre products. If G has height h we find

$$(\textstyle\bigwedge^h E(G))_k \sim E_{\dim(G)/1}$$

$$(\textstyle\bigwedge^h E(G))_K \sim E_{\dim(G)/1} \quad \text{over } \overline{K} .$$

Over $K^{\mathrm{perf.}}$ the general fibre of $\bigwedge^h E(G)$ is isogenous to $E_{\dim(G)/1}$ twisted by the character $\varepsilon = \det(\rho)$. But the F-crystal $E_{\dim(G)/1}$ has only the <u>trivial</u> lifting from k to $\mathcal{O}$ [3]. As $\bigwedge^h E(G)$ is such a lifting, its general fibre is isomorphic to its special fibre and $\varepsilon = 1$.

<u>Notes</u>: 1) Suppose G has height h over $\mathcal{O}$ and its general fibre decomposes as in (2.5); then $\sum d_\lambda s_\lambda = h$ where $s_\lambda = \mathrm{denom}(\lambda)$. If C is the completion of the maximal unramified extension of $\mathbb{Q}_p$ (which splits all the algebras D_λ) , we have an embedding

$$\prod_\lambda GL(d_\lambda, D_\lambda) \hookrightarrow GL(h,C) .$$

Now let U be the open set $\mathrm{Spec}\,\mathcal{O} - \mathrm{Spec}\,k$, so $\mathcal{G} = \pi_1(U)$. Then (2.6) gives us a "monodromy representation"

$$\rho : \pi_1(U) \longrightarrow GL(h,C) . \tag{2.8}$$

In the geometric case when K has characteristic p , Theorem 2.7 asserts that the monodromy representation factors through $SL(h,C)$.

2) Theorem 2.7 may be formulated for K of arbitrary characteristic. Let χ be the cyclotomic character giving the action of $\mathcal{G}$ on p-power roots of unity in $\overline{K}$. Then

$$\varepsilon = \chi^{\dim(G)} \quad \text{in } \mathrm{Hom}(\mathcal{G}, \mathbb{Q}_p^*) . \tag{2.9}$$

For K of characteristic zero this is due to Raynaud [6]; for K of characteristic p it is a restatement of (2.7) .

§3. Formal A-modules of dimension 1 .

Let G be a connected p-divisible group of dimension 1 over $\mathcal{O}$. Then G can be identified with a formal group on one parameter, and we can make the representation ρ of (2.6) more explicit by using Lazard's one-dimensional theory. When G has additional endomorphisms it is convenient to analyse this situation using the language of formal A-modules [2] [4].

Let A be the ring of integers in a finite extension F of $\mathbb{Q}_p$, let π be a prime of A and $q = \mathrm{Card}(A/\pi A)$. Suppose R is a ring over A and $\gamma : A \longrightarrow R$ is the natural morphism. Then a formal A-module of dimension n over R is a pair $G = (\hat{G},i)$, where $\hat{G}$ is a formal group of dimension n over R and $i : A \longrightarrow \mathrm{End}_R(\hat{G})$ is an injective ring homomorphism such that $i(a)$ induces multiplication by $\gamma(a)$ on $\mathrm{Lie}\ (\hat{G})$. We write $[a]_G$ for the element $i(a)$ in $\mathrm{End}_R(\hat{G})$. If G and H are two formal A-modules over R , we define

$$\mathrm{Hom}_R(G,H) = \{\phi \in \mathrm{Hom}_R(\hat{G},\hat{H}) : \phi \circ [a]_G = [a]_H \circ \phi \quad \text{all } a \in A\} .$$

We shall henceforth only consider formal A-modules and formal groups of dimension one.

It is quite easy to describe the category of formal A-modules over a field K of characteristic p ; if $A = \mathbb{Z}_p$ this is equivalent to the category of formal groups. Choosing a model for $\hat{G}$ over K we have

$$[\pi]_G\ (x) = f(x^{q^h})$$

where $f(x)$ is a power series over K with $f'(0) \neq 0$, and h is a strictly positive integer, the <u>height</u> of G . (The height of $\hat{G}$, as a formal group, is then $h\cdot[A:\mathbb{Z}_p]$, and we shall assume the height is finite.) If K is separably closed there is one isomorphism class of formal A-modules for each finite height.

As a representative, we can take the formal A-module $G_{1/h}$, which is defined over $A/\pi A$ and characterized by

$$[\pi]_{G_{1/h}}(x) = x^{q^h} . \tag{3.1}$$

This formal A-module achieves all of its endomorphisms over the field $\mathbb{F}_{q^h}$; there we have

$$\mathrm{End}_{\mathbb{F}_{q^h}}(G_{1/h}) = B_{1/h}$$

where $B_{1/h}$ is the maximal order in the central division algebra over $F = A\otimes\mathbb{Q}_p$ with invariant $1/h \pmod{\mathbb{Z}}$. When K is not separably closed G is classified over K by its height and a representation

$$\rho : \mathrm{Gal}(\overline{K}_s/K) \longrightarrow B^*_{1/h}$$

as in §2.

We can now apply this to formal A-modules G over $\mathcal{O}$ whose special fibre is isomorphic to $G_{1/h}$ over k. Let $G_{0/1}$ denote the constant étale A-module F/A. When $\mathrm{char}(K) = 0$ we have $G_K \underset{\overline{K}}{\simeq} (G_{0/1})^h$. When $\mathrm{char}(K) = p$ the general fibre of G must also have dimension 1, therefore

$$G_K \underset{\overline{K}}{\sim} G_{1/g} \times (G_{0/1})^d$$

where $1 \leq g \leq h$ and $g + d = h$. Define the Tate modules

$$\begin{aligned} T^{1/g}(G) &= \mathrm{Hom}_{\overline{K}}(G_{1/g}, G_K) \quad \text{of rank 1 over } B_{1/g} \\ T^{0/1}(G) &= \mathrm{Hom}_{\overline{K}}(G_{0/1}, G_K) \quad \text{of rank } d \text{ over } A . \end{aligned} \tag{3.2}$$

These afford Galois representations:

$$\begin{aligned} \rho^{1/g} &: \mathcal{G} \longrightarrow B^*_{1/g} = B^* \\ \rho^{0/1} &: \mathcal{G} \longrightarrow GL(d,A) \end{aligned} \tag{3.3}$$

as in (2.4). We shall restrict our study to the equicharacteristic case ($g \geq 1$), as the ramification of $\rho^{0/1}$ when $\mathrm{char}(K) = 0$ is well-known [7].

Choosing a model for $\hat{G}$ over $\mathcal{O}$ we have

$$[\pi]_G\,(x) = f(x^{q^g}) \tag{3.4}$$

where $f(x) = a_1 x + a_2 x^2 + \ldots$ has coefficients in $\mathcal{O}$ and $a_1 \neq 0$. If we insist on a model lifting the standard model of $G_{1/h}$, then all the a_i lie in the maximal ideal except for a_{q^d}. The integer $e = v_K(a_1)$ is independent of the model chosen; it is zero if and only if $d = 0$. In that case the representation $\rho = \rho^{1/g} \oplus \rho^{0/1}$ is <u>trivial</u> [5]. The simplest nontrivial case is when $d = e = 1$; **here** we have complete results.

<u>Theorem 3.5</u>

Let G be a formal A-module of dimension 1 and height $h = g + d$ over $\mathcal{O}$. Assume $d = e = 1$ and for $n \geq 0$ define the rational numbers

$$a(n) = \frac{q^h - 1}{(q^g-1)(q^d-1)}\,(q^n - 1)$$

1) a) *The representation $\rho^{1/g} : \mathcal{G} \longrightarrow B^*$ is surjective, so B^* inherits an upper ramification filtration.*

   b) *The upper breaks in this filtration are precisely at the points $a(n)$, $n \geq 0$ (or $n \geq 1$ if $q^g = 2$).*

   c) *For $n \geq 1$ $(B^*)^{a(n)} = 1 + \pi_B^n B$, where π_B is a prime of B.*

2) a) *The representation $\rho^{0/1} : \mathcal{G} \longrightarrow A^*$ is surjective, so A^* inherits an upper ramification filtration.*

   b) *The upper breaks in this filtration are precisely at the points $a(gn)$, $n \geq 0$ (or $n \geq 1$ if $q = 2$).*

   c) *For $n \geq 1$ $(A^*)^{a(gn)} = 1 + \pi_A^n A$.*

We will prove this result in the following section. First we shall make a few remarks on its contents and provide a concrete example.

Example: Let E be the elliptic curve over $\mathcal{O} = \overline{\mathbb{F}}_2[[t]]$ with plane equation

$$y^2 + txy + y = x^3$$

and origin at the inflection point $(x,y) = (0,0)$. Then E_K is ordinary, but E_k is supersingular. The formal group $\hat{E}$ associated to this model, using x as a local parameter at the origin, gives a formal A-module G with $A = \mathbb{Z}_2$ and

$$[-2]_G(x) = tx^2 + (1+t^3)x^4 + \ldots + (t^{2n-4}+t^{2n-1})x^{2n} + \ldots.$$

Thus $h = 2$ and $g = d = e = 1$. Applying (3.5) we see the upper breaks in $\rho^{0/1}(\mathcal{G}) = A^*$ occur at the points $a(gn) = 3(2^n-1)$, and $(A^*)^{3(2^n-1)} = 1 + 2^n A$ for $n \geq 1$. These are the breaks in the separable quotient of the 2-division field of E_K.

Notes: 1) The breaks in the upper filtration of $\rho^{1/g}(\mathcal{G})$ are integral if and only if $g = 1$, i.e. if and only if $B^*_{1/g}$ is abelian.

2) Since $(\pi_B)^g = (\pi_A)$ in $B_{1/g}$, we find

$$(B^*)^{a(gn)} = 1 + \pi_A^n B \quad \text{for } n \geq 1$$

and the function $a(gn)$ relates the ramification filtration to the π_A-filtration in both $\rho^{0/1}$ and $\rho^{1/g}$. Let H^* denote the elements in $B^* \times A^*$ whose reduced norm down to A^* is 1, and H_n the elements of H^* congruent to 1 $(\text{mod } \pi_A^n)$. I suspect that $\rho = \rho^{1/g} \oplus \rho^{0/1}$ maps $\mathcal{G}$ surjectively onto H^* and that for $n \geq 1$,

$$(H^*)^{a(gn)} = H_n .$$

Theorem 2.7, combined with (3.5), shows that this holds at least when $A = \mathbb{Z}_p$.

3) When $d = 1$ but $e > 1$ we can prove a slightly weaker result. Let e_s be the separable degree of K over $L = k((a_1))$. Then there are positive constants c and N such that, for all $n > N$,

$$(3.6) \qquad \begin{aligned} \rho^{1/g}(\mathcal{G})^{e_s a(n)+c} &\subseteq 1 + \pi_B^n B \subseteq \rho^{1/g}(\mathcal{G})^{e_s a(n)-c} \\ \rho^{0/1}(\mathcal{G})^{e_s a(gn)+c} &\subseteq 1 + \pi_A^n A \subseteq \rho^{0/1}(\mathcal{G})^{e_s a(gn)-c} \end{aligned} .$$

Indeed, by Drinfeld's moduli theory [2], we can find a model for G over $k[[a_1]]$ where we can apply (3.5). Then (3.6) follows from a comparison of the upper numbering on $\mathrm{Gal}(\overline{L}_s/L)$ with that on its subgroup $\mathcal{G} = \mathrm{Gal}(\overline{K}_s/K)$ of index e_s.

Thus the breaks in the π_A-filtration of $\rho(\mathcal{G})$ occur near the upper breaks $e_s \cdot a(gn)$. The breaks in the p-saturated filtration therefore occur near the upper breaks $e_s \cdot a(g \cdot e_F \cdot n)$, where $F = A \otimes \mathbb{Q}_p$ and $e_F = v_F(p)$. This result bears an eerie formal relation to a theorem of Sen in characteristic zero. By definition

$$\begin{aligned} e_s a(g e_F n) &= e_s \frac{(q^h-1)}{(q^g-1)(q^d-1)} (q^{g e_F n}-1) \\ &= e_s \frac{q^h-1}{q^d-1} (1 + q^g + q^{2g} + \dots + q^{(e_F n-1)g}) . \end{aligned}$$

When $\mathcal{O}$ has mixed characteristic, $g = 0$, $d = h$, and $e_s = v_K(\pi_A)$. Thus, arguing <u>purely formally</u>, we might expect that in this case the breaks in the p-saturated filtration of $\rho(\mathcal{G})$ would be near the upper breaks $e_s e_F n = e_K n$. But this is precisely Sen's result [7] : is there a general theory which can obtain both results simultaneously?

4) When $d > 1$ the situation becomes more complicated. It seems that the upper breaks in $\rho(\mathcal{G})$ are determined by the valuations of the d moduli that classify the lifting of G over $G_{1/h}$ [2], [5]. When $d = 1$, a_1 is the unique modulus of the lifting; it might be interesting to study maximal 1-dimensional families in general.

§4. The proof of Theorem 3.5

To prove part 1) we start with the representation

$$\rho^{1/g} : \mathcal{G} \longrightarrow B^* .$$

Recall that the prime π_B gives a filtration on the image:

$$B^* \supsetneq 1 + \pi_B B \supset 1 + \pi_B^2 B \supset \ldots$$

with successive quotients:

$$B^* / 1 + \pi_B B \simeq \mathbb{F}_{q^g}^*$$

$$1 + \pi_B^n B / 1 + \pi_B^{n+1} B \simeq \mathbb{F}_{q^g}^+ \quad \text{for } n \geq 1 .$$

For $n \geq 0$ let H_n be the kernel of the composed homomorphism:

$$\rho_n : \mathcal{G} \longrightarrow B^* \longrightarrow (B^*/1+\pi_B^{n+1}B) \simeq (B/\pi_B^{n+1}B)^* ,$$

and let K_n be the fixed field of H_n in $\overline{K}_s$. Then $(\mathcal{G}/H_n) \simeq \mathrm{Gal}(K_n/K)$ and we have a tower of fields:

$$\begin{array}{l} \overline{K}_s \\ \quad \ddots \\ \qquad K_1 \\ \qquad | \\ \qquad K_0 \\ \qquad | \\ \qquad K = K_{-1} \end{array}$$

If we choose an isomorphism of formal A-modules over $\overline{K}_s$:

$$\phi : G \longrightarrow G_{1/g}$$

we have, for $\sigma \in \mathcal{G}$,

$$\rho^{1/g}(\sigma) = \phi \circ \phi^{-\sigma} \in \mathrm{Aut}(G_{1/g}) \simeq B^* .$$

Choosing models for G and $G_{1/g}$ over $\mathcal{O}$, we may write ϕ as a power series:

$$\phi(x) = k_1 x + k_2 x^2 + \dots$$

with coefficients in $\overline{K}_s$. Similarly, we have the power series over $\mathcal{O}$:

$$[\pi]_G(x) = a_1 x^{q^g} + a_2 x^{2q^g} + \dots$$

$$[\pi]_{G_{1/g}}(x) = x^{q^g} .$$

Since ϕ is an isomorphism of formal A-modules, these series satisfy:

$$\phi \circ [\pi]_G(x) = [\pi]_{G_{1/g}} \circ \phi(x) = \phi^{q^g}(x^{q^g}) . \tag{4.1}$$

Lemma 4.2

1) *The coefficients* k_j *in* $\phi(x)$ *are integral in* $\overline{K}_s$.

2) *One has* $k_j \in K_{n-1}$ *for all* $j < q^n$, *and* $K_n = K_{n-1}(k_{q^n})$.

Proof. The integrality of the k_j follows from the identity (4.1), which may be used to define them successively. Since $\sigma \in H_0$ if and only if $k_1^\sigma = k_1$, we have $K_0 = K(k_1)$. But for $\sigma \in H_0$:

$$\phi \circ \phi^{-\sigma}(x) = x + kx^{q^m} + \dots ;$$

furthermore, $\sigma \in H_n$ if and only if $m > n$. This gives part 2) .

Lemma 4.3

Assume that $d = e = 1$. *Then for* $n \geq 0$,

1) ρ_n *induces an isomorphism* $\mathrm{Gal}(K_n/K) \simeq (B/\pi_B^{n+1}B)^*$.

2) k_{q^n} *is a uniformizing parameter of* K_n .

3) $\mathrm{Gal}(K_n/K_{n-1})$ *has a unique upper and lower break at the point* $m = q^{hn} - 1$.

4) *The lower filtration of* $G = \mathrm{Gal}(K_n/K)$ *is given by:*

$$\begin{aligned}
G_0 &= G \\
G_x &= \mathrm{Gal}(K_n/K_0) && \text{for} \quad 0 < x \le q^h - 1 \\
G_x &= \mathrm{Gal}(K_n/K_1) && \text{for} \quad q^h - 1 < x \le q^{2h} - 1 \\
&\vdots \\
G_x &= \mathrm{Gal}(K_n/K_{n-1}) && \text{for} \quad q^{(n-1)h} - 1 < x \le q^{nh} - 1 \\
G_x &= (1) && \text{for} \quad q^{nh} - 1 < x .
\end{aligned}$$

5) The upper filtration of $G = \mathrm{Gal}(K_n/K)$ is given by:

$$\begin{aligned}
G^0 &= G \\
G^x &= \mathrm{Gal}(K_n/K_0) && \text{for} \quad 0 < x \le a(1) \\
G^x &= \mathrm{Gal}(K_n/K_1) && \text{for} \quad a(1) < x \le a(2) \\
&\vdots \\
G^x &= \mathrm{Gal}(K_n/K_{n-1}) && \text{for} \quad a(n-1) < x \le a(n) \\
G^x &= (1) && \text{for} \quad a(n) < x ,
\end{aligned}$$

where $a(1), a(2), \dots, a(n)$ are defined in Theorem 3.5.

Proof. We use an induction on n. For $n = 0$ look at the coefficient of x^{q^g} in the identity (4.1). This gives the equation:

$$k_1 a_1 = k_1^{q^g} .$$

Since $e = v_K(a_1) = 1$, this shows that $K_0 = K(k_1)$ has degree $q^g - 1$ over K and that k_1 is a uniformizing parameter. By counting we see that the injection

$$\rho_0 : \ \mathrm{Gal}(K_0/K_1) \longrightarrow (B/\pi_B B)^*$$

is an isomorphism. The only upper and lower break is at 0, as K_0 is a tamely ramified extension of K.

Now assume that the lemma holds for K_{n-1}/K. Look at the coefficient of $x^{q^{g+n}}$ in the identity (4.1). This gives the equation:

$$k_1 a_{q^n} + \ldots + k_{q^{n-1}} a_q^{q^{n-1}} + \ldots + k_{q^n} a_1^{q^n} = (k_{q^n})^{q^g} .$$

But I claim this is an Eisenstein equation:

$$b + a_1^{q^n} y = y^{q^g} \tag{4.4}$$

for $y = k_{q^n}$ over K_{n-1}. It is clear that b is integral, by (4.2). Since G lifts $G_{1/h}$ and $d = 1$, we know $v_K(a_i) \geq 1$ for $i \neq q$. Consequently, $v_{K_{n-1}}(a_i) > 1$ for $i \neq q$ and

$$v_{K_{n-1}}(b) = v_{K_{n-1}}(k_{q^{n-1}} a_q^{q^{n-1}}) = 1$$

by our inductive hypothesis that $k_{q^{n-1}}$ is a uniformizing parameter in K_{n-1}. Therefore $K_n = K_{n-1}(k_{q^n})$ has degree q^g over K_{n-1} and uniformizing parameter k_{q^n}. By induction, we know that $[K_{n-1}:K] = (q^g-1)q^{g(n-1)}$; hence the injection

$$\rho_n : \operatorname{Gal}(K_n/K) \longrightarrow (B/\pi_B^{n+1}B)^*$$

is surjective by counting. By applying corollary (1.6) to the equation (4.4) we see that $\operatorname{Gal}(K_n/K_{n-1})$ has a unique upper and lower break at the point:

$$m = v_{K_{n-1}}(a_1^{q^n})\, q^g/q^g-1 - 1 = q^{hn} - 1 .$$

The calculation of the filtrations on $\operatorname{Gal}(K_n/K)$ is now accomplished using the identity $\phi_{K_n/K} = \phi_{K_{n-1}/K} \circ \phi_{K_n/K_{n-1}}$, the inductive hypothesis, and the fact that

$$\phi_{K_n/K_{n-1}}(x) = x \qquad \text{for } x \leq q^{nh} - 1 .$$

This lemma yields part 1) of Theorem 3.5 as an immediate corollary. Given an adequate theory of π-divisible A-modules, we can see how part 2) of this Theorem would follow formally from part 1). We can define the character:

$$\varepsilon_A = \det_A(\rho) : \mathcal{G} \longrightarrow A^*$$

where $\det_A : B^*_{1/g} \times GL(d,A) \longrightarrow A^*$ is the reduced norm in the category of F-algebras. In analogy with (2.7) one would expect:

$$\varepsilon_A \overset{?}{=} 1 \quad \text{in } \mathrm{Hom}(\mathcal{G}, A^*) . \tag{4.5}$$

When $d = 1$ this would imply:

$$\rho_{0/1} \overset{?}{=} (\mathrm{Nm}_{1/g} \circ \rho_{1/g})^{-1} , \tag{4.6}$$

from which we could easily derive its ramification filtration. Since the full theory of "A-crystals" is not available to prove (4.5), we shall prove part 2) independently, and check that the results are consistent with (4.6).

First we must identify the representation

$$\rho_{0/1} : \mathcal{G} \longrightarrow GL(d,A) = M^*$$

where $M = \mathrm{Mat}(d,A)$. We appropriate our previous notation: for $n \geq 0$ let H_n be the kernel of the composed homomorphism:

$$\rho_n : \mathcal{G} \longrightarrow M^* \longrightarrow M^*/\, 1 + \pi^{n+1}M \simeq (M/\pi^{n+1}M)^*$$

and let K_n be the fixed field of H_n in $\overline{K}_s$.

If $\overline{m} = \{x \in \overline{K} : v_K(x) > 0\}$, then the set of points of G in $\overline{m}$ give a genuine A-module $G(\overline{m})$. Let $G(\overline{m})_{\pi^{n+1}}$ be the finite submodule of π^{n+1}-torsion. This module is free of rank d over $A/\pi^{n+1}A$ and is stable under the action of $\mathcal{G}$. The resulting representation:

$$\mathcal{G} \longrightarrow \mathrm{Aut}_{A/\pi^{n+1}A}(G(\overline{m})_{\pi^{n+1}}) \simeq (M/\pi^{n+1}M)^*$$

may be identified with ρ_n . Consequently, K_n is just the separable subfield of the field of π^{n+1}-division points.

<u>Lemma 4.14</u>

<u>Assume that</u> $d = e = 1$. <u>Then for</u> $n \geq 0$,

1) ρ_n induces an isomorphism $\mathrm{Gal}(K_n/K) \simeq (A/\pi^{n+1}A)^*$.

2) If $\alpha_n \in G(\overline{m})_{\pi^{n+1}}$ and $[\pi^n]_G(\alpha) \neq 0$, then $\beta_n = \alpha_n^{q^{g(n+1)}}$ is a uniformizing parameter in K_n .

3) $\mathrm{Gal}(K_n/K_{n-1})$ has a unique upper and lower break at the point $m = q^{hn} - 1$.

4) The lower filtration of $G = \mathrm{Gal}(K_n/K)$ is given by:

$$\begin{array}{lll} G_0 = G & & \\ G_x = \mathrm{Gal}(K_n/K_0) & \text{for} & 0 < x \leq q^h - 1 \\ G_x = \mathrm{Gal}(K_n/K_1) & \text{for} & q^h - 1 < x \leq q^{2h} - 1 \\ \vdots & & \\ G_x = \mathrm{Gal}(K_n/K_{n-1}) & \text{for} & q^{(n-1)h} - 1 < x \leq q^{nh} - 1 \\ G_x = (1) & \text{for} & q^{nh} - 1 < x . \end{array}$$

5: The upper filtration of $G = \mathrm{Gal}(K_n/K)$ is given by:

$$\begin{array}{lll} G^0 = G & & \\ G^x = \mathrm{Gal}(K_n/K_0) & \text{for} & 0 < x \leq a(g) \\ G^x = \mathrm{Gal}(K_n/K_1) & \text{for} & a(g) < x \leq a(2g) \\ \vdots & & \\ G^x = \mathrm{Gal}(K_n/K_{n-1}) & \text{for} & a(g(n-1)) < x \leq a(gn) \\ G^x = (1) & \text{for} & a(gn) < x , \end{array}$$

where $a(g), a(2g), \ldots, a(ng)$ are defined in Theorem 3.5.

Proof. We use an induction on n . For $n = 0$ the extension K_0 is generated by the non-zero roots of the polynomial $f(x)$, where

$$[\pi]_G(x) = f(x^{q^g}) .$$

Since $d = e = 1$ each non-zero root β_0 has K-valuation $1/(q-1)$. Consequently the injection:

$$\rho_0 : \mathrm{Gal}(K_0/K) \longrightarrow (A/\pi A)^*$$

is an isomorphism, and β_0 is a uniformizing element. The break sequence is obvious as K_0 is tamely ramified over K.

Now assume the result holds for the layer K_{n-1}/K. Let α_n be an element in $G(\bar{m})_{\pi^{n+1}}$ not killed by π^n, and put

$$\alpha_{n-1} = [\pi]_G(\alpha_n) = f(\alpha_n{}^{q^g}) .$$

Raising this identity to the q^{ng} power, we obtain:

$$\beta_{n-1} = \alpha_{n-1}^{q^{ng}} = f^{q^{ng}}(\alpha_n^{q^{(n+1)g}}) = f^{q^{ng}}(\beta_n) .$$

By our induction hypothesis, β_{n-1} is a uniformizing parameter in K_{n-1}. Applying the Weierstrass preparation theorem to the power series

$$f^{q^{ng}}(x) = a_1^{q^{ng}} x + a_2^{q^{ng}} x^2 + \ldots + a_q^{q^{ng}} x^q + \ldots$$

we see that β_n satisfies an Eisenstein polynomial over K_{n-1} :

$$g(x) = x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0$$

with

$$v_{K_{n-1}}(b_0) = 1 \quad v_{K_{n-1}}(b_i) \geq v_{K_{n-1}}(b_1) .$$

We may therefore apply corollary (1.6) to conclude that $K_{n-1}(\beta_n)$ has degree q over K_{n-1} and a unique upper break at the point

$$m = qv_{K_{n-1}}(b_1)/q\text{-}1 - 1 = q^{nh} - 1 ,$$

as

$$v_{K_{n-1}}(b_1) = v_{K_{n-1}}(a_1^{q^{ng}}) = q^{ng}(q-1)q^{n-1} .$$

Clearly β_n is a uniformizing parameter in $K_{n-1}(\beta_n)$; counting degrees shows that $K_n = K_{n-1}(\beta_n)$ and that the injection

$$\rho_n : \mathrm{Gal}(K_n/K) \longrightarrow (A/\pi A)^*$$

is an isomorphism. One can now calculate the entire break sequence using the induction hypothesis and the identity

$$\phi_{K_n/K} = \phi_{K_{n-1}/K} \circ \phi_{K_n/K_{n-1}} .$$

This lemma immediately yields part 2) of Theorem 3.5 as a corollary. It is easy to check that **parts** 1) and 2) are consistent with (4.6) using the identities:

$$\mathrm{Nm}_A(1+\pi_B^{gn}B) = 1 + \pi_A^n A$$

$$\mathrm{Nm}_A(1+\pi_B^{gn+1}B) = 1 + \pi_A^{n+1}A .$$

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