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ON FORMAL GROUPS. THE FUNCTIONAL EQUATION
 LEMMA AND SOME OF ITS APPLICATIONS.

par

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1. INTRODUCTION. Let R be a ring and let $F(X,Y)$ be an n -dimensional commutative formal group law over R . Assume that R is torsion free and let $f(X)$ over $R \otimes \mathbb{Q}$ be the logarithm of $F(X,Y)$. Roughly, the functional equation lemma to be discussed below says what kind of regularity $f(X) \in R \otimes \mathbb{Q}[[X]]^n$ must exhibit in order that it be the logarithm of a formal group law with coefficients in R . The precise statement of the lemma is in section 2 below. The lemma turns out to have many more applications (then just the construction of universal formal group laws). It is the purpose of the present paper to outline a few of these and to try to convince the reader of the power of the lemma in proving a large variety of integrality statements. (Because commutative formal group laws over $\mathbb{Q}$ -algebras are trivial, the theory of commutative formal group laws over torsion free rings is largely a matter of integrality statements). To cite a few instances: the integrality of the addition and multiplication polynomials of the Witt vectors, the Atkin-Swinnerton Dyer congruences, the construction of generalized Lubin-Tate formal group laws ("tapis de Cartier") can all be seen as applications of the functional equation lemma. Many more applications of the functional equation lemma can be found in [7] and [8]. This paper contains no new results or proofs which are not also in [7], with the exception of the proof of " $v(M,\eta)(X)$ reduces to $V(X)$ " in section 6 below, which in [7] is done in a needlessly cumbersome fashion.

2. THE FUNCTIONAL EQUATION LEMMA. The ingredients we need are the following

$$(2.1) \quad B \subset L, \mathfrak{a} \subset B, \sigma : L \rightarrow L, p, q, s_1, s_2, \dots$$

Here B is a subring of a ring L , $\mathfrak{a}$ is an ideal in B , σ a ring endomorphism of L , p is a prime number, q is a power of p and the s_i , $i = 1, 2, \dots$ are $m \times m$ matrices with their coefficients in L . These ingredients are supposed to satisfy the following conditions

$$(2.2) \quad p \in \mathfrak{a}, \sigma(b) \equiv b^q \pmod{\mathfrak{a}} \text{ for all } b \in B, \sigma^r(s_i(j,k)) \mathfrak{a} \subset B \text{ for all } i, j, k, r$$

Here $s_i(j,k)$ is the (j,k) -entry of the matrix s_i . For example if $\mathfrak{a} = B$ then the last condition means that $s_i(j,k) \in B$; and if e.g. $B = \mathbb{Z}$, $L = \mathbb{Q}$, $\sigma = \text{id}$, $q = p$ then the conditions are satisfied iff $s_i(j,k) \in p^{-1}\mathbb{Z}$ for all i, j, k .

If $g(X)$ is an m -tuple of power series in $X_1, \dots, X_n$ with coefficients in L then we denote with $\sigma_* g(X)$ the m -tuple of power series obtained by applying σ to the coefficients of $g(X)$.

2.3. Functional Equation Lemma. Let $f(X) \in L[[X]]^m$ be an m -tuple of power series in m indeterminates $X_1, \dots, X_m$ and $\bar{f}(\bar{X})$ an m -tuple of power series in n indeterminates $\bar{X}_1, \dots, \bar{X}_n$. Suppose that $f(X) \equiv b_1 X \pmod{(\text{degree } 2)}$ where b_1 is a matrix with coefficients in B which is invertible (over B). Suppose moreover that

$$(2.4) \quad f(X) - \sum_{i=1}^{\infty} s_i \sigma_*^i f(X^q)^i \in B[[X]]^m, \quad \bar{f}(\bar{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \bar{f}(\bar{X}^q)^i \in B[[\bar{X}]]^m$$

where X^q and $\bar{X}^q$ are short for $(X_1^q, \dots, X_m^q)$ and $(\bar{X}_1^q, \dots, \bar{X}_n^q)$. Then we have

$$(2.5) \quad F(X, Y) = f^{-1}(f(X) + f(Y)) \in B[[X; Y]]^m$$

$$(2.6) \quad f^{-1}(\bar{f}(\bar{X})) \in B[[\bar{X}]]^m.$$

Let $\hat{h}(\hat{X}) \in B[[\hat{X}]]^m$ be an m -tuple of power series with coefficients in B in yet another set of indeterminates and let $\hat{f}(\hat{X}) = f(\hat{h}(\hat{X}))$. Then

$$(2.7) \quad \hat{f}(\hat{X}) - \sum_{i=1}^{\infty} s_i \sigma_*^i \hat{f}(\hat{X}^q)^i \in B[[\hat{X}]]$$

Finally let $\alpha(\hat{X}) \in B[[\hat{X}]]^m$, $\beta(\hat{X}) \in L[[\hat{X}]]^m$, $r \in \mathbb{N} = \{1, 2, \dots\}$. Then

$$(2.8) \quad \alpha(\hat{X}) \equiv \beta(\hat{X}) \pmod{\mathfrak{A}^r} \iff f(\alpha(\hat{X})) \equiv f(\beta(\hat{X})) \pmod{\mathfrak{A}^r}$$

For a proof cf. [7], sections 2 and 10.

3. SOME ALMOST TRIVIAL APPLICATIONS. Let $H(X) = X + p^{-1}X^p + p^{-2}X^{p^2} + \dots$ and $\ell(X) = \log(1+X) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} X^n$. One notes that $H(X) - p^{-1}H(X^p) = X$ and $\ell(X) - p^{-1}\ell(X^p) \in \mathbb{Z}_{(p)}[X]$. So taking $B = \mathbb{Z}_{(p)}$, $\mathfrak{A} = pB$, $L = \mathbb{Q}$, $q = p$, $s_1 = p^{-1}$, $s_2 = s_3 = \dots = 0$ and $\sigma = \text{id}$, we obtain from (2.6) Hasse's old result that $\exp(H(X))$ has its coefficients in $\mathbb{Z}_{(p)}$.

More generally let $d(X) = d_0X + d_1X^p + \dots$, $d_i \in \mathbb{Q}$. Using the same ingredients and combining (2.6) and (2.7) above one finds that $\exp(d(X)) \in \mathbb{Z}_{(p)}[[X]]$ if and only if $d_i - p^{-1}d_{i-1} \in \mathbb{Z}_{(p)}$ for all i (where one takes $d_{-1} = 0$). This is a lemma of Dieudonné [3].

An easy application with σ non trivial is the following. Let B be the ring of integers of the completed maximal unramified extension T of $\mathbb{Q}_p$; let $L = T$, $p = q$, $s_1 = p^{-1}$, $s_2 = s_3 = \dots = 0$, and σ the Frobenius automorphism of T . Let $h(X) = 1 + a_1X + a_2X^2 + \dots \in T[[X]]$. In this setting the combination of (2.6) and (2.7) yields that $h(X) \in B[[X]]$ if and only if $\sigma_* h(X^p)/h(X)^p \in 1 + pXB[[X]]$, which is lemma 1 of Dwork [6].

For an easy more dimensional application consider the slightly modified Witt vector polynomials $\bar{w}_0(X) = X_0$, $\bar{w}_1(X) = X_1 + p^{-1}X_0^p$, $\dots$,

$\bar{w}_n(X) = X_n + p^{-1}X_{n-1}^p + \dots + p^{-n}X_0^{p^n}$. Take $B = \mathbb{Z}$, $\mathfrak{A} = p\mathbb{Z}$, $L = \mathbb{Q}$, $\sigma = \text{id}$, $q = p$, $s_2, s_3, \dots = 0$ and let s_1 be the $(n+1) \times (n+1)$ matrix with p^{-1} on the first subdiagonal and zero's elsewhere; i.e. $s_1(j, k) = 0$ unless $j = k + 1$ and $s_1(k+1, k) = p^{-1}$, $k = 1, 2, \dots, n$. Let $\bar{w}(X)$ be the column vector $(\bar{w}_0(X), \dots, \bar{w}_n(X))$. Then, obviously, $\bar{w}(X) = X + s_1 \bar{w}(X^p)$. It now follows from (2.5) that $\Sigma(X) = \bar{w}^{-1}(\bar{w}(X) + \bar{w}(Y))$ has integral coefficients; or, multiplying both sides of $\bar{w}(\Sigma(X)) = \bar{w}(X) + \bar{w}(Y)$ with p^n , we see that we have shown that the addition polynomials of the Witt vectors have integral coefficients.

4. ATKIN-SWINNERTON DYER CONGRUENCES. Let E be an elliptic curve over $\mathbb{Q}$ and let $L(s) = \prod_p (1 - a_p p^{-s} + b_p p^{1-2s})^{-1}$ be its global L-function, where the local factors $(1 - a_p p^{-s} + b_p p^{1-2s})^{-1}$ are defined as follows in terms of the

reductions mod p of a global minimal model D over $\mathbb{Z}$ for E :

- (i) if p is good, i.e. if $D \otimes \mathbb{Z}/(p)$ is nonsingular then $(1 - a_p^{-s} + b_p^{-1-2s})$ is the numerator of the zetafunction of the elliptic curve $D \otimes \mathbb{Z}/(p)$ over $\mathbb{Z}/(p)$;
- (ii) if $D \otimes \mathbb{Z}/(p)$ has an ordinary doublepoint then $1 - a_p^{-s} + b_p^{-1-2s} = 1 - \varepsilon_p^{-s}$ where $\varepsilon_p = \pm 1$ depending on whether the tangents in the double point are rational over $\mathbb{Z}/(p)$ or not;
- (iii) if $D \otimes \mathbb{Z}/(p)$ has a cusp $1 - a_p^{-s} + b_p^{-1-2s} = 1$.

Now let $f_E(X) = \sum_{n=1}^{\infty} n^{-1} a_n X^n$ where $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then an immediate and

obvious consequence of the Euler product structure of $L(s)$ is that for all p

$$(4.1) \quad f_E(X) - p^{-1} a_p f_E(X^p) + p^{-1} b_p f_E(X^{p^2}) \in \mathbb{Z}_{(p)}[X].$$

It now follows from (2.5) that $F_E(X, Y) = f_E^{-1}(f_E(X) + f_E(Y))$ is a formal group law over $\mathbb{Z}$. Let $G_E(X, Y)$ be the formal completion along the identity of the minimal model D over $\mathbb{Z}$. The formal group law $G_E(X, Y)$ can be explicitly described as follows. Let D be given by $y^2 + c_1 XY + c_3 Y = X^3 + c_2 X^2 + c_4 X + c_6$; let $\omega = (2Y + c_1 X + c_3)^{-1} dX$ be the invariant differential and $z = (2Y)^{-1} X$ a local parameter at zero. Let, locally, $\omega = \sum \beta(n) z^{n-1} dz$ and define $g_E(X) = \sum_{n=1}^{\infty} n^{-1} \beta(n) X^n$,

then $G_E(X, Y) = g_E^{-1}(g_E(X) + g_E(Y))$. This comes from the fact that if $f(X)$ is the logarithm of a formal group law $F(X, Y)$ over a torsion free ring R then $df(X)$ is an invariant differential for $F(X, Y)$.

4.2. Theorem (Honda, Hill; [11], [10] and [12]). The formal group laws $F_E(X, Y)$ and $G_E(X, Y)$ are strictly isomorphic over $\mathbb{Z}$ (i.e. there exists a power series $\phi(X) = X + b_2 X^2 + \dots$, $b_i \in \mathbb{Z}$ such that $\phi(F_E(X, Y)) = G_E(\phi(X), \phi(Y))$).

It follows that $g_E(X) = f_E(\phi^{-1}(X))$. So that by (2.7) we have that $g_E(X)$ also satisfies the integrality conditions (4.1). Writing this out in terms of coefficients one finds the Atkin Swinnerton-Dyer congruences.

$$(4.3) \quad \beta(np) - a_p \beta(n) + b_p \beta(n/p) \equiv 0 \pmod{p^s} \text{ if } n \equiv 0 \pmod{p^{s-1}}$$

where $\beta(n//p) = \beta(n/p)$ if $p|n$ and $\beta(n//p) = 0$ otherwise.

5. LUBIN-TATE FORMAL GROUP LAWS. The so-called Lubin-Tate formal group laws are constructed as follows in [13]. Let K be a local field with finite residue field (i.e. K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(x)$); let A be the ring of integers of K , let π be a uniformizing element and let q be the number of

elements of k , the residue field of K . Let $e(X) \in A[[X]]$ be any power series in one variable such that

$$(5.1) \quad e(X) \equiv \pi X \pmod{\text{degree } 2}, \quad e(X) \equiv X^q \pmod{\pi}$$

Then there is a unique power series $F_e(X, Y)$ such that $F_e(e(X), e(Y)) = e(F_e(X, Y))$ and $F_e(X, Y) \equiv X + Y \pmod{\text{degree } 2}$. This is a formal group law over A . Moreover for all $a \in A$ there is a unique power series $[a]_e(X)$ such that $e([a]_e(X)) = [a]_e(e(X))$ and $[a]_e(X) \equiv aX \pmod{\text{degree } 2}$; the map $a \mapsto [a]_e(X)$ defines a ring homomorphism $A \rightarrow \text{End}_A(F(X, Y))$ and $[\pi]_e(X) = e(X)$. Finally if both $e(X)$ and $e'(X)$ satisfy (5.1) (with respect to the same π) then $F_e(X, Y)$ and $F_{e'}(X, Y)$ are strictly isomorphic over A .

In the ingredients (2.1) for the functional equation lemma now take $B = A$, $L = K$, $\sigma = \pi A$, $p = \text{char}(k)$, $q = \frac{1}{p}k$, $\sigma = \text{id}$, $s_1 = \pi^{-1}$, $0 = s_2 = s_3 = \dots$. Then the conditions (2.2) are satisfied. Let $g(X) \in A[[X]]$ be any power series such that $g(X) \equiv X \pmod{\text{degree } 2}$, and consider $f(X) \in K[[X]]$ defined (recursively) by the functional equation

$$(5.2) \quad f(X) = g(X) + \pi^{-1} f(X^q)$$

Then parts (2.5) and (2.6) of the functional equation lemma say that the power series

$$(5.3) \quad F(X, Y) = f^{-1}(f(X) + f(Y)), \quad [a](X) = f^{-1}(af(X)), \quad a \in A$$

have their coefficients in A and hence define a formal A -module over A . (A formal A -module, where A is as above, over an A -algebra R is a formal group law $F(X, Y)$ over R together with a ring endomorphism $\rho_F: A \rightarrow \text{End}_R(F(X, Y))$ such that $\rho_F(a) \equiv aX \pmod{\text{degree } 2}$ for all $a \in A$). Now consider $[\pi](X)$. We have

$$(5.4) \quad f([\pi](X)) = \pi f(X) = \pi g(X) + f(X^q) \equiv f(X^q) \pmod{\pi}$$

It follows by part (2.8) of the functional equation lemma that $[\pi](X) \equiv X^q \pmod{\pi}$. Also of course (cf. (5.3)) $F([\pi](X), [\pi](Y)) = [\pi](F(X, Y))$ so that $F(X, Y)$ is a Lubin-Tate formal group law with $e(X) = [\pi](X)$. As all Lubin-Tate formal group laws constructed via the same uniformizing element π are strictly isomorphic, it follows from part (2.7) of the functional equation lemma that all Lubin-Tate formal group laws are obtained by the construction (5.2), (5.3) by varying

$g(X)$.

Finally we use the functional equation lemma to show that Lubin-Tate formal group laws constructed via different uniformizing elements π and $\bar{\pi}$ become isomorphic over $\hat{A}_{nr}$, the completion of the ring of integers of the completion $\hat{K}_{nr}$ of the maximal unramified extension K_{nr} of K . Let therefore $f(X), \bar{f}(X) \in A[[X]]$ satisfy

$$(5.5) \quad f(X) - \pi^{-1}f(X^q) \in A[[X]], \quad \bar{f}(X) - \bar{\pi}^{-1}\bar{f}(X^q) \in A[[X]]$$

Now take as functional equation ingredients $B = \hat{A}_{nr}$, $\sigma = \pi B$, $L = \hat{K}_{nr}$, σ the Frobenius substitution in $\text{Gal}(\hat{K}_{nr}/K)$ extended by continuity to $\hat{K}_{nr}$, $p, q, s_1, s_2, \dots$ as before. Let $u \in \hat{A}_{nr}^*$, the units of $\hat{A}_{nr}$, be such that $u^{-1}\sigma(u) = \pi^{-1}\bar{\pi}$. (Such a u exists). Then we have

$$(5.6) \quad \begin{aligned} uf(X) - \pi^{-1}\sigma_*(uf(X^q)) &= uf(X) - \bar{\pi}^{-1}\sigma(u)f(X^q) = \\ &= u(f(X) - \pi^{-1}f(X^q)) \in \hat{A}_{nr}[[X]] \end{aligned}$$

and also of course $\bar{f}(X) - \bar{\pi}^{-1}\sigma_*(\bar{f}(X^q)) = \bar{f}(X) - \pi^{-1}\bar{f}(X^q) \in A[[X]] \subset \hat{A}_{nr}[[X]]$, so that by part (2.6) of the functional equation lemma we have that

$$(5.7) \quad \phi(X) = \bar{f}^{-1}(uf(X)) \in \hat{A}_{nr}[[X]]$$

which defines as an isomorphism $\phi(X)$ between the formal A -modules defined by $f(X)$ and $\bar{f}(X)$ as in (5.3).

6. TAPIS DE CARTIER. Let A be the ring of integers of an unramified extension K of $\mathbb{Q}_p$. Let $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ be the Frobenius automorphism. Now suppose we have given a free A -module M of finite rank $h < \infty$ together with a semilinear endomorphism $\eta : M \rightarrow M$ (i.e. $\eta(m+am') = \eta(m) + \eta(m')$, $\eta(am) = \sigma(a)\eta(m)$). To these data we associate a formal group law over A as follows. Let $D(\eta)$ be the matrix of η with respect to some basis for M . Define $g(M, \eta)(X) \in K[[X_1, \dots, X_h]]^h$ by the equation

$$(6.1) \quad g(M, \eta)(X) = X + p^{-1}D(\eta)\sigma_*g(M, \eta)(X^p)$$

By part (2.5) of the functional equation lemma (with $B = A$, $L = K$, $\sigma = pA$, σ as above, $q = p$, $s_1 = p^{-1}D(\eta)$, $s_2 = s_3 = \dots = 0$) it follows that $G(M, \eta)(X, Y) = g(M, \eta)^{-1}(g(M, \eta)(X) + g(M, \eta)(Y))$ is a formal group law over A . This

construction is functorial in the following sense. Let $\alpha : (M, \eta) \rightarrow (M', \eta')$ be a morphism. This means that $\alpha : M \rightarrow M'$ is A -linear and that $\eta'\alpha = \alpha\eta$. Let $E(\alpha)$ be the matrix of α with respect to the chosen bases of M and M' . Then we have $E(\alpha)g(M, \eta)(X) - p^{-1}D(\eta')\sigma_*(E(\alpha)g(M, \eta)(X^p)) = E(\alpha)X \in A[[X]]^h$, because $\eta'\alpha = \alpha\eta$, together with the semilinearity of η and η' , precisely means that $D(\eta')\sigma_*(E(\alpha)) = E(\alpha)D(\eta)$. It follows in particular that $G(M, \eta)(X, Y)$ does not depend (up to isomorphism) on the choice of a basis for M .

For each (M, η) as above let (M^σ, η) be the pair obtained by leaving the additive group M and the map η unchanged but by changing the A -action to $a \cdot m = \sigma^{-1}(a)m$. One easily checks that $G(M^\sigma, \eta) = \sigma_*G(M, \eta)$. There is an obvious morphism $(M^\sigma, \eta) \rightarrow (M, \eta)$, viz. η itself. Let $v(M, \eta) : \sigma_*G(M, \eta) \rightarrow G(M, \eta)$ be the corresponding morphism of formal groups. We claim that $v(M, \eta)$ reduces mod p to the Verschiebung morphism $V(X) : \sigma_*\bar{G}(X, Y) \rightarrow \bar{G}(X, Y)$ over k where the bar denotes reduction mod p and where we omitted to write (M, η) . (If $F(X, Y)$ is a formal group law over k , then $V(X) : \sigma_*F(X, Y) \rightarrow F(X, Y)$ is the power series over k defined by $V(X^p) = [p](X)$ (because $\text{char}(k) = p$, $[p](X)$ is necessarily a power series in X^p)). This is seen as follows. We have

$$g(M, \eta)v(X^q) = D(\eta)g(M^\sigma, \eta)(X^q) = D(\eta)\sigma_*g(M, \eta)(X^q) \equiv pg(M, \eta)(X) \pmod{pA}$$

It follows by part (2.8) of the functional equation lemma that $v(X^q) \equiv g(M, \eta)^{-1}(pg(M, \eta)(X)) = [p](X) \pmod{pB}$, proving our claim.

Thus we have a functor $(M, \eta) \mapsto (G(M, \eta), v(M, \eta))$. There is an obvious functor in the inverse direction, viz. taking Lie-algebras. And we clearly have $\text{Lie}(G(M, \eta)) = M$, $\text{Lie}(v(M, \eta)) = \eta$. The Tapis de Cartier ([1], [2], [7]) now says that these functors are inverse equivalence of categories. To prove this we have to show that every formal group law $F(X, Y)$ together with a morphism $v : \sigma_*F(X, Y) \rightarrow F(X, Y)$ over A which reduces to $V(X) \pmod{pA}$ comes from a pair (M, η) .

To prove this we first remark that, because A is unramified, every $F(X, Y)$ over A is of functional equation type (Honda [12], cf. [7], section 20.3) i.e. if $f(X)$ is the logarithm of $F(X, Y)$ then there are $s_1, s_2, \dots$ such that $f(X) - \sum s_i \sigma_i^* f(X^{p^i}) \in A[[X]]^h$, where $h = \dim(F(X, Y))$. Now a homomorphism $v(X) : \sigma_*F(X, Y) \rightarrow F(X, Y)$ is necessarily of the form $v(X) = f^{-1}(E\sigma_*f(X))$ for some matrix E .

Hence $f^{-1}(pf(X)) = [p](X) \equiv v(X^p) = f^{-1}(E\sigma_*f(X^p))$. It follows by part (2.8) of the functional equation lemma that $pf(X) \equiv E\sigma_*f(X^p) \pmod{pA}$, i.e. that $f(X) - p^{-1}E\sigma_*f(X^p) \in A[[X]]$, so that by part (2.6) of the functional equation

lemma $F(X,Y)$ is strictly isomorphic to the formal group law with logarithm defined by $g(X) = X + p^{-1}\text{Eg}(X^p)$ which is of the form $g(M,\eta)(X)$.

For some details about the rôle which the tapis de Cartier plays in the theory of lifting formal group laws cf. [7], section 30, as well as for an analogous theory for formal A -modules, where A is a finite extension of $\mathbb{F}_p$ or $\mathbb{F}_p(x)$.

7. RAMIFIED WITT VECTORS. Let A be the ring of integers of a finite (not necessarily unramified) extension K of $\mathbb{F}_p$ or $\mathbb{F}_p(x)$. Let k be the residue field of K , $q = \#k = p^r$, π a uniformizing element. Consider the power series

$$(7.1) \quad g_\pi(X) = X + \pi^{-1}X^q + \pi^{-2}X^{q^2} + \dots, \quad G_\pi(X,Y) = g_\pi^{-1}(g_\pi(X) + g_\pi(Y))$$

Then $g_\pi(X) = X + \pi^{-1}g_\pi(X^q)$ so that by section 5 above, $G_\pi(X,Y)$ is a Lubin-Tate formal group law over A . For every A -torsion free A -algebra B let $W_{q,\infty}^A(B)$ be the following set of power series in one variable t

$$(7.2) \quad W_{q,\infty}^A(B) = \{\gamma(t) \in B[[t]] \mid \gamma(0) = 0, g_\pi \gamma(t) = \sum_{i=0}^{\infty} x_i t^{q^i} \text{ for certain } x_i \in B \otimes_A K\}$$

For arbitrary A -algebras B one can define $W_{q,\infty}^A(B) = \{\phi_* \gamma(t) \mid \gamma(t) \in W_{q,\infty}^A(B')\}$ where B' is any A -torsion free A -algebra with a surjective A -algebra homomorphism $\phi : B' \rightarrow B$. The sets $W_{q,\infty}^A(B)$ have a natural group structure defined by $\gamma(t) + \delta(t) = G_\pi(\gamma(t), \delta(t))$ and a topology defined by the subgroups

$\{\gamma(t) \in W_{q,\infty}^A(B) \mid \gamma(t) \equiv 0 \pmod{t^{q^n}}\}$. There is an obvious morphism

$W_{q,\infty}^A(B_1) \rightarrow W_{q,\infty}^A(B_2)$ attached to an A -algebra homomorphism $\phi : B_1 \rightarrow B_2$, viz.

$\gamma(t) \mapsto \phi_* \gamma(t)$. So that we have a complete topological group valued functor $B \mapsto W_{q,\infty}^A(B)$.

We are now going to define a functorial ring structure on $W_{q,\infty}^A(B)$. The definition for A -torsion free A -algebras B is:

$$(7.3) \quad \text{if } g_\pi \gamma(t) = \sum x_i t^{q^i}, g_\pi \delta(t) = \sum y_i t^{q^i}, \text{ then } \gamma(t)\delta(t) = g_\pi^{-1}(\sum \pi^i x_i y_i t^{q^i})$$

To show that this is welldefined we must show that the coefficients of $\gamma(t)\delta(t)$ are in B (and not just in $B \otimes_A K$). This is seen as follows.

Assume that B is A -torsion free and admits an A -algebra endomorphism σ such that $\sigma(b) \equiv b^q \pmod{\pi B}$ for all $b \in B$. By part (2.7) of lemma 2.3 we then

have $x_i - \pi^{-1} x_{i-1} = a_i \in B$, $y_i - \pi^{-1} x_{i-1} = b_i \in B$ for all i (with $x_{-1} = y_{-1} = 0$).
 Hence $\pi^i x_i, \pi^i y_i \in B$ for all i . It follows that $\pi^i x_i y_i - \pi^{-1} (\pi^{i-1} x_{i-1} y_{i-1}) =$
 $= \pi^i a_i b_i + \pi^{i-1} a_i y_{i-1} + \pi^{i-1} b_i x_{i-1} \in B$, so that by part (2.6) of lemma 2.3 we
 have indeed that $g_{\pi}^{-1}(\sum \pi^i x_i y_i t^{q^i})$ has its coefficients in B . To extend this
 definition to the case of arbitrary A -algebras B use an argument similar as
 just below (7.2) using that every A -algebra B is a quotient of an A -algebra B'
 which satisfies our assumptions, e.g. $B' = A[Z_b | b \in B]$. There is also a natural
 A -module structure on $W_{q,\infty}^A(B)$ defined by $\gamma(t) \mapsto [a](\gamma(t))$ where
 $[a](X) = g_{\pi}^{-1}(ag_{\pi}(X))$, $a \in A$, cf. also section 5. All in all this defines a functor
 $W_{q,\infty}^A: \underline{\text{Alg}}_A \rightarrow \underline{\text{Alg}}_A$, which, we claim, possibly deserves the name "ramified Witt
 vector functor". To bolster this claim we remark the following

- There is an additive Verschiebung morphism $\underline{V}_q$ defined by $\underline{V}_q \gamma(t) = \gamma(t^q)$
 and a Frobenius A -algebra functor endomorphism $\underline{f}_{\pi}$. The latter is defined for
 A -torsion free A -algebras B by the formula $\underline{f}_{\pi} \gamma(t) = g_{\pi}^{-1}(\sum_{i=0}^{\infty} \pi x_{i+1} t^{q^i})$ where the
 x_i are as in (7.3). Of course the integrality of $\underline{f}_{\pi} \gamma(t)$ is proved by means of
 the functional equation lemma. We have $\underline{f}_{\pi} \underline{V}_q = [\pi]$, $\underline{f}_{\pi} \gamma(t) \equiv \gamma(t)^q \pmod{[\pi]} W_{q,\infty}^A(B)$.
 - Let A' be the ring of integers of an unramified extension K' of K . Let k'
 be the residue field of K' and let $\sigma \in \text{Gal}(K'/K)$ be the Frobenius automorphism.
 For each $a' \in A'$ let $\Delta(a') = g_{\pi}^{-1}(\sum_{i=0}^{\infty} \pi^{-i} \sigma^i(a') t^{q^i}) \in W_{q,\infty}^A(B)$. (Integrality of
 $\Delta(a')$ is of course proved by means of the functional equation lemma). Then
 $a' \mapsto \Delta(a')$ is a homomorphism of A -algebras and the composite
 $A' \xrightarrow{\Delta} W_{q,\infty}^A(A') \rightarrow W_{q,\infty}^A(k')$ is an isomorphism. In particular $W_{q,\infty}^A(k') = A'$ with σ
 corresponding to $\underline{f}_{\pi}$, generalizing a wellknown property of the Witt vectors.
 - There is an A -algebra homomorphism $\Delta: W_{q,\infty}^A(-) \rightarrow W_{q,\infty}^A(W_{q,\infty}^A(-))$, the ramified
 Artin-Hasse exponential, characterized by $w_{q,i}^A \circ \Delta = \underline{f}_{\pi}^i$, where $w_{q,i}^A: W_{q,\infty}^A(B) \rightarrow B$
 is the functorial A -algebra homomorphism $w_{q,i}^A(\gamma(t)) = \pi^i$ times the coefficient of
 t^{q^i} in $g_{\pi}(\gamma(t))$.

For more details concerning this construction cf. [7], section 25 ; for a
 twisted version of these constructions which also works for local fields with not
 necessarily finite residue field cf. also [9]. Another construction of the functors
 $W_{q,\infty}^A$ has independently been given by Ditters [4] and Drinfel'd [5].

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