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THETA FUNCTIONS IN POSITIVE CHARACTERISTIC

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(Padova)

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1. The theta function of a divisor ; a special case.

The minimal requirement for something which has a claim to being called the theta function Θ_X of a divisor X on an abelian variety A (over a field k), is that Θ_X should be a power series in a finite number of indeterminates, or a quotient of two such power series, and that it should be possible to treat X as the "divisor" of Θ_X much in the same manner as X is the divisor of an element $z \in k(A)$ when X is linearly equivalent to zero. Such theta functions were developed ten years ago in [2] for the case in which k is of characteristic zero ; they were defined in a purely algebraic and local manner (no periods required), and naturally turned out to coincide with classical theta functions when k is the complex field. There seemed to be strong technical difficulties to the extension of the method to characteristic p , but it is now clear that it was only a matter of picking the right end of the rope. This has now been done, and we finally have theta functions also in characteristic p ; typically, no new tool born in the meantime has been necessary. I will now briefly describe the underlying ideas and sketch the method and results ; details, complete proofs, and further developments will be found in a forthcoming paper by V. Cristante ; it is lucky that (Witt) covectors, which I have been using since 1958, have now become popular [3] ; I hope that the same will soon be true of bivectors, of which covectors are only a homomorphic image. Finally, I must apologize for the use of only those concepts with which I am familiar, thus

barring sheaves, schemes, spectra, and other complicated simplifications.

Let k be a perfect field of characteristic $p \neq 0$, and let A be an abelian variety of dimension n over k . It may not be useless to say that by that expression I mean a particular set of points of a projective space over the algebraic closure of k ; a point, in turn, means a point with coordinates in that algebraic closure; on the other hand, if the listener has a different picture in mind, the results will apply equally well to that picture. We could start with a commutative group-variety (without periodic subvarieties) instead of an abelian one, as I did in [2], but the most interesting case arises when A is abelian.

Set $C = k(A)$ = field of rational functions on A , and let $x = \{x_1, \dots, x_n\}$ be a regular set of parameters of the completion R of the local ring of the identity point $\underline{0}$ on A ; the maximal prime of R will be denoted by R^+ ; thus, $R = k[x] = k[x_1, \dots, x_n]$, this being the ring of power series in $x_1, \dots, x_n$, with coefficients in k and with integral nonnegative exponents. The field C can be canonically embedded in the quotient field of R , and we shall consider it so embedded. We shall also use an affine ring $k[y_1, \dots, y_m]$ of A , with $m \geq n$, such that $C = k(y)$ and that the identity point be at finite distance for y , say at $y = 0$. Let X be a divisor on A ; I shall tacitly assume, whenever a divisor is considered, that none of its components go through $\underline{0}$; naturally, this condition must be eliminated from a complete theory, but the elimination is an easy trick which adds nothing to the substance of the method. If $X \sim 0$ (linearly equivalent to zero), then $X = \text{div } z$ for some $z \in C$; the condition on X entails that $z \in R$, and clearly this z is entitled to be called the theta element of X , and to be denoted by $\theta_X = \theta_X(x)$. It is uniquely defined but for a nonzero factor in k , and it can be normalized by requiring that $z \equiv 1 \pmod{R^+}$.

Next step is the case $X \equiv 0$ (algebraically equivalent to zero); before describing it I must recall that R is a hyperalgebra over k , with its coproduct $\mathbb{P}$ which maps R algebra-isomorphically into the completed tensor product $R \overline{\otimes} R$ (over k). A regular set of parameters of $R \overline{\otimes} R$ is the set

$$\{x \overline{1}, 1 \overline{x}\} = \{\dots, x_i \overline{1}, \dots, \dots, 1 \overline{x}_i, \dots\};$$

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since many copies of R , A , C , etc. will be needed, I would rather index them ; thus, P may map R into $R_1 \overline{\times} R_2$, and this will have regular parameters $\{x_1, x_2\}$; the individual x 's , indexed from 1 to n , will never be used specifically again, so that no confusion arises. Px is a set of regular parameters of $PR \subset R_1 \overline{\times} R_2$, and it is expedient to denote it by $x_1 + x_2$.

The condition $X \equiv 0$ means that $\sigma_P X \sim X$ for each $P \in A$; here, σ_P is the translation by P . It also means the following : on $A \times A = A_1 \times A_2$ consider the divisors $X_1 = X \times A_2$, $X_2 = A_1 \times X$, and $X_{12} = (\text{div } \mu)X = \text{counterimage of } X \text{ if } \mu : A_1 \times A_2 \longrightarrow A$ is the law of composition ; then $Y = X_{12} - X_1 - X_2 \sim 0$ on $A_1 \times A_2$. Thus, Y has a theta element in the previous sense, namely an $f(x_1, x_2) \in R_1 \overline{\times} R_2$, symmetric in x_1, x_2 ; we select $f \equiv 1 \pmod{(R_1 \overline{\times} R_2)^+}$, and then it is easily verified that

$$f(x_1 + x_3, x_2) f(x_1, x_3) = f(x_1, x_3 + x_2) f(x_3, x_2).$$

In other words, f is a symmetric factor set of R into R_m , if R_m denotes the hyperalgebra $k[t]$ (bialgebra really : no inversion) with coproduct $Pt = t \otimes t$ [one indeterminate ; algebraic torus of dimension 1 ; multiplicative straight line]. It produces an extension of R by R_m (or viceversa, depending on the language you use), and it is well known that the only such extension is $R \otimes R_m$ (trivial extension). Therefore f itself is "trivial" as we now say, or associated to 1 as we once said (some H^2 is equal to 1), this meaning that for a suitable $g(x) \in R$ we have $f(x_1, x_2) = g(x_1 + x_2) / g(x_1)g(x_2)$. This $g(x)$ is a theta element $\Theta_X(x)$ of X ; it is unique but for a nonvanishing (constant) factor in k , and for a nonzero factor $h(x) \in R$ which satisfies the condition $h(x_1 + x_2) = h(x_1)h(x_2)$. Such an h is a multiplicative element of R , and it can exist if and only if R has a block of slope 1 (by now everybody knows this meaning of the word slope, introduced in chapter 5 of [MA] ; anyhow, slope 1 is present if and only if there are points $P \neq \underline{0}$ on A such that $pP = \underline{0}$).

So we now have Θ_X when $X \equiv 0$ (which includes $X \sim 0$) ; it belongs to R ; more generally, if X has poles through $\underline{0}$ it belongs to the quotient field $k\{x\}$ of R ; and it can be chosen in C if and only if $X \sim 0$.

2. Continuation ; general case.

We now relinquish any special condition on X ; we shall denote by C^∞ the perfect closure of C , by $\mathcal{R}^\circ$ the perfect closure of R , and by $\mathcal{R}$ its completion ; in the notation of [MA], the last three symbols would have been ${}^\pi\mathcal{R}^\circ$, ${}^\pi R$, ${}^\pi\mathcal{R}$, while C^∞ would have meant the union of the $(p\iota)^{-r}C$ for $r = 1, 2, \dots$, after denoting by ι the identity mapping ; this former C^∞ , which contains our present C^∞ , is still important, and will be used (and called C') in section 5 ; it is automatically embedded in the quotient field of $\mathcal{R}$ when our C^∞ is so embedded.

Let C_1 be a copy of C , extend A over C_1^∞ , and consider the point P of the extension at which the coordinates y assume the values y_1 (copy of y in C_1). It is known that $\sigma_P X - X \equiv 0$ if X denotes also the extension of X over C_1^∞ ; since C_1^∞ is perfect, the discussion of section 1 applies, and $\sigma_P X - X$ has a theta element

$$(1) \quad \varphi(x_1, x) \in C_1^\infty\{x\} ,$$

which we assume normalized by $\varphi(x_1, 0) = 1$. It is not difficult to prove that $\varphi(x_1, x) \in \mathcal{R}^\circ\{x\} \subset \mathcal{R} \overline{\mathcal{R}}$, and that we can also require $\varphi(0, x) = 1$. The meaning of X_i , X_{ij} being as in section 1, and that of X_{123} being similar, consider the divisor

$$Y = X_{123} + X_1 + X_2 + X_3 - X_{12} - X_{13} - X_{23}$$

on $A_1 \times A_2 \times A_3$; it is known that $Y \sim 0$, so that Y has a theta element $F(x_1, x_2, x_3)$ in the quotient field of $C_1 \otimes C_2 \otimes C_3$, normalized by

$$F(0, x_2, x_3) = F(x_1, 0, x_3) = F(x_1, x_2, 0) = 1.$$

The relation between φ and F is

$$F(x_1, x_2, x_3) = \varphi(x_1, x_2 + x_3) / \varphi(x_1, x_2) \varphi(x_1, x_3) = \varphi(x_1 + x_2, x_3) / \varphi(x_1, x_3) \varphi(x_2, x_3)$$

(not immediate, but not very hard either) ; from this, and from the symmetry of F in x_1, x_2, x_3 follows that $\varphi(x_1, x_2) / \varphi(x_2, x_1)$ is a skew-symmetric bi-multiplicative element of $\mathcal{R} \overline{\mathcal{R}}$ (it is the Riemann form of X on the radical part of R ; see section 5) ; as a consequence, there exists a bi-multiplicative element

$\chi(x_1, x_2) \in \mathcal{R} \overline{\mathcal{R}}$ such that $\psi(x_1, x_2) = \varphi(x_1, x_2) \chi(x_1, x_2)$ is symmetric ; it is in fact a symmetric factor set of $\mathcal{R}$ (not of R) into R_m . It must again be asso-

ciated to 1, and again, as in section 1, it provides a theta element $\theta_X(x)$ of X by the relation $\Psi(x_1, x_2) = \theta_X(x_1 + x_2) / \theta_X(x_1) \theta_X(x_2)$. In this general case θ_X belongs to $\mathcal{R}$ and not necessarily to R ; when it does not belong to R it does not belong to $\mathcal{R}^0$ either. The relation between $\theta = \theta_X$ and F is

$$(2) \quad F(x_1, x_2, x_3) = \theta(x_1 + x_2 + x_3) \theta(x_1) \theta(x_2) \theta(x_3) / \theta(x_1 + x_2) \theta(x_1 + x_3) \theta(x_2 + x_3) .$$

This θ_X is uniquely defined by (2) but for a quadratic exponential factor ; this expression shall mean the product of :

- i) a nonzero element of k ;
- ii) a multiplicative element of $\mathcal{R}$ (which is necessarily contained in the block of slope 1) ;
- iii) an element $e \neq 0$ of $\mathcal{R}$ such that $\mathbb{P}e / e \bar{x} e$ is bimultiplicative (such e 's are necessarily contained in the product of the blocks of slopes $\neq 1$). The theta element satisfies the usual relation $\theta_{X+Y} = \theta_X \theta_Y$, and it identifies X uniquely.

3. The case of characteristic zero.

The contents of sections 1 and 2 can be applied, with slight modifications, to the case of characteristic zero, and they afford a simplification of the method adopted in section 1 of [2], as they do not use classes of repartitions ($H^0(A, C/\theta_A)$ for the connoisseurs). The modifications are the following :

- 1) R is perfect, hence $\mathcal{R} = \mathcal{R}^0 = R$ and $C^\infty = C$;
- 2) one can select for x a set of integrals of the first kind ; in this case the $+$ of $\varphi(x_1 + x_2)$ is a true addition of sets of indeterminates.

In the case of characteristic zero the name "theta functions" is appropriate, since the arguments of which they are functions are canonically selected (see 2 above) ; in the case of characteristic p , on the contrary, $\theta(x)$ is a special element of $\mathcal{R}$, not a special power series in the x 's ; hence the use of the expression "theta element" rather than "theta function".

4. Theta functions in a given hyperalgebra.

We now start from an n -dimensional local equidimensional hyperalgebra $R = k\{x\}$

over k ; equidimensional means that $\mathfrak{p} \cap R$ is also of dimension n . If $\mathcal{R}^\circ$, $\mathcal{R}$ are related to R as in section 2, a nonzero element $\theta \in \mathcal{R}$ is of type theta on R if the function F of (2) belongs to the quotient field of $R_1 \otimes R_2 \otimes R_3$ (tensor product over k , not completion of ...). This is the same definition adopted in [2], except that θ is sought in $\mathcal{R}$ rather than R . As in section 3 of [2], and by similar arguments, there exists a smallest subfield C , finitely generated over k , of the quotient field of R , with the property that F belongs to the quotient field of $C \otimes C \otimes C$; the field C inherits $\mathbb{P}$ from R , and is therefore a hyperfield ; in other words, $C = k(A)$ for a suitable commutative group-variety A (a sketchy treatment of hyperfields is given in section 2 of [2] ; a developed theory is contained in [4]). More details about C can be found with the analytic machinery of bivectors : consider the bivector $\{\theta\} = (\dots, 0, 0; \theta, 0, 0, \dots)$; its logarithm exists and is of the type $\log\{\theta\} = (\dots, v, v; v, v, \dots)$, where v is the Artin-Hasse logarithm of θ . I will next recall the definition of $\mathcal{E}\tilde{\mathcal{R}}$: the discrete hyperalgebra $\tilde{\mathcal{R}}$ is the dual of $\mathcal{R}$; $\tilde{\mathcal{R}}$ is the completion of the dual of $\mathcal{R}$; $\mathcal{E}\tilde{\mathcal{R}}$ is the set of the elements d of $\text{biv } \tilde{\mathcal{R}}$ which are canonical, namely satisfy $\mathbb{P}d = d\overline{x}1 + 1\overline{x}d$ (it is a sort of Dieudonné module) ; $\mathcal{E}\tilde{\mathcal{R}}$ is the subset of $\mathcal{E}\tilde{\mathcal{R}}$ formed by those $d = (\dots, d_{-1}; d_0, d_1, \dots)$ having the property that $d_{-1}R = 0$, or, equivalently, that d_0 , viewed as an element of $\text{End}_k \mathcal{R}$, induces an (invariant) derivation in R . For such d 's I defined in chapter 5 of [MA] an element d^* of $\text{End}_K \text{Biv } \mathcal{R}$, where $K = \text{vect } k$, which is not quite a derivation on $\text{Biv } \mathcal{R}$ (it turns out to be a component of a covector whose ghost components are derivations on a subring of $\text{Biv } \mathcal{R}$). Well, C contains all the components of all the bivectors $d \cdot d^* \cdot \log\{\theta\}$ (which are really vectors), when d, d' range over $\mathcal{E}\tilde{\mathcal{R}}$; if D is the field generated, over k , by these components and by their hyperderivatives, D itself is a hyperfield, and we strongly suspect that $C = D$; so far it is only proved that the embedding of D into C is a purely inseparable isogeny. (Added Nov. 78 : $C = D$ now proved).

Two elements of type theta are associated if their ratio is a quadratic exponential ; the dimension of θ is the dimension of the smallest subhyperalgebra of $\mathcal{R}$ which contains some element associated to θ ; the transcendency of C over k

turns out to be not less than the dimension of Θ ; if it is equal to this dimension, Θ is a theta element on R , and it is nondegenerate when its dimension is n . The discussion in the next section will show that there are theta elements on R if and only if : for each block of slope α and dimension n_α of R , with $0 < \alpha < 1$, R has a block of slope $1-\alpha$ and codimension n_α [this condition reflects the symmetry theorem satisfied by hyperalgebras arising from abelian varieties ; it would be interesting to establish the condition directly within the theory of theta elements, thus supplying a third proof of the symmetry theorem]. Picking theta elements (on R) in $\mathcal{R}$ is equivalent to picking rational composition laws for R ; or also to viewing R as an algebraic group.

Now that we have a theta element defined a priori, we naturally want to know whether $\Theta = \Theta_X$ for some X on A ; the answer is the same as in [2] : X is the only divisor on A (with no component on the degeneration locus in case A is not abelian) such that $\text{div } F - X \times A \times A$ has no component of the form $Y \times A \times A$; this X is very strongly ample on A (if Θ is holomorphic) , namely : $\sigma_P X = X$ only when $P = \underline{0}$; by the way : X is effective (= positive) if and only if Θ is holomorphic, this meaning that $\Theta(x_1+x_2)\Theta(x_1-x_2) \in \mathcal{R} \otimes \mathcal{R}$.

5. The abelian case and the Riemann form.

For a deeper discussion we must make full use of the tools provided by [MA], in particular those of chapter 6 ; let us go back to the case discussed in sections 1 and 2, where $\Theta = \Theta_X$ for some X on A . Let us denote by C' the union of the fields $(p\iota)^{-r}C$ described at the opening of section 2 ; theorems 6.12, 6.13, 6.14 (and others) of [MA] provide, for each $d \in \mathcal{C}\tilde{R}$, an element z_d of $\text{vect } C'$, uniquely determined, but for a summand in $K = \text{vect } k$, by the following property : for each prime divisor Y on A^∞ (inverse limit of $A \xleftarrow{p\iota} A \xleftarrow{p\iota} A \dots$), let x_Y be a representative of X at Y ; then all the components of the vector $d \cdot \log\{x_Y\} - z_d$ belong to the local ring of Y on A^∞ . After a suitable choice of the arbitrary summand it can be proved (easily) that for suitable elements $\eta_d \in \mathcal{C}R$ and $c_d \in \text{biv } k$ (this being the quotient field of K) we have

$$(3) \quad z_d = d * \log\{\theta\} + \eta_d - c_d .$$

The mapping $d \longrightarrow \eta_d$ is K -linear and commutes with π (Frobenius) and t (shift) ; in particular, if d has slope 0 also η_d must have slope 0 ; but $\mathcal{C}R$ contains no element of slope 0 (those elements all come from the separable = totally disconnected = etale block of a hyperalgebra) ; hence $\eta_d = 0$ if d has slope 0, a fact which shows that for such d 's, the bivector $d * \log\{\theta\} - c_d$ is in $\text{vect } R$. Assume instead that d has no direct summand of slope 0 ; more precisely, in what follows d will range over $\mathcal{C}\tilde{R}_R$, where $\tilde{R}_R$ is the radical part of $\tilde{R}$, made up of all the blocks of slope $\neq 0$; then $\eta_d \in \mathcal{C}R_R$ (and this R_R is made up of blocks of slope $\neq 1$), and we would like to know more about it. If χ has the meaning of section 2, define the elements ζ_d, ξ_d of $\mathcal{C}R$ (actually of $\mathcal{C}R_R$) by : $\zeta_d = (d \otimes 1) * \log\{\chi\}$, and $\xi_d = (1 \otimes d) * \log\{\chi\}$; consider also the operators $\alpha = \lim_{r \rightarrow \infty} p^r (p\iota)^{-r}$, and $\beta = \lim_{r \rightarrow \infty} p^{-r} (p\iota)^r$, and remember that $\beta z_d \in \mathcal{C}\mathcal{R}_R$ is an old acquaintance, namely $\varphi_X d$, where φ_X is the Riemann form of X (see chapters 6 and 7 of [MA] ; do not confuse with the mapping of A into its dual denoted by φ_X in Lang's book : this mapping I had christened λ_X in 1954, and I haven't changed since). Application of β to (3) gives $\varphi_X d = \beta(d * \log\{\theta\} - c_d) + \eta_d$, while application of α gives $0 = \alpha(d * \log\{\theta\}) + \eta_d$. On the other hand, from the definition of θ_X and from (1) we can derive that $\alpha(d * \log\{\theta\}) = \xi_d$ and that $\beta(d * \log\{\theta\} - c_d) = \zeta_d$. So $\eta_d = -\xi_d$, and $\varphi_X d = \zeta_d + \eta_d = \zeta_d - \xi_d$. We conclude that (3) gives the decomposition of the mapping

$$(4) \quad (d, d') \longrightarrow d' * (z_d + \zeta_d)$$

into the alternating part

$$(5) \quad (d, d') \longrightarrow d' * \varphi_X d$$

and the "symmetric" part

$$(6) \quad (d, d') \longrightarrow d' * d * \log\{\theta\} .$$

The word " symmetric " is in quotations because mappings (4) and (6) are not K -bilinear ; with this limitation, due to the imperfect nature of $d *$ as a derivation, (5) is a holomorphic differential of dimension 2, while (6) is a metric .

We can now go back to a question left open in section 4 ; given a nondegene-

rate theta element Θ , when is the group-variety A abelian ? The discussion of this section provides the answer. To begin with, the equidimensionality of R rules out the possibility of periodic group-varieties ; hence A can only be the extension of an abelian variety by multiplicative lines (tori) ; the answer can thus be sought by "counting" p -division points, which are lacking on multiplicative lines. Equivalently, let f be the dimension of the block of slope 1 of R ; then, in order that A be abelian it is necessary and sufficient that when d ranges over the elements of $\mathcal{C}\tilde{R}$ of slope 0, the vectors $d \cdot \log\{\Theta\} = z_d$, modulo vect C , span a K -module with f free generators ; when this is the case we say that Θ is an abelian theta element.

6. Conclusion.

This exposition starts with an article of faith (about Θ having to be a power series, albeit with non-integral exponents, as we have later seen) which I am about to abjure, with a warning that the following free-wheeling considerations are more than wishful thinking but less than a description of work accomplished.

For a general setting we start from a group G which I choose to call "analytic", and which is the candidate for being the "completion" of a commutative algebraic group A (this A is assumed to be an abelian variety in the description which follows) ; "completion" is the accepted word, but a very poor choice for something which is usually smaller than A . Anyhow, since groups are only dimly present, while hyperalgebras of "analytic" functions on groups are very much present, it is better to speak of $C = k(A)$ and of $R =$ functions on G : in C we select an array (= order) S such that $\mathbb{P}S$ is a subring of the quotient field of $S \otimes S$; on S we place a suitable topology T_0 , and denote by R the T_0 -completion of S (naturally T_0 comes from a metric) ; we then seek a "universal covering" $\mathcal{U}$ of G , which in terms of rings of functions means a "maximal embedding" of R into a hyperalgebra $\mathcal{R}$. Essentially, $\mathcal{U}$ must give enough information on the universal covering of A , which algebraically seems to be the maximal covering of A whose ramification arises only from inseparability. In the first five sections S has been the local ring of $\underline{0}$ on A and T_0 has been its natural local topology ; however, the same R , which I will now revert to

calling ${}^\pi R$, can be reached, as explained in chapter 6 of [MA] (modulo some silly mistake), by taking for S the intersection S_p of the local rings of all the p -division points on A , and for T_0 the π -topology (this S_p is an array in C if k is not too small) ; hence the use of ${}^\pi R$ rather than R . I will give a list of four possible selections of R and $\mathcal{R}$, of which number 1 is the one just described, i.e. the one adopted in the preceding sections :

1) S is S_p ; T_0 is the π -topology (see chapter 6 of [MA]) ; R is ${}^\pi R$, and $\mathcal{R} = {}^\pi \mathcal{R}$ is the completion of the direct limit ${}^\pi R \xrightarrow{\pi} {}^\pi R \xrightarrow{\pi} \dots$. This ${}^\pi \mathcal{R}$ is the completed tensor product ${}^\pi \mathcal{R}_\pi \overline{\times} {}^\pi \mathcal{R}_r$, where ${}^\pi \mathcal{R}_r$ is the radical part of ${}^\pi \mathcal{R}$ (slopes < 1 , and certainly > 0), while ${}^\pi \mathcal{R}_\pi$ is the logarithmic, or toroidal, part (slope 1). If $f = \text{sep codim } A = \dim {}^\pi \mathcal{R}_\pi$, it is not idle to remark that ${}^\pi \mathcal{R}_\pi$ is isomorphic and homeomorphic to the hyperalgebra of certain measures on $\mathbb{Q}_p^f$ with values in k (at least when k is algebraically closed) ; the topology is that of uniform convergence on balls of bounded radius (k is taken to be discrete). This interpretation of ${}^\pi \mathcal{R}_\pi$, as well as similar interpretations in the cases which follow, are the object of [5].

2) S is S_p ; T_0 is the t -topology ; R is ${}^t R$, and $\mathcal{R} = {}^t \mathcal{R}$ is the completion of the direct limit ${}^t R \xrightarrow{t} {}^t R \xrightarrow{t} \dots$. Now ${}^t \mathcal{R} = {}^t \mathcal{R}_r \overline{\times} {}^t \mathcal{R}_t$, where ${}^t \mathcal{R}_r \cong {}^\pi \mathcal{R}_r$ and ${}^t \mathcal{R}_t$ is the separable, or etale, part of ${}^t \mathcal{R}$ (slope 0) ; it is isomorphic and homeomorphic to the hyperalgebra of continuous functions on $\mathbb{Q}_p^f$, with values in k ; the topology is that of uniform convergence on compacts.

3) S is S_p ; T_0 is the $p\iota$ -topology (remember that $p\iota = \pi t$) ; R is R , and $\mathcal{R} = \mathcal{R}$ is the completion of the direct limit $R \xrightarrow{p\iota} R \xrightarrow{p\iota} \dots$. Now $\mathcal{R} = \mathcal{R}_\pi \overline{\times} \mathcal{R}_r \overline{\times} \mathcal{R}_t$, with $\mathcal{R}_\pi \cong {}^\pi \mathcal{R}_\pi$, $\mathcal{R}_r \cong {}^\pi \mathcal{R}_r \cong {}^t \mathcal{R}_r$, $\mathcal{R}_t \cong {}^t \mathcal{R}_t$.

4) S is S_q for a prime $q \neq p$ (usually called ell) ; T_0 is the $q\iota$ -topology, where a basis for neighbourhoods of 0 consists of the maximal primes of S_q ; R can be called R_q , and $\mathcal{R} = \mathcal{R}_q$ is the completion of the direct limit $R_q \xrightarrow{q\iota} R_q \xrightarrow{q\iota} \dots$. If $n = \dim A$, $\mathcal{R}_q$ is isomorphic and homeomorphic to the hyperalgebra of continuous functions on $\mathbb{Q}_q^{2n}$, with values in k ; the topology is

that of uniform convergence on compacts.

If k has characteristic zero the only two known possibilities, barring special fields such as the complex field, are :

1') (similar to 1) S is the local ring of $\underline{0}$; R is its completion ; $\mathcal{R} = R$ (see section 3).

4') Same as 4, q being any prime.

After deciding on an $\mathcal{R}$, a theta element on R (or element of type theta as the case may be) is simply a $\theta \in \mathcal{R}$ such that the F of formula (2) is the ratio of two elements of $R \otimes R \otimes R$; notice the $\otimes$ rather than $\bar{\times}$. Naturally now F must be written as

$$((\mathbb{C} \bar{\times} \mathbb{P})\mathbb{P}\theta)(\theta \bar{\times} \theta \bar{\times} \theta)/(\mathbb{P}\theta \bar{\times} 1)(1 \bar{\times} \mathbb{P}\theta)(sc_{12}(1 \bar{\times} \mathbb{P}\theta)) .$$

The field C is then retrieved as the quotient field of the smallest subring U of R having the property that the quotient field of $U \otimes U \otimes U$ contains F . The existence of such θ 's, for instance in case 4', is essentially due to the fact that the only crossed product of $\mathbb{Q}_q$ by the multiplicative group of nonzero elements of k is the direct product. Knowledge of $\theta = \theta_X$ must entail knowledge of the restriction ρ_X of the Riemann form of X to $\mathcal{R} \bar{\times} \mathcal{R}$; the recipe is as follows : find a bimultiplicative element $\chi \in \mathcal{R} \bar{\times} \mathcal{R}$ such that $\mathbb{P}\theta/(\theta \bar{\times} \theta)\chi \in C^\infty \bar{\times} \mathcal{R}$; then ρ_X is simply $\chi/sc\chi$, or its reciprocal according to taste. This is ρ_X viewed as a skew-symmetric bi-multiplicative element ; in order to view it as a bilinear element one must take its "logarithm" according to some suitable definition of the term.

Examples : in case 1 (the object of this exposition) χ is the χ of section 2, and the logarithm is $\log\{ \}$; in case 4, χ is given by $\chi(r,v) = \theta(0)\theta(v+r)/\theta(v)\theta(r)$ for $v \in \mathbb{Q}_q^{2n}$ and $r \in \mathbb{Z}_q^{2n}$; the logarithm is the inverse of a standard homomorphism of $\mathbb{Q}_p$ into the group of q^∞ -th roots of 1 in the algebraic closure of k .

It is now only fair to ask whether Mumford's thetas [6] fit into this scheme. The work of comparison is a tall order, except that in 1970-71, when I still refused to consider thetas which were not power series, I devoted some time and effort to the construction of (illegal) theta elements which fall under case 4 above ; they turned out to be very similar to Mumford's thetas as described in section 8

of [6], modulo the fact that I had not selected $q = 2$. Thus, the answer to the question should be yes.

BIBLIOGRAPHY

- [1] V. CRISTANTE, to appear.
- [2] I. BARSOTTI, *Considerazioni sulle funzioni theta*, Symp. Math., 3, 1970, p. 247.
- [3] P. BERTHELOT, *Théorie de Dieudonné sur un anneau de valuation parfait*, mimeographed notes, University of Rennes, 1977-78.
- [4] G. GEROTTO, *Alcuni elementi di teoria degli ipercorpi*, Ann. Mat. Pura e Appl., 115, 1977, p. 349.
- [MA] I. BARSOTTI, *Metodi analitici per varietà abeliane in caratteristica positiva. Capitoli 1,2 ; Capitoli 3,4 ; Capitolo 5 ; Capitolo 6 ; Capitolo 7*, Ann. Scuola Norm Sup., 18, 1964, p. 1 ; 19, 1965, p. 277 ; 19, 1965, p. 481 ; 20, 1966, p. 101 ; 20, 1966, p. 331.
- [5] F. BALDASSARRI, *Interpretazione funzionale di certe iperalgebre e dei loro anelli di bivettori di Witt*, Tesi facoltà di scienze, Univ. di Padova, 1972-73.
- [6] D. MUMFORD, *On the equations defining abelian varieties. I ; II ; III*, Inv. Math. 1, 1966, p. 287 ; 3, 1967, p. 75 ; 3, 1967, p. 215.

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