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150 years after Gauss' « disquisitiones generales circa superficies curvas »

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The necessity of coming back to original mathematical texts of utmost importance is nowadays well recognized. This issue of Astérisque is devoted to the presentation of Gauss' Disquisitiones Generales Circa Superficies Curvas and other texts related to it, together with a discussion by Peter Dombrowski of its contents, presentation and historical importance, written at the occasion of the bicentenary of Gauss' birthday.

This would not have been made possible without the help of Michael Keane and David Trotman who translated Peter Dombrowski's original German manuscript, the authorization of the Braunschweigische Wissenschaftliche Gesellschaft (who organized the bicentenary celebration), the assistance of J. P. Bourguignon, and the beautiful typing and page setting of Marie-José Lécuyer.

We thank them, and also Prof. P. Dombrowski for having provided the scientific community with a mathematical text of a type which, in our opinion, is indispensable and was still missing.

The editors

TABLE OF CONTENTS

DISQUISITIONES GENERALES

CIRCA

SUPERFICIES CURVAS

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE OBLATAE D. 8. OCTOB. 1827

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DISQUISITIONES GENERALES

CIRCA SUPERFICIES CURVAS.

1.

Disquisitiones, in quibus de directionibus variarum rectarum in spatio agitur. plerumque ad maius perspicuitatis et simplicitatis fastigium evehuntur, in auxilium vocando superficiem sphaericam radio $= 1$ circa centrum arbitrarium descriptam, cuius singula puncta repraesentare censebuntur directiones rectarum radiis ad illa terminatis parallelarum. Dum situs omnium punctorum in spatio per tres coordinatas determinatur, puta per distantias a tribus planis fixis inter se normalibus, ante omnia considerandae veniunt directiones axium his planis normalium: puncta superficiei sphaericae, quae has directiones repraesentant, per (1) (2), (3) denotabimus; mutua igitur horum distantia erit quadrans. Ceterum axium directiones versus eas partes acceptas supponemus, versus quas coordinatae respondentes crescunt.

2.

Haud inutile erit, quasdam propositiones, quae in huiusmodi quaestionibus usum frequentem offerunt, hic in conspectum producere.

I. Angulus inter duas rectas se secantes mensuratur per arcum inter puncta, quae in superficie sphaerica illarum directionibus respondent.

II. Situs cuiuslibet plani repraesentari potest per circulum maximum in superficie sphaerica, cuius planum illi est parallelum.

GENERAL INVESTIGATIONS

OF

CURVED SURFACES

BY

KARL FRIEDRICH GAUSS

PRESENTED TO THE ROYAL SOCIETY, OCTOBER 8, 1827

1.

Investigations, in which the directions of various straight lines in space are to be considered, attain a high degree of clearness and simplicity if we employ, as an auxiliary, a sphere of unit radius described about an arbitrary centre, and suppose the different points of the sphere to represent the directions of straight lines parallel to the radii ending at these points. As the position of every point in space is determined by three coordinates, that is to say, the distances of the point from three mutually perpendicular fixed planes, it is necessary to consider, first of all, the directions of the axes perpendicular to these planes. The points on the sphere, which represent these directions, we shall denote by (1), (2), (3). The distance of any one of these points from either of the other two will be a quadrant; and we shall suppose that the directions of the axes are those in which the corresponding coordinates increase.

2.

It will be advantageous to bring together here some propositions which are frequently used in questions of this kind.

I. The angle between two intersecting straight lines is measured by the arc between the points on the sphere which correspond to the directions of the lines.

II. The orientation of any plane whatever can be represented by the great circle on the sphere, the plane of which is parallel to the given plane.

Disquisitiones Generales circa Superficies Curvas

III. Angulus inter duo plana aequalis est angulo sphaerico inter circulos maximos illa repraesentantes, et proin etiam per arcum inter horum circulorum maximorum polos interceptum mensuratur. Et perinde inclinatio rectae ad planum mensuratur per arcum, a puncto, quod respondet directioni rectae, ad circulum maximum, qui plani situm repraesentat, normaliter ductum.

IV. Denotantibus x, y, z; x', y', z' coordinatas duorum punctorum, r eorundem distantiam, atque L punctum, quod in superficie sphaerica repraesentat directionem rectae a puncto priore ad posterius ductae, erit

$$x' = x + r\cos(1)L, \quad y' = y + r\cos(2)L, \quad z' = z + r\cos(3)L$$

V. Hinc facile sequitur, haberi generaliter

$$\cos(1)L^2 + \cos(2)L^2 + \cos(3)L^2 = 1$$

nec non, denotante L' quodcunque aliud punctum superficiei sphaericae, esse

$$\cos(1)L.\cos(1)L' + \cos(2)L.\cos(2)L' + \cos(3)L.\cos(3)L' = \cos LL'$$

VI. Theorema. *Denotantibus L, L', L'', L''' quatuor puncta in superficie sphaerae, atque A angulum, quem arcus LL', $L''L'''$ in puncto concursus sui formant, erit*

$$\cos LL''.\cos L'L''' - \cos LL'''.\cos L'L'' = \sin LL'.\sin L''L'''.\cos A$$

Dem. Denotet litera A insuper punctum concursus ipsum, statuaturque

$$AL = t, \quad AL' = t', \quad AL'' = t'', \quad AL''' = t'''$$

Habemus itaque:

$$\begin{aligned}
\cos LL'' &= \cos t\cos t'' + \sin t\sin t''\cos A\\
\cos L'L''' &= \cos t'\cos t''' + \sin t'\sin t'''\cos A\\
\cos LL''' &= \cos t\cos t''' + \sin t\sin t'''\cos A\\
\cos L'L'' &= \cos t'\cos t'' + \sin t'\sin t''\cos A
\end{aligned}$$

et proin

$$\begin{aligned}
&\cos LL''.\cos L'L''' - \cos LL'''.\cos L'L''\\
&= \cos A(\cos t\cos t''\sin t'\sin t''' + \cos t'\cos t'''\sin t\sin t''\\
&\qquad - \cos t\cos t'''\sin t'\sin t'' - \cos t'\cos t''\sin t\sin t''')\\
&= \cos A(\cos t\sin t' - \sin t\cos t')(\cos t''\sin t''' - \sin t''\cos t''')\\
&= \cos A.\sin(t'-t).\sin(t'''-t'')\\
&= \cos A.\sin LL'.\sin L''L'''
\end{aligned}$$

General Investigations of Curved Surfaces

III. The angle between two planes is equal to the spherical angle between the great circles representing them, and, consequently, is also measured by the arc intercepted between the poles of these great circles. And, in like manner, the angle of inclination of a straight line to a plane is measured by the arc drawn from the point which corresponds to the direction of the line, perpendicular to the great circle which represents the orientation of the plane.

IV. Letting $x, y, z; x', y', z'$ denote the coordinates of two points, r the distance between them, and L the point on the sphere which represents the direction of the line drawn from the first point to the second, we shall have

$$x' = x + r \cos(1)L$$
$$y' = y + r \cos(2)L$$
$$z' = z + r \cos(3)L$$

V. From this it follows at once that, generally,

$$\cos^2(1)L + \cos^2(2)L + \cos^2(3)L = 1$$

and also, if L' denote any other point on the sphere,

$$\cos(1)L \,.\, \cos(1)L' + \cos(2)L \,.\, \cos(2)L' + \cos(3)L \,.\, \cos(3)L' = \cos LL'.$$

VI. Theorem. *If L, L', L'', L''' denote four points on the sphere, and A the angle which the arcs LL', $L''L'''$ make at their point of intersection, then we shall have*

$$\cos LL'' \,.\, \cos L'L''' - \cos LL''' \,.\, \cos L'L'' = \sin LL' \,.\, \sin L''L''' \,.\, \cos A$$

Demonstration. Let A denote also the point of intersection itself, and set

$$AL = t, \quad AL' = t', \quad AL'' = t'', \quad AL''' = t'''$$

Then we shall have

$$\cos LL'' = \cos t \,.\cos t'' + \sin t \, \sin t'' \, \cos A$$
$$\cos L'L''' = \cos t' \cos t''' + \sin t' \sin t''' \cos A$$
$$\cos LL''' = \cos t \cos t''' + \sin t \sin t''' \cos A$$
$$\cos L'L'' = \cos t' \cos t'' + \sin t' \sin t'' \cos A$$

and consequently,

$$\begin{aligned}\cos LL'' \,.\, \cos L'L''' - \cos LL''' .\ \cos L'L''\\ = \cos A\ (\cos t \cos t'' \sin t' \sin t''' + \cos t' \cos t''' \sin t \sin t''\\ - \cos t \cos t''' \sin t' \sin t'' - \cos t' \cos t'' \sin t \sin t''')\\ = \cos A\ (\cos t \sin t' - \sin t \cos t')\ (\cos t'' \sin t''' - \sin t'' \cos t''')\\ = \cos A \,.\, \sin(t' - t) \,.\, \sin(t''' - t'')\\ = \cos A \,.\, \sin LL' \,.\, \sin L''L'''\end{aligned}$$

Disquisitiones Generales circa Superficies Curvas

Ceterum quum inde a puncto A bini rami utriusque circuli maximi proficiscantur, duo quidem ibi anguli formantur, quorum alter alterius complementum ad 180^0: sed analysis nostra monstrat, eos ramos adoptandos esse, quorum directiones cum sensu progressionis a puncto L ad L', et a puncto L'' ad L''' consentiunt: quibus intellectis simul patet, quum circuli maximi duobus punctis concurrant, arbitrarium esse, utrum eligatur. Loco anguli A etiam arcus inter polos circulorum maximorum, quorum partes sunt arcus LL', $L''L'''$, adhiberi potest: manifesto autem polos tales accipere oportet, qui respectu horum arcuum similiter iacent, puta vel uterque polus ad dextram iacens, dum a L versus L' atque ab L'' versus L''' procedimus, vel uterque ad laevam.

VII. Sint L, L', L'' tria puncta in superficie sphaerica, statuamusque brevitatis caussa

$$\begin{array}{lll} \cos(1)L = x, & \cos(2)L = y, & \cos(3)L = z \\ \cos(1)L' = x', & \cos(2)L' = y', & \cos(3)L' = z' \\ \cos(1)L'' = x'', & \cos(2)L'' = y'', & \cos(3)L'' = z'' \end{array}$$

nec non

$$xy'z'' + x'y''z + x''yz' - xy''z' - x'yz'' - x''y'z = \Delta$$

Designet λ polum circuli maximi, cuius pars est arcus LL', et quidem eum, qui respectu huius arcus similiter iacet, ac punctum (1) respectu arcus (2)(3). Tunc erit, ex theoremate praecedente, $yz' - y'z = \cos(1)\lambda . \sin(2)(3) . \sin LL'$, sive, propter $(2)(3) = 90^0$,

$$\begin{aligned} yz' - y'z &= \cos(1)\lambda . \sin LL', \text{ et perinde} \\ zx' - z'x &= \cos(2)\lambda . \sin LL' \\ xy' - x'y &= \cos(3)\lambda . \sin LL' \end{aligned}$$

Multiplicando has aequationes resp. per x'', y'', z'' et addendo, obtinemus adiumento theorematis secundi in Y prolati

$$\Delta = \cos\lambda L'' . \sin LL'$$

Iam tres casus sunt distinguendi. *Primo*, quoties L'' iacet in eodem circulo maximo, cuius pars est arcus LL', erit $\lambda L'' = 90^0$, adeoque $\Delta = 0$. Quoties vero L'' iacet extra circulum illum maximum, aderit casus *secundus*, si est ab eadem parte, a qua est λ, *tertius*, si ab opposita: in his casibus puncta L, L', L''

General Investigations of Curved Surfaces

But as there are for each great circle two branches going out from the point A, these two branches form at this point two angles whose sum is 180°. But our analysis shows that those branches are to be taken whose directions are in the sense from the point L to L', and from the point L'' to L'''; and since great circles intersect in two points, it is clear that either of the two points can be chosen arbitrarily. Also, instead of the angle A, we can take the arc between the poles of the great circles of which the arcs $L\,L'$, $L''\,L'''$ are parts. But it is evident that those poles are to be chosen which are similarly placed with respect to these arcs; that is to say, when we go from L to L' and from L'' to L''', both of the two poles are to be on the right, or both on the left.

VII. Let L, L', L'' be the three points on the sphere and set, for brevity,

$$\cos(1)L = x,\quad \cos(2)L = y,\quad \cos(3)L = z$$
$$\cos(1)L' = x',\quad \cos(2)L' = y',\quad \cos(3)L' = z'$$
$$\cos(1)L'' = x'',\quad \cos(2)L'' = y'',\quad \cos(3)L'' = z''$$

and also

$$x\,y'\,z'' + x'\,y''\,z + x''\,y\,z' - x\,y''\,z' - x'\,y\,z'' - x''\,y'\,z = \Delta$$

Let λ denote the pole of the great circle of which LL' is a part, this pole being the one that is placed in the same position with respect to this arc as the point (1) is with respect to the arc (2)(3). Then we shall have, by the preceding theorem,

$$y\,z' - y'\,z = \cos(1)\lambda\ .\ \sin(2)(3)\ .\ \sin LL',$$

or, because (2)(3) = 90°,

$$y\,z' - y'\,z = \cos(1)\lambda\ .\ \sin LL',$$

and similarly,

$$z\,x' - z'\,x = \cos(2)\lambda\ .\ \sin LL'$$
$$x\,y' - x'\,y = \cos(3)\lambda\ .\ \sin LL'$$

Multiplying these equations by x'', y'', z'' respectively, and adding, we obtain, by means of the second of the theorems deduced in V,

$$\Delta = \cos\lambda\,L''\,.\,\sin LL'$$

Now there are three cases to be distinguished. *First*, when L'' lies on the great circle of which the arc LL' is a part, we shall have $\lambda L'' = 90°$, and consequently, $\Delta = 0$. If L'' does not lie on that great circle, the *second* case will be when L'' is on the same side as λ; the *third* case when they are on opposite sides. In the last two cases the points L, L', L'' will form a spherical triangle, and in the second case these points will lie in the same order as the points (1), (2), (3), and in the opposite order in the third case.

formabunt triangulum sphaericum, et quidem iacebunt in casu secundo eodem ordine quo puncta (1), (2), (3), in casu tertio vero ordine opposito. Denotando angulos illius trianguli simpliciter per L, L', L'', atque perpendiculum in superficie sphaerica a puncto L'' ad latus LL' ductum per p, erit

$$\sin p = \sin L . \sin LL'' = \sin L' . \sin L'L'', \quad \text{atque} \quad \lambda L'' = 90^0 \mp p$$

valente signo superiori pro casu secundo, inferiori pro tertio. Hinc itaque colligimus

$$\pm \Delta = \sin L . \sin LL' . \sin LL'' = \sin L' . \sin LL' . \sin L'L'' = \sin L'' . \sin LL'' . \sin L'L''$$

Ceterum manifesto casus primus in secundo vel tertio comprehendi censeri potest, nulloque negotio perspicitur, $\pm \Delta$ exhibere sextuplum soliditatis pyramidis inter puncta L, L', L'' atque centrum sphaerae formatae. Denique hinc facillime colligitur, eandem expressionem $\pm \frac{1}{6} \Delta$ generaliter exprimere soliditatem cuiusvis pyramidis inter initium coordinatarum atque puncta quorum coordinatae sunt x, y, z; x', y', z'; x'', y'', z'', contentae.

3.

Superficies curva apud punctum A in ipsa situm curvatura continua gaudere dicitur, si directiones omnium rectarum ab A ad omnia puncta superficiei ab A infinite parum distantia ductarum infinite parum ab uno eodemque plano per A transiente deflectuntur: hoc planum superficiem curvam in puncto A *tangere* dicitur. Quodsi huic conditioni in aliquo puncto satisfieri nequit, continuitas curvaturae hic interrumpitur, uti e.g. evenit in cuspide coni. Disquisitiones praesentes ad tales superficies curvas, vel ad tales superficiei partes, restringentur, in quibus continuitas curvaturae nullibi interrumpitur. Hic tantummodo observamus, methodos, quae positioni plani tangentis determinandae inserviunt, pro punctis singularibus, in quibus continuitas curvaturae interrumpitur, vim suam perdere, et ad indeterminata perducere debere.

4.

Situs plani tangentis commodissime e situ rectae ipsi in puncto A normalis cognoscitur, quae etiam ipsi superficiei curvae normalis dicitur. Directionem huius normalis per punctum L in superficie sphaerae auxiliaris repraesentabimus, atque statuemus

General Investigations of Curved Surfaces

Denoting the angles of this triangle simply by L, L', L'' and the perpendicular drawn on the sphere from the point L'' to the side LL' by p, we shall have

$$\sin p = \sin L \,.\, \sin L L'' = \sin L' \,.\, \sin L' L'',$$

and

$$\lambda\, L'' = 90^\circ \mp p,$$

the upper sign being taken for the second case, the lower for the third. From this it follows that

$$\pm \Delta = \sin L \,.\, \sin LL' \,.\, \sin L L'' = \sin L' \,.\, \sin L L' \,.\, \sin L' L''$$
$$= \sin L'' \,.\, \sin L L'' \,.\, \sin L' L''$$

Moreover, it is evident that the first case can be regarded as contained in the second or third, and it is easily seen that the expression $\pm \Delta$ represents six times the volume of the pyramid formed by the points L, L', L'' and the centre of the sphere. Whence, finally, it is clear that the expression $\pm \frac{1}{6} \Delta$ expresses generally the volume of any pyramid contained between the origin of coordinates and the three points whose coordinates are z, y, z ; x', y', z' ; x'', y'', z''.

3.

A curved surface is said to possess continuous curvature at one of its points A, if the directions of all the straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A. This plane is said to *touch* the surface at the point A. If this condition is not satisfied for any point, the continuity of the curvature is here interrupted, as happens, for example, at the vertex of a cone. The following investigations will be restricted to such surfaces, or to such parts of surfaces, as have the continuity of their curvature nowhere interrupted. We shall only observe now that the methods used to determine the position of the tangent plane lose their meaning at singular points, in which the continuity of the curvature is interrupted, and must lead to indeterminate solutions.

4.

The orientation of the tangent plane is most conveniently studied by means of the direction of the straight line normal to the plane at the point A, which is also called the normal to the curved surface at the point A. We shall represent the direction of this normal by the point L on the auxiliary sphere, and we shall set

$$\cos(1)L = X, \quad \cos(2)L = Y, \quad \cos(3)L = Z$$

coordinatas puncti A per x, y, z denotamus. Sint porro $x + dx$, $y + dy$, $z + dz$ coordinatae alius puncti in superficie curva A'; ds ipsius distantia infinite parva ab A; denique λ punctum superficiei sphaericae repraesentans directionem elementi AA'. Erit itaque

$$dx = ds.\cos(1)\lambda, \quad dy = ds.\cos(2)\lambda, \quad dz = ds.\cos(3)\lambda$$

et quum esse debeat $\lambda L = 90^0$,

$$X\cos(1)\lambda + Y\cos(2)\lambda + Z\cos(3)\lambda = 0$$

E combinatione harum aequationum derivamus

$$Xdx + Ydy + Zdz = 0$$

Duae habentur methodi generales ad exhibendam indolem superficiei curvae. Methodus *prima* utitur aequatione inter coordinatas x, y, z, quam reductam esse supponemus ad formam $W = 0$, ubi W erit functio indeterminatarum x, y, z. Sit differentiale completum functionis W

$$dW = Pdx + Qdy + Rdz$$

eritque in superficie curva

$$Pdx + Qdy + Rdz = 0$$

et proin

$$P\cos(1)\lambda + Q\cos(2)\lambda + R\cos(3)\lambda = 0$$

Quum haec aequatio, perinde ut ea quam supra stabilivimus, valere debeat pro directionibus omnium elementorum ds in superficie curva, facile perspiciemus, X, Y, Z proportionales esse debere ipsis P, Q, R et proin, quum fiat

$$XX + YY + ZZ = 1$$

erit vel

$$X = \frac{P}{\sqrt{(PP + QQ + RR)}}, \quad Y = \frac{Q}{\sqrt{(PP + QQ + RR)}}, \quad Z = \frac{R}{\sqrt{(PP + QQ + RR)}}$$

vel

$$X = \frac{-P}{\sqrt{(PP + QQ + RR)}}, \quad Y = \frac{-Q}{\sqrt{(PP + QQ + RR)}}, \quad Z = \frac{-R}{\sqrt{(PP + QQ + RR)}}$$

General Investigations
of
Curved Surfaces

$$\cos(1)L = X, \quad \cos(2)L = Y, \quad \cos(3)L = Z;$$

and denote the coordinates of the point A by x, y, z. Also let $x + dx, y + dy, z + dz$ be the coordinates of another point A' on the curved surface; ds its distance from A, which is infinitely small; and finally, let λ be the point on the sphere representing the direction of the element AA'. Then we shall have

$$dx = ds \,.\, \cos(1)\lambda, \quad dy = ds \,.\, \cos(2)\lambda, \quad dz = ds \,.\, \cos(3)\lambda$$

and, since $\lambda\, L$ must be equal to $90°$,

$$X\cos(1)\lambda + Y\cos(2)\lambda + Z\cos(3)\lambda = 0$$

By combining these equations we obtain

$$X\,dx + Y\,dy + Z\,dz = 0.$$

There are two general methods for defining the nature of a curved surface. The *first* uses the equation between the coordinates x, y, z, which we may suppose reduced to the form $W = 0$, where W will be a function of the indeterminates x, y, z. Let the complete differential of the function W be

$$dW = P\,dx + Q\,dy + R\,dz$$

and on the curved surface we shall have

$$P\,dx + Q\,dy + R\,dz = 0$$

and consequently,

$$P\cos(1)\lambda + Q\cos(2)\lambda + R\cos(3)\lambda = 0$$

Since this equation, as well as the one we have established above, must be true for the directions of all elements ds on the curved surface, we easily see that X, Y, Z must be proportional to P, Q, R respectively, and consequently, since

$$X^2 + Y^2 + Z^2 = 1,$$

we shall have either

$$X = \frac{P}{\sqrt{(P^2 + Q^2 + R^2)}}, \quad Y = \frac{Q}{\sqrt{(P^2 + Q^2 + R^2)}}, \quad Z = \frac{R}{\sqrt{(P^2 + Q^2 + R^2)}}$$

or

$$X = \frac{-P}{\sqrt{(P^2 + Q^2 + R^2)}}, \quad Y = \frac{-Q}{\sqrt{(P^2 + Q^2 + R^2)}}, \quad Z = \frac{-R}{\sqrt{(P^2 + Q^2 + R^2)}}$$

Disquisitiones Generales circa Superficies Curvas

Methodus *secunda* sistit coordinatas in forma functionum duarum variabilium p, q. Supponamus per differentiationem harum functionum prodire

$$\mathrm{d}x = a\,\mathrm{d}p + a'\mathrm{d}q$$
$$\mathrm{d}y = b\,\mathrm{d}p + b'\mathrm{d}q$$
$$\mathrm{d}z = c\,\mathrm{d}p + c'\mathrm{d}q$$

quibus valoribus in formula supra data substitutis, obtinemus

$$(aX + bY + cZ)\,\mathrm{d}p + (a'X + b'Y + c'Z)\,\mathrm{d}q = 0$$

Quum haec aequatio locum habere debeat independenter a valoribus differentialium $\mathrm{d}p$, $\mathrm{d}q$, manifesto esse debebit

$$aX + bY + cZ = 0, \quad a'X + b'Y + c'Z = 0$$

unde colligimus, X, Y, Z proportionales esse debere quantitatibus

$$bc' - cb', \quad ca' - ac', \quad ab' - ba'$$

Statuendo itaque brevitatis caussa

$$\sqrt{((bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2)} = \Delta$$

erit vel

$$X = \frac{bc' - cb'}{\Delta}, \quad Y = \frac{ca' - ac'}{\Delta}, \quad Z = \frac{ab' - ba'}{\Delta}$$

vel

$$X = \frac{cb' - bc'}{\Delta}, \quad Y = \frac{ac' - ca'}{\Delta}, \quad Z = \frac{ba' - ab'}{\Delta}$$

His duabus methodis generalibus accedit *tertia*, ubi una coordinatarum, e. g. z exhibetur in forma functionis reliquarum x, y: haec methodus manifesto nihil aliud est, nisi casus specialis vel methodi primae, vel secundae. Quodsi hic statuitur

$$\mathrm{d}z = t\,\mathrm{d}x + u\,\mathrm{d}y$$

erit vel

$$X = \frac{-t}{\sqrt{(1 + tt + uu)}}, \quad Y = \frac{-u}{\sqrt{(1 + tt + uu)}}, \quad Z = \frac{1}{\sqrt{(1 + tt + uu)}}$$

vel

$$X = \frac{t}{\sqrt{(1 + tt + uu)}}, \quad Y = \frac{u}{\sqrt{(1 + tt + uu)}}, \quad Z = \frac{-1}{\sqrt{(1 + tt + uu)}}$$

General Investigations of Curved Surfaces

The *second* method expresses the coordinates in the form of functions of two variables, p, q. Suppose that differentiation of these functions gives

$$dx = a\,dp + a'\,dq$$
$$dy = b\,dp + b'\,dq$$
$$dz = c\,dp + c'\,dq$$

Substituting these values in the formula given above, we obtain

$$(a\,X + b\,Y + c\,Z)\,dp + (a'\,X + b'\,Y + c'\,Z)\,dq = 0$$

Since this equation must hold independently of the values of the differentials dp, dq, we evidently shall have

$$a\,X + b\,Y + c\,Z = 0, \quad a'\,X + b'\,Y + c'\,Z = 0$$

From this we see that X, Y, Z will be proportioned to the quantities

$$bc' - cb', \quad ca' - ac', \quad ab' - ba'$$

Hence, on setting, for brevity,

$$\sqrt{\left((bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2\right)} = \Delta$$

we shall have either

$$X = \frac{bc' - cb'}{\Delta}, \quad Y = \frac{ca' - ac'}{\Delta}, \quad Z = \frac{ab' - ba'}{\Delta}$$

or

$$X = \frac{cb' - bc'}{\Delta}, \quad Y = \frac{ac' - ca'}{\Delta}, \quad Z = \frac{ba' - ab'}{\Delta}$$

With these two general methods is associated a *third*, in which one of the coordinates, z, say, is expressed in the form of a function of the other two, x, y. This method is evidently only a particular case either of the first method, or of the second. If we set

$$dz = t\,dx + u\,dy$$

we shall have either

$$X = \frac{-t}{\sqrt{(1 + t^2 + u^2)}}, \quad Y = \frac{-u}{\sqrt{(1 + t^2 + u^2)}}, \quad Z = \frac{1}{\sqrt{(1 + t^2 + u^2)}}$$

or

$$X = \frac{t}{\sqrt{(1 + t^2 + u^2)}}, \quad Y = \frac{u}{\sqrt{(1 + t^2 + u^2)}}, \quad Z = \frac{-1}{\sqrt{(1 + t^2 + u^2)}}$$

Disquisitiones Generales circa Superficies Curvas

5.

Duae solutiones in art. praec. inventae manifesto ad puncta superficiei sphaericae opposita, sive ad directiones oppositas referuntur, quod cum rei natura quadrat, quum normalem ad utramvis plagam superficiei curvae ducere liceat. Quodsi duas plagas, superficiei contiguas, inter se distinguere, alteramque exteriorem alteram interiorem vocare placet, etiam utrique normali suam solutionem rite tribuere licebit adiumento theorematis in art. 2 (VII) evoluti, simulatque criterium stabilitum est ad plagam alteram ab altera distinguendam.

In methodo prima tale criterium petendum erit a signo valoris quantitatis W. Scilicet generaliter loquendo superficies curva eas spatii partes, in quibus W valorem positivum obtinet, ab iis dirimet, in quibus valor ipsius W fit negativus. E theoremate illo vero facile colligitur, si W valorem positivum obtineat versus plagam exteriorem, normalisque extrorsum ducta concipiatur, solutionem priorem adoptandam esse. Ceterum in quovis casu facile diiudicabitur, utrum per superficiem integram eadem regula respectu signi ipsius W valeat, an pro diversis partibus diversae: quamdiu coëfficientes P, Q, R valores finitos habent, nec simul omnes tres evanescunt, lex continuitatis vicissitudinem vetabit.

Si methodum secundam sequimur, in superficie curva duo systemata linearum curvarum concipere possumus, alterum, pro quo p est variabilis, q constans; alterum, pro quo q variabilis, p constans: situs mutuus harum linearum respectu plagae exterioris decidere debet, utram solutionem adoptare oporteat. Scilicet quoties tres lineae, puta ramus lineae prioris systematis a puncto A proficiscens crescente p, ramus posterioris systematis a puncto A egrediens crescente q, atque normalis versus plagam exteriorem ducta *similiter* iacent, ut, inde ab origine abscissarum, axes ipsarum x, y, z resp. (e. g. si tum e tribus lineis illis, tum e tribus his, prima sinistrorsum, secunda dextrorsum, tertia sursum directa concipi potest), solutio prima adoptari debet; quoties autem situs mutuus trium linearum oppositus est situi mutuo axium ipsarum x, y, z, solutio secunda valebit.

In methodo tertia dispiciendum est, utrum, dum z incrementum positivum accipit, manentibus x et y invariatis, transitus fiat versus plagam exteriorem an interiorem. In casu priore, pro normali extrorsum directa, solutio prima valet, in posteriore secunda.

General Investigations of Curved Surfaces

5.

The two solutions found in the preceding article evidently refer to opposite points of the sphere, or to opposite directions, as one would expect, since the normal may be drawn toward either of the two sides of the curved surface. If we wish to distinguish between the two regions bordering upon the surface, and call one the exterior region and the other the interior region, we can then assign to each of the two normals its appropriate solution by aid of the theorem derived in Art. 2 (VII), and at the same time establish a criterion for distinguishing the one region from the other.

In the first method, such a criterion is to be drawn from the sign of the quantity W. Indeed, generally speaking, the curved surface divides those regions of space in which W keeps a positive value from those in which the value of W becomes negative. In fact, it is easily seen from this theorem that, if W takes a positive value toward the exterior region, and if the normal is supposed to be drawn outwardly, the first solution is to be taken. Moreover, it will be easy to decide in any case whether the same rule for the sign of W is to hold throughout the entire surface, or whether for different parts there will be different rules. As long as the coefficients P, Q, R have finite values and do not all vanish at the same time, the law of continuity will prevent any change.

If we follow the second method, we can imagine two systems of curved lines on the curved surface, one system for which p is variable, q constant; the other for which q is variable, p constant. The respective positions of these lines with reference to the exterior region will decide which of the two solutions must be taken. In fact, whenever the three lines, namely, the branch of the line of the former system going out from the point A as p increases, the branch of the line of the latter system going out from the point A as q increases, and the normal drawn toward the exterior region, are *similarly* placed as the x, y, z axes respectively from the origin of abscissas (*e. g.*, if, both for the former three lines and for the latter three, we can conceive the first directed to the left, the second to the right, and the third upward), the first solution is to be taken. But whenever the relative position of the three lines is opposite to the relative position of the x, y, z axes, the second solution will hold.

In the third method, it is to be seen whether, when z receives a positive increment, x and y remaining constant, the point crosses toward the exterior or the interior region. In the former case, for the normal drawn outward, the first solution holds; in the latter case, the second.

Disquisitiones Generales circa Superficies Curvas

6.

Sicuti, per translatam directionem normalis in superficiem curvam ad superficiem sphaerae, cuius puncto determinato prioris superficiei respondet punctum determinatum in posteriore, ita etiam quaevis linea, vel quaevis figura in illa repraesentabitur per lineam vel figuram correspondentem in hac. In comparatione duarum figurarum hoc modo sibi mutuo correspondentium, quarum altera quasi imago alterius erit, duo momenta sunt respicienda, alterum, quatenus sola quantitas consideratur, alterum, quatenus abstrahendo a relationibus quantitativis solum situm contemplamur.

Momentum primum basis erit quarundam notionum, quas in doctrinam de superficiebus curvis recipere utile videtur. Scilicet cuilibet parti superficiei curvae limitibus determinatis cinctae *curvaturam totalem* seu *integram* adscribemus, quae per aream figurae illi in superficie sphaerica respondentem exprimetur. Ab hac curvatura integra probe distinguenda est curvatura quasi specifica, quam nos *mensuram curvaturae* vocabimus: haec posterior ad *punctum* superficiei refertur, et denotabit quotientem qui oritur, dum curvatura integra elementi superficialis puncto adiacentis per aream ipsius elementi dividitur, et proin indicat rationem arearum infinite parvarum in superficie curva et in superficie sphaerica sibi mutuo respondentium. Utilitas harum innovationum per ea, quae in posterum a nobis explicabuntur, abunde, ut speramus, sancietur. Quod vero attinet ad terminologiam, imprimis prospiciendum esse duximus, ut omnis ambiguitas arceatur, quapropter haud congruum putavimus, analogiam terminologiae in doctrina de lineis curvis planis vulgo receptam (etsi non omnibus probatam) stricte sequi, secundum quam mensura curvaturae simpliciter audire debuisset curvatura, curvatura integra autem amplitudo. Sed quidni in verbis faciles esse liceret, dummodo res non sint inanes, neque dictio interpretationi erroneae obnoxia?

Situs figurae in superficie sphaerica vel similis esse potest situi figurae respondentis in superficie curva, vel oppositus (inversus); casus prior locum habet, ubi binae lineae in superficie curva ab eodem puncto directionibus inaequalibus sed non oppositis proficiscentes repraesentantur in superficie sphaerica per lineas similiter iacentes, puta ubi imago lineae ad dextram iacentis ipsa est ad dextram; casus posterior, ubi contrarium valet. Hos duos casus per *signum* mensurae curvaturae vel positivum vel negativum distinguemus. Sed manifesto haec distinctio eatenus tantum locum habere potest, quatenus in utraque superficie pla-

General Investigations of Curved Surfaces

6.

Just as each definite point on the curved surface is made to correspond to a definite point on the sphere, by the direction of the normal to the curved surface which is transferred to the surface of the sphere, so also any line whatever, or any figure whatever, on the latter will be represented by a corresponding line or figure on the former. In the comparison of two figures corresponding to one another in this way, one of which will be as the map of the other, two important points are to be considered, one when quantity alone is considered, the other when, disregarding quantitative relations, position alone is considered.

The first of these important points will be the basis of some ideas which it seems judicious to introduce into the theory of curved surfaces. Thus, to each part of a curved surface inclosed within definite limits we assign a *total* or *integral curvature*, which is represented by the area of the figure on the sphere corresponding to it. From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the *measure of curvature*. The latter refers to a *point* of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; and hence it denotes the ratio of the infinitely small areas which correspond to one another on the curved surface and on the sphere. The use of these innovations will be abundantly justified, as we hope, by what we shall explain below. As for the terminology, we have thought it especially desirable that all ambiguity be avoided. For this reason we have not thought it advantageous to follow strictly the analogy of the terminology commonly adopted (though not approved by all) in the theory of plane curves, according to which the measure of curvature should be called simply curvature, but the total curvature, the amplitude. But why not be free in the choice of words, provided they are not meaningless and not liable to a misleading interpretation ?

The position of a figure on the sphere can be either similar to the position of the corresponding figure on the curved surface, or opposite (inverse). The former is the case when two lines going out on the curved surface from the same point in different, but not opposite directions, are represented on the sphere by lines similarly placed, that is, when the map of the line to the right is also to the right; the latter is the case when the contrary holds. We shall distinguish these two cases by the positive or negative *sign* of the measure of curvature. But evidently this distinction can hold only when on each surface we choose a definite face on which we suppose the figure to lie. On the auxiliary sphere we shall use always the exterior face, that is, that turned away from the centre; on the curved surface also there may be taken for the exterior face the one already considered, or rather that face from which the normal is supposed to be drawn. For, evidently, there is no change in regard to the similitude of the figures, if on the curved surface both the gam determinatam eligimus, in qua figura concipi debet. In sphaera auxiliari semper plagam exteriorem, a centro aversam, adhibebimus: in superficie curva etiam plaga exterior sive quae tamquam exterior consideratur, adoptari potest, vel potius plaga eadem, a qua normalis erecta concipitur; manifesto enim respectu similitudinis figurarum nihil mutatur, si in superficie curva tum figura ad plagam oppositam transfertur, tum normalis, dummodo ipsius imago semper in eadem plaga superficiei sphaericae depingatur.

Signum positivum vel negativum, quod pro situ figurae infinite parvae *mensurae* curvaturae adscribimus, etiam ad curvaturam integram figurae finitae in superficie curva extendimus. Attamen si argumentum omni generalitate amplecti suscipimus, quaedam dilucidationes requiruntur, quas hic breviter tantum attingemus. Quamdiu figura in superficie curva ita comparata est, ut singulis punctis intra ipsam puncta *diversa* in superficie sphaerica respondeant, definitio ulteriore explicatione non indiget. Quoties autem conditio ista locum non habet, necesse erit, quasdam partes figurae in superficie sphaerica bis vel pluries in computum ducere, unde, pro situ simili vel opposito, vel accumulatio vel destructio oriri poterit. Simplicissimum erit in tali casu, figuram in superficie curva in partes tales divisam concipere, quae singulae per se spectatae conditioni illi satisfaciant, singulis tribuere curvaturam suam integram, quantitate per aream figurae in superficie sphaerica respondentis, signo per situm determinatis, ac denique figurae toti adscribere curvaturam integram ortam per additionem curvaturarum integrarum, quae singulis partibus respondent. Generaliter itaque curvatura integra figurae est $= \int k \, d\sigma$, denotante $d\sigma$ elementum areae figurae, k mensuram curvaturae in quovis puncto. Quod vero attinet ad repraesentationem geometricam huius integralis, praecipua huius rei momenta ad sequentia redeunt. Peripheriae figurae in superficie curva (sub restrictione art. 3) semper respondebit in superficie sphaerica linea in se ipsam rediens. Quae si se ipsam nullibi intersecat, totam superficiem sphaericam in duas partes dirimet, quarum altera respondebit figurae in superficie curva, et cuius area, positive vel negative accipienda, prout respectu peripheriae suae similiter iacet ut figura in superficie curva respectu suae, vel inverse, exhibebit posterioris curvaturam integram. Quoties vero linea ista se ipsam semel vel pluries secat, exhibebit figuram complicatam, cui tamen area certa aeque legitime tribui potest, ac figuris absque nodis, haecque area, rite intellecta, semper valorem iustum curvaturae integrae exhibebit. Attamen uberiofigure and the normal be transferred to the opposite side, so long as the image itself is represented on the same side of the sphere.

The positive or negative sign, which we assign to the *measure* of curvature according to the position of the infinitely small figure, we extend also to the integral curvature of a finite figure on the curved surface. However, if we wish to discuss the general case, some explanations will be necessary, which we can only touch here briefly. So long as the figure on the curved surface is such that to *distinct* points on itself there correspond distinct points on the sphere, the definition needs no further explanation. But whenever this condition is not satisfied, it will be necessary to take into account twice or several times certain parts of the figure on the sphere. Whence for a similar, or inverse position, may arise an accumulation of areas, or the areas may partially or wholly destroy each other. In such a case, the simplest way is to suppose the curved surface divided into parts, such that each part, considered separately, satisfies the above condition; to assign to each of the parts its integral curvature, determining this magnitude by the area of the corresponding figure on the sphere, and the sign by the position of this figure; and, finally, to assign to the total figure the integral curvature arising from the addition of the integral curvatures which correspond to the single parts. So, generally, the integral curvature of a figure is equal to $\int k\, d\sigma$, $d\sigma$ denoting the element of area of the figure, and k the measure of curvature at any point. The principal points concerning the geometric representation of this integral reduce to the following. To the perimeter of the figure on the curved surface (under the restriction of Art. 3) will correspond always a closed line on the sphere. If the latter nowhere intersect itself, it will divide the whole surface of the sphere into two parts, one of which will correspond to the figure on the curved surface; and its area (taken as positive or negative according as, with respect to its perimeter, its position is similar, or inverse, to the position of the figure on the curved surface) will represent the integral curvature of the figure on the curved surface. But whenever this line intersects itself once or several times, it will give a complicated figure, to which, however, it is possible to assign a definite area as legitimately as in the case of a figure without nodes; and this area, properly interpreted, will give always an exact value for the rem huius argumenti de figuris generalissime conceptis expositionem ad aliam occasionem nobis reservare debemus.

7.

Investigemus iam formulam ad exprimendam mensuram curvaturae pro quovis puncto superficiei curvae. Denotante $\mathrm{d}\sigma$ aream elementi huius superficiei, $Z\mathrm{d}\sigma$ erit area proiectionis huius elementi in planum coordinatarum x, y; et perinde, si $\mathrm{d}\Sigma$ est area elementi respondentis in superficie sphaerica, erit $Z\mathrm{d}\Sigma$ area proiectionis ad idem planum: signum positivum vel negativum ipsius Z vero indicabit situm proiectionis similem vel oppositum situi elementi proiecti: manifesto itaque illae proiectiones eandem rationem quoad quantitatem, simulque eandem relationem quoad situm, inter se tenent, ut elementa ipsa. Consideremus iam elementum triangulare in superficie curva, supponamusque coordinatas trium punctorum, quae formant ipsius proiectionem, esse

$$\begin{array}{ll} x, & y \\ x+\mathrm{d}x, & y+\mathrm{d}y \\ x+\delta x, & y+\delta y \end{array}$$

Duplex area huius trianguli exprimetur per formulam

$$\mathrm{d}x.\delta y - \mathrm{d}y.\delta x$$

et quidem in forma positiva vel negativa, prout situs lateris a puncto primo ad tertium respectu lateris a puncto primo ad secundum similis vel oppositus est situi axis coordinatarum y respectu axis coordinatarum x.

Perinde si coordinatae trium punctorum, quae formant proiectionem elementi respondentis in superficie sphaerica, a centro sphaerae inchoatae, sunt

$$\begin{array}{ll} \mathbf{X}, & \boldsymbol{Y} \\ \mathbf{X}+\mathrm{d}\mathbf{X}, & \boldsymbol{Y}+\mathrm{d}\boldsymbol{Y} \\ \mathbf{X}+\delta\mathbf{X}, & \boldsymbol{Y}+\delta\boldsymbol{Y} \end{array}$$

duplex area huius proiectionis exprimetur per

$$\mathrm{d}\mathbf{X}.\delta\boldsymbol{Y} - \mathrm{d}\boldsymbol{Y}.\delta\mathbf{X}$$

de cuius expressionis signo eadem valent quae supra. Quocirca mensura curva-

General Investigations of Curved Surfaces

integral curvature. However, we must reserve for another occasion the more extended exposition of the theory of these figures viewed from this very general standpoint.

7.

We shall now find a formula which will express the measure of curvature for any point of a curved surface. Let $d\sigma$ denote the area of an element of this surface; then $Z\,d\sigma$ will be the area of the projection of this element on the plane of the coordinates x, y; and consequently, if $d\Sigma$ is the area of the corresponding element on the sphere, $Z\,d\Sigma$ will be the area of its projection on the same plane. The positive or negative sign of Z will, in fact, indicate that the position of the projection is similar or inverse to that of the projected element. Evidently these projections have the same ratio as to quantity and the same relation as to position as the elements themselves. Let us consider now a triangular element on the curved surface, and let us suppose that the coordinates of the three points which form its projection are

$$\begin{array}{ll} x, & y \\ x + dx, & y + dy \\ x + \delta x, & y + \delta y \end{array}$$

The double area of this triangle will be expressed by the formula

$$dx\,.\,\delta y - dy\,.\,\delta x$$

and this will be in a positive or negative form according as the position of the side from the first point to the third, with respect to the side from the first point to the second, is similar or opposite to the position of the y-axis of coordinates with respect to the x-axis of coordinates.

In like manner, if the coordinates of the three points which form the projection of the corresponding element on the sphere, from the centre of the sphere as origin, are

$$\begin{array}{ll} X, & Y \\ X + dX, & Y + dY \\ X + \delta X, & Y + \delta Y \end{array}$$

the double area of this projection will be expressed by

$$dX\,.\,\delta Y - dY\,.\,\delta X$$

and the sign of this expression is determined in the same manner as above. Whereturae in hoc loco superficiei curvae erit

$$k = \frac{dX.\delta Y - dY.\delta X}{dx.\delta y - dy.\delta x}$$

Quodsi iam supponimus, indolem superficiei curvae datam esse secundum modum tertium in art. 4 consideratum, habebuntur X et Y in forma functionum quantitatum x, y, unde erit

$$\begin{aligned} dX &= (\tfrac{dX}{dx})dx + (\tfrac{dX}{dy})dy \\ \delta X &= (\tfrac{dX}{dx})\delta x + (\tfrac{dX}{dy})\delta y \\ dY &= (\tfrac{dY}{dx})dx + (\tfrac{dY}{dy})dy \\ \delta Y &= (\tfrac{dY}{dx})\delta x + (\tfrac{dY}{dy})\delta y \end{aligned}$$

Substitutis his valoribus, expressio praecedens transit in hanc:

$$k = (\tfrac{dX}{dx})(\tfrac{dY}{dy}) - (\tfrac{dX}{dy})(\tfrac{dY}{dx})$$

Statuendo ut supra

$$\tfrac{dz}{dx} = t, \quad \tfrac{dz}{dy} = u$$

atque insuper

$$\tfrac{ddz}{dx^2} = T, \quad \tfrac{ddz}{dx.dy} = U, \quad \tfrac{ddz}{dy^2} = V$$

sive

$$dt = Tdx + Udy, \quad du = Udx + Vdy$$

habemus ex formulis supra datis

$$X = -tZ, \quad Y = -uZ, \quad (1+tt+uu)ZZ = 1$$

atque hinc

$$\begin{aligned} dX &= -Zdt - tdZ \\ dY &= -Zdu - udZ \\ (1+tt+uu)dZ &+ Z(tdt+udu) = 0 \end{aligned}$$

sive

$$\begin{aligned} dZ &= -Z^3(tdt+udu) \\ dX &= -Z^3(1+uu)dt + Z^3tudu \\ dY &= +Z^3tudt - Z^3(1+tt)du \end{aligned}$$

fore the measure of curvature at this point of the curved surface will be

$$k = \frac{dX \,.\, \delta Y - dY \,.\, \delta X}{dx \,.\, \delta y - dy \,.\, \delta x}$$

If now we suppose the nature of the curved surface to be defined according to the third method considered in Art. 4, X and Y will be in the form of functions of the quantities x, y. We shall have, therefore,

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy$$

$$\delta X = \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y$$

$$dY = \frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy$$

$$\delta Y = \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y$$

When these values have been substituted, the above expression becomes

$$k = \frac{\partial X}{\partial x}\cdot\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\cdot\frac{\partial Y}{\partial x}$$

Setting, as above,

$$\frac{\partial z}{\partial x} = t, \quad \frac{\partial z}{\partial y} = u$$

and also

$$\frac{\partial^2 z}{\partial x^2} = T, \quad \frac{\partial^2 z}{\partial x \,.\, \partial y} = U, \quad \frac{\partial^2 z}{\partial y^2} = V$$

or

$$dt = T\,dx + U\,dy, \quad du = U\,dx + V\,dy$$

we have from the formulæ given above

$$X = -tZ, \quad Y = -uZ, \quad (1 + t^2 + u^2)\,Z^2 = 1$$

and hence

$$dX = -Z\,dt - t\,dZ$$
$$dY = -Z\,du - u\,dZ$$
$$(1 + t^2 + u^2)\,dZ + Z\,(t\,dt + u\,du) = 0$$

or

$$dZ = -Z^3\,(t\,dt + u\,du)$$
$$dX = -Z^3\,(1 + u^2)\,dt + Z^3\,tu\,du$$
$$dY = +Z^3\,tu\,dt - Z^3\,(1 + t^2)\,du$$

adeoque

$$\frac{dX}{dx} = Z^3(-(1+uu)T + tuU)$$
$$\frac{dX}{dy} = Z^3(-(1+uu)U + tuV)$$
$$\frac{dY}{dx} = Z^3(tuT - (1+tt)U)$$
$$\frac{dY}{dy} = Z^3(tuU - (1+tt)V)$$

quibus valoribus in expressione praecedente substitutis, prodit

$$k = Z^6(TV - UU)(1+tt+uu) = Z^4(TV - UU) = \frac{TV - UU}{(1+tt+uu)^2}$$

8.

Per idoneam electionem initii et axium coordinatarum facile effici potest, ut pro puncto determinato A valores quantitatum t, u, U evanescant. Scilicet duae priores conditiones iam adimplentur, si planum tangens in hoc puncto pro plano coordinatarum x, y adoptatur. Quarum initium si insuper in puncto A ipso collocatur, manifesto expressio coordinatarum z adipiscitur formam talem

$$z = \tfrac{1}{2}T^0xx + U^0xy + \tfrac{1}{2}V^0yy + \Omega$$

ubi Ω erit ordinis altioris quam secundi. Mutando dein situm axium ipsarum x, y angulo M tali ut habeatur

$$\operatorname{tang} 2M = \frac{2U^0}{T^0 - V^0}$$

facile perspicitur, prodituram esse aequationem huius formae

$$z = \tfrac{1}{2}Txx + \tfrac{1}{2}Vyy + \Omega$$

quo pacto etiam tertiae conditioni satisfactum est. Quibus ita factis, patet

I. Si superficies curva secetur plano ipsi normali et per axem coordinatarum x transeunte, oriri curvam planam, cuius radius curvaturae in puncto A fiat $= \frac{1}{T}$, signo positivo vel negativo indicante concavitatem vel convexitatem versus plagam eam, versus quam coordinatae z sunt positivae.

II. Simili modo $\frac{1}{V}$ erit in puncto A radius curvaturae curvae planae, quae oritur per sectionem superficiei curvae cum plano per axes ipsarum y, z transeunte.

and so

$$\frac{\partial X}{\partial x} = Z^3(-(1+u^2)\,T + tu\,U)$$

$$\frac{\partial X}{\partial y} = Z^3(-(1+u^2)\,U + tu\,V)$$

$$\frac{\partial Y}{\partial x} = Z^3(tu\,T - (1+t^2)U)$$

$$\frac{\partial Y}{\partial y} = Z^3(tu\,U - (1+t^2)V)$$

Substituting these values in the above expression, it becomes

$$k = Z^6(TV - U^2)(1+t^2+u^2) = Z^4(TV - U^2)$$
$$= \frac{TV - U^2}{(1+t^2+u^2)^2}$$

8.

By a suitable choice of origin and axes of coordinates, we can easily make the values of the quantities t, u, U vanish for a definite point A. Indeed, the first two conditions will be fulfilled at once if the tangent plane at this point be taken for the xy-plane. If, further, the origin is placed at the point A itself, the expression for the coordinate z evidently takes the form

$$z = \tfrac{1}{2}\,T^\circ x^2 + U^\circ xy + \tfrac{1}{2}\,V^\circ y^2 + \Omega$$

where Ω will be of higher degree than the second. Turning now the axes of x and y through an angle M such that

$$\tan 2\,M = \frac{2\,U^\circ}{T^\circ - V^\circ}$$

it is easily seen that there must result an equation of the form

$$z = \tfrac{1}{2}\,Tx^2 + \tfrac{1}{2}\,Vy^2 + \Omega$$

In this way the third condition is also satisfied. When this has been done, it is evident that

I. If the curved surface be cut by a plane passing through the normal itself and through the x-axis, a plane curve will be obtained, the radius of curvature of which at the point A will be equal to $\frac{1}{T}$, the positive or negative sign indicating that the curve is concave or convex toward that region toward which the coordinates z are positive.

II. In like manner $\frac{1}{V}$ will be the radius of curvature at the point A of the plane curve which is the intersection of the surface and the plane through the y-axis and the z-axis.

Disquisitiones Generales circa Superficies Curvas

III. Statuendo $x = r\cos\varphi$, $y = r\sin\varphi$, fit

$$z = \tfrac{1}{2}(T\cos\varphi^2 + V\sin\varphi^2)rr + \Omega$$

unde colligitur, si sectio fiat per planum superficiei in A normale et cum axe ipsarum x angulum φ efficiens, oriri curvam planam, cuius radius curvaturae in puncto A sit

$$= \frac{1}{T\cos\varphi^2 + V\sin\varphi^2}$$

IV. Quoties itaque habetur $T = V$, radii curvaturae in *cunctis* planis normalibus aequales erunt. Si vero T et V sunt inaequales, manifestum est, quum $T\cos\varphi^2 + V\sin\varphi^2$ pro quovis valore anguli φ cadat intra T et V, radios curvaturae in sectionibus principalibus, in I et II consideratis, referri ad curvaturas extremas, puta alterum ad curvaturam maximam, alterum ad minimam, si T et V eodem signo affectae sint, contra alterum ad maximam convexitatem, alterum ad maximam concavitatem, si T et V signis oppositis gaudeant. Hae conclusiones omnia fere continent, quae ill. Euler de curvatura superficierum curvarum primus docuit.

V. Mensura curvaturae superficiei curvae in puncto A autem nanciscitur expressionem simplicissimam $k = TV$, unde habemus

Theorema. *Mensura curvaturae in quovis superficiei puncto aequalis est fractioni, cuius numerator unitas, denominator autem productum duorum radiorum curvaturae extremorum in sectionibus per plana normalia.*

Simul patet, mensuram curvaturae fieri positivam pro superficiebus concavo-concavis vel convexo-convexis (quod discrimen non est essentiale), negativam vero pro concavo-convexis. Si superficies constat e partibus utriusque generis, in earum confiniis mensura curvaturae evanescens esse debebit. De indole superficierum curvarum talium, in quibus mensura curvaturae ubique evanescit, infra pluribus agetur.

9.

Formula generalis pro mensura curvaturae in fine art. 7 proposita, omnium simplicissima est, quippe quae quinque tantum elementa implicat; ad magis complicatam, scilicet novem elementa involventem, deferimur, si adhibere volumus

III. Setting $x = r\cos\phi$, $y = r\sin\phi$, the equation becomes

$$z = \tfrac{1}{2}(T\cos^2\phi + V\sin^2\phi)\,r^2 + \Omega$$

from which we see that if the section is made by a plane through the normal at A and making an angle ϕ with the x-axis, we shall have a plane curve whose radius of curvature at the point A will be

$$\frac{1}{T\cos^2\phi + V\sin^2\phi}$$

IV. Therefore, whenever we have $T = V$, the radii of curvature in *all* the normal planes will be equal. But if T and V are not equal, it is evident that, since for any value whatever of the angle ϕ, $T\cos^2\phi + V\sin^2\phi$ falls between T and V, the radii of curvature in the principal sections considered in I. and II. refer to the extreme curvatures; that is to say, the one to the maximum curvature, the other to the minimum, if T and V have the same sign. On the other hand, one has the greatest convex curvature, the other the greatest concave curvature, if T and V have opposite signs. These conclusions contain almost all that the illustrious Euler was the first to prove on the curvature of curved surfaces.

V. The measure of curvature at the point A on the curved surface takes the very simple form

$$k = TV,$$

whence we have the

THEOREM. *The measure of curvature at any point whatever of the surface is equal to a fraction whose numerator is unity, and whose denominator is the product of the two extreme radii of curvature of the sections by normal planes.*

At the same time it is clear that the measure of curvature is positive for concavo-concave or convexo-convex surfaces (which distinction is not essential), but negative for concavo-convex surfaces. If the surface consists of parts of each kind, then on the lines separating the two kinds the measure of curvature ought to vanish. Later we shall make a detailed study of the nature of curved surfaces for which the measure of curvature everywhere vanishes.

9.

The general formula for the measure of curvature given at the end of Art. 7 is the most simple of all, since it involves only five elements. We shall arrive at a more complicated formula, indeed, one involving nine elements, if we wish to use the modum primum indolem superficiei curvae exprimendi. Retinendo notationes art. 4 insuper statuemus:

$$\frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x^2} = P', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}y^2} = Q', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}z^2} = R'$$
$$\frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}y\,.\,\mathrm{d}z} = P'', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x\,.\,\mathrm{d}z} = Q'', \qquad \frac{\mathrm{d}\,\mathrm{d}W}{\mathrm{d}x\,.\,\mathrm{d}y} = R''$$

ita ut fiat

$$\begin{aligned} \mathrm{d}P &= P'\mathrm{d}x + R''\mathrm{d}y + Q''\mathrm{d}z \\ \mathrm{d}Q &= R''\mathrm{d}x + Q'\mathrm{d}y + P''\mathrm{d}z \\ \mathrm{d}R &= Q''\mathrm{d}x + P''\mathrm{d}y + R'\mathrm{d}z \end{aligned}$$

Iam quum habeatur $t = -\frac{P}{R}$, invenimus per differentiationem

$$RR\,\mathrm{d}t = -R\mathrm{d}P + P\mathrm{d}R = (PQ'' - RP')\,\mathrm{d}x + (PP'' - RR'')\mathrm{d}y + (PR' - RQ'')\mathrm{d}z$$

sive, eliminata $\mathrm{d}z$ adiumento aequationis $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0$,

$$R^3\mathrm{d}t = (-RRP' + 2PRQ'' - PPR')\mathrm{d}x + (PRP'' + QRQ'' - PQR' - RRR'')\mathrm{d}y$$

Prorsus simili modo obtinemus

$$R^3\mathrm{d}u = (PRP'' + QRQ'' - PQR' - RRR'')\mathrm{d}x + (-RRQ' + 2QRP'' - QQR')\mathrm{d}y$$

Hinc itaque colligimus

$$\begin{aligned} R^3\,T &= -RRP' + 2PRQ'' - PPR' \\ R^3\,U &= PRP'' + QRQ'' - PQR' - RRR'' \\ R^3\,V &= -RRQ' + 2QRP'' - QQR' \end{aligned}$$

Substituendo hos valores in formula art. 7, obtinemus pro mensura curvaturae k expressionem symmetricam sequentem:

$$\begin{aligned} &(PP + QQ + RR)^2 k \\ &\quad = PP(Q'R' - P''P'') + QQ(P'R' - Q''Q'') + RR(P'Q' - R''R'') \\ &\qquad + 2QR(Q''R'' - P'P'') + 2PR(P''R'' - Q'Q'') + 2PQ(P''Q'' - R'R'') \end{aligned}$$

10.

Formulam adhuc magis complicatam, puta e quindecim elementis conflatam, obtinemus, si methodum generalem secundam, indolem superficierum

General Investigations of Curved Surfaces

first method of representing a curved surface. Keeping the notation of Art. 4, let us set also

$$\frac{\partial^2 W}{\partial x^2} = P', \quad \frac{\partial^2 W}{\partial y^2} = Q', \quad \frac{\partial^2 W}{\partial z^2} = R'$$
$$\frac{\partial^2 W}{\partial y . \partial z} = P'', \quad \frac{\partial^2 W}{\partial x . \partial z} = Q'', \quad \frac{\partial^2 W}{\partial x . \partial y} = R''$$

so that

$$\begin{aligned} dP &= P'\, dx + R''\, dy + Q''\, dz \\ dQ &= R''\, dx + Q'\, dy + P''\, dz \\ dR &= Q''\, dx + P''\, dy + R'\, dz \end{aligned}$$

Now since $t = -\frac{P}{R}$, we find through differentiation

$$R^2\, dt = -R\, dP + P\, dR = (P Q'' - R P')\, dx + (P P'' - R R'')\, dy + (P R' - R Q'')\, dz$$

or, eliminating dz by means of the equation

$$P\, dx + Q\, dy + R\, dz = 0,$$
$$R^3\, dt = (-R^2 P' + 2 P R Q'' - P^2 R')\, dx + (P R P'' + Q R Q'' - P Q R' - R^2 R'')\, dy.$$

In like manner we obtain

$$R^3\, du = (P R P'' + Q R Q'' - P Q R' - R^2 R'')\, dx + (-R^2 Q' + 2\, Q R P'' - Q^2 R')\, dy$$

From this we conclude that

$$\begin{aligned} R^3 T &= -R^2 P' + 2\, P R Q'' - P^2 R' \\ R^3 U &= P R P'' + Q R Q'' - P Q R' - R^2 R'' \\ R^3 V &= -R^2 Q' + 2\, Q R P'' - Q^2 R' \end{aligned}$$

Substituting these values in the formula of Art. 7, we obtain for the measure of curvature k the following symmetric expression:

$$\begin{aligned} (P^2 + Q^2 + R^2)^2 k = {} & P^2 (Q' R' - P''^2) + Q^2 (P' R' - Q''^2) + R^2 (P' Q' - R''^2) \\ & + 2\, Q R (Q'' R'' - P' P'') + 2\, P R (P'' R'' - Q' Q'') + 2\, P Q\, (P'' Q'' - R' R'') \end{aligned}$$

10.

We obtain a still more complicated formula, indeed, one involving fifteen elements, if we follow the second general method of defining the nature of a curved surface. It

Disquisitiones Generales circa Superficies Curvas

curvarum exprimendi, sequimur. Magni tamen momenti est, hanc quoque elaborare. Retinendo signa art. 4, insuper statuemus

$$\frac{\mathrm{dd}x}{\mathrm{d}p^2} = \alpha, \quad \frac{\mathrm{dd}x}{\mathrm{d}p\,.\,\mathrm{d}q} = \alpha', \quad \frac{\mathrm{dd}x}{\mathrm{d}q^2} = \alpha''$$
$$\frac{\mathrm{dd}y}{\mathrm{d}p^2} = \beta, \quad \frac{\mathrm{dd}y}{\mathrm{d}p\,.\,\mathrm{d}q} = \beta', \quad \frac{\mathrm{dd}y}{\mathrm{d}q^2} = \beta''$$
$$\frac{\mathrm{dd}z}{\mathrm{d}p^2} = \gamma, \quad \frac{\mathrm{dd}z}{\mathrm{d}p\,.\,\mathrm{d}q} = \gamma', \quad \frac{\mathrm{dd}z}{\mathrm{d}q^2} = \gamma''$$

Praeterea brevitatis caussa faciemus

$$bc' - cb' = A$$
$$ca' - ac' = B$$
$$ab' - ba' = C$$

Primo observamus, haberi $A\mathrm{d}x + B\mathrm{d}y + C\mathrm{d}z = 0$, sive $\mathrm{d}z = -\frac{A}{C}\mathrm{d}x - \frac{B}{C}\mathrm{d}y$; quatenus itaque z spectatur tamquam functio ipsarum x, y, fit

$$\frac{\mathrm{d}z}{\mathrm{d}x} = t = -\frac{A}{C}$$
$$\frac{\mathrm{d}z}{\mathrm{d}y} = u = -\frac{B}{C}$$

Porro deducimus, ex $\mathrm{d}x = a\mathrm{d}p + a'\mathrm{d}q$, $\mathrm{d}y = b\mathrm{d}p + b'\mathrm{d}q$,

$$C\mathrm{d}p = b'\mathrm{d}x - a'\mathrm{d}y$$
$$C\mathrm{d}q = -b\mathrm{d}x + a\mathrm{d}y$$

Hinc obtinemus differentialia completa ipsarum t, u

$$C^3\mathrm{d}t = \left(A\frac{\mathrm{d}C}{\mathrm{d}p} - C\frac{\mathrm{d}A}{\mathrm{d}p}\right)(b'\mathrm{d}x - a'\mathrm{d}y) + \left(C\frac{\mathrm{d}A}{\mathrm{d}q} - A\frac{\mathrm{d}C}{\mathrm{d}q}\right)(b\mathrm{d}x - a\mathrm{d}y)$$
$$C^3\mathrm{d}u = \left(B\frac{\mathrm{d}C}{\mathrm{d}p} - C\frac{\mathrm{d}B}{\mathrm{d}p}\right)(b'\mathrm{d}x - a'\mathrm{d}y) + \left(C\frac{\mathrm{d}B}{\mathrm{d}q} - B\frac{\mathrm{d}C}{\mathrm{d}q}\right)(b\mathrm{d}x - a\mathrm{d}y)$$

Iam si in his formulis substituimus

$$\frac{\mathrm{d}A}{\mathrm{d}p} = c'\beta + b\gamma' - c\beta' - b'\gamma$$
$$\frac{\mathrm{d}A}{\mathrm{d}q} = c'\beta' + b\gamma'' - c\beta'' - b'\gamma'$$
$$\frac{\mathrm{d}B}{\mathrm{d}p} = a'\gamma + c\alpha' - a\gamma' - c'\alpha$$
$$\frac{\mathrm{d}B}{\mathrm{d}q} = a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha'$$
$$\frac{\mathrm{d}C}{\mathrm{d}p} = b'\alpha + a\beta' - b\alpha' - a'\beta$$
$$\frac{\mathrm{d}C}{\mathrm{d}q} = b'\alpha' + a\beta'' - b\alpha'' - a'\beta'$$

is, however, very important that we develop this formula also. Retaining the notations of Art. 4, let us put also

$$\frac{\partial^2 x}{\partial p^2} = \alpha, \quad \frac{\partial^2 x}{\partial p \,.\, \partial q} = \alpha', \quad \frac{\partial^2 x}{\partial q^2} = \alpha''$$

$$\frac{\partial^2 y}{\partial p^2} = \beta, \quad \frac{\partial^2 y}{\partial p \,.\, \partial q} = \beta', \quad \frac{\partial^2 y}{\partial q^2} = \beta''$$

$$\frac{\partial^2 z}{\partial p^2} = \gamma, \quad \frac{\partial^2 z}{\partial p \,.\, \partial q} = \gamma', \quad \frac{\partial^2 z}{\partial q^2} = \gamma''$$

and let us put, for brevity,

$$bc' - cb' = A$$
$$ca' - ac' = B$$
$$ab' - ba' = C$$

First we see that

$$A\,dx + B\,dy + C\,dz = 0,$$

or

$$dz = -\frac{A}{C}dx - \frac{B}{C}dy.$$

Thus, inasmuch as z may be regarded as a function of x, y, we have

$$\frac{\partial z}{\partial x} = t = -\frac{A}{C}$$

$$\frac{\partial z}{\partial y} = u = -\frac{B}{C}$$

Then from the formulæ

$$dx = a\,dp + a'\,dq, \quad dy = b\,dp + b'\,dq,$$

we have

$$C\,dp = \quad b'\,dx - a'\,dy$$
$$C\,dq = -b\,dx + a\,dy$$

Thence we obtain for the total differentials of t, u

$$C^3\,dt = \left(A\frac{\partial C}{\partial p} - C\frac{\partial A}{\partial p}\right)(b'\,dx - a'\,dy) + \left(C\frac{\partial A}{\partial q} - A\frac{\partial C}{\partial q}\right)(b\,dx - a\,dy)$$

$$C^3\,du = \left(B\frac{\partial C}{\partial p} - C\frac{\partial B}{\partial p}\right)(b'\,dx - a'\,dy) + \left(C\frac{\partial B}{\partial q} - B\frac{\partial C}{\partial q}\right)(b\,dx - a\,dy)$$

If now we substitute in these formulæ

$$\frac{\partial A}{\partial p} = c'\beta + b\gamma' - c\beta' - b'\gamma$$

$$\frac{\partial A}{\partial q} = c'\beta' + b\gamma'' - c\beta'' - b'\gamma'$$

$$\frac{\partial B}{\partial p} = a'\gamma + c\alpha' - a\gamma' - c'\alpha$$

$$\frac{\partial B}{\partial q} = a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha'$$

$$\frac{\partial C}{\partial p} = b'\alpha + a\beta' - b\alpha' - a'\beta$$

$$\frac{\partial C}{\partial q} = b'\alpha' + a\beta'' - b\alpha'' - a'\beta'$$

atque perpendimus, valores differentialium $\mathrm{d}t$, $\mathrm{d}u$ sic prodeuntium, aequales esse debere, independenter a differentialibus $\mathrm{d}x$, $\mathrm{d}y$, quantitatibus $T\mathrm{d}x + U\mathrm{d}y$, $U\mathrm{d}x + V\mathrm{d}y$ resp. inveniemus, post quasdam transformationes satis obvias:

$$\begin{aligned}
C^3 T = &\ \alpha A b'b' + \beta B b'b' + \gamma C b'b' \\
&- 2\alpha' A b b' - 2\beta' B b b' - 2\gamma' C b b' \\
&+ \alpha'' A b b + \beta'' B b b + \gamma'' C b b \\
C^3 U = &- \alpha A a'b' - \beta B a'b' - \gamma C a'b' \\
&+ \alpha' A (ab' + ba') + \beta' B (ab' + ba') + \gamma' C (ab' + ba') \\
&- \alpha'' A ab - \beta'' B ab - \gamma'' C ab \\
C^3 V = &\ \alpha A a'a' + \beta B a'a' + \gamma C a'a' \\
&- 2\alpha' A a a' - 2\beta' B a a' - 2\gamma' C a a' \\
&+ \alpha'' A a a + \beta'' B a a + \gamma'' C a a
\end{aligned}$$

Si itaque brevitatis caussa statuimus

$$A\alpha + B\beta + C\gamma = D \quad \ldots\ldots\ldots\ldots \quad (1)$$
$$A\alpha' + B\beta' + C\gamma' = D' \quad \ldots\ldots\ldots\ldots \quad (2)$$
$$A\alpha'' + B\beta'' + C\gamma'' = D'' \quad \ldots\ldots\ldots\ldots \quad (3)$$

fit

$$\begin{aligned}
C^3 T &= \ D b'b' - 2D' b b' + D'' b b \\
C^3 U &= - D a'b' + D'(ab' + ba') - D'' ab \\
C^3 V &= \ D a'a' - 2D' a a' + D'' a a
\end{aligned}$$

Hinc invenimus, evolutione facta,

$$C^6(TV - UU) = (DD'' - D'D')(ab' - ba')^2 = (DD'' - D'D')CC$$

et proin formulam pro mensura curvaturae

$$k = \frac{DD'' - D'D'}{(AA + BB + CC)^2}$$

11.

Formulae modo inventae iam aliam superstruemus, quae inter fertilissima theoremata in doctrina de superficiebus curvis referenda est. Introducamus sequentes notationes:

and if we note that the values of the differentials dt, du thus obtained must be equal, independently of the differentials dx, dy, to the quantities $T\,dx + U\,dy$, $U\,dx + V\,dy$ respectively, we shall find, after some sufficiently obvious transformations,

$$\begin{aligned}
C^3\,T = {} & \alpha A b'^2 + \beta B b'^2 + \gamma C b'^2 \\
& - 2\alpha' A b b' - 2\beta' B b b' - 2\gamma' C b b' \\
& + \alpha'' A b^2 + \beta'' B b^2 + \gamma'' C b^2 \\
C^3\,U = {} & -\alpha A a' b' - \beta B a' b' - \gamma C a' b' \\
& + \alpha' A (a b' + b a') + \beta' B (a b' + b a') + \gamma' C (a b' + b a') \\
& - \alpha'' A a b - \beta'' B a b - \gamma'' C a b \\
C^3\,V = {} & \alpha A a'^2 + \beta B a'^2 + \gamma C a'^2 \\
& - 2\alpha' A a a' - 2\beta' B a a' - 2\gamma' C a a' \\
& + \alpha'' A a^2 + \beta'' B a^2 + \gamma'' C a^2
\end{aligned}$$

Hence, if we put, for the sake of brevity,

$$A\alpha + B\beta + C\gamma = D \quad \ldots\ldots\ldots \quad (1)$$

$$A\alpha' + B\beta' + C\gamma' = D' \quad \ldots\ldots\ldots \quad (2)$$

$$A\alpha'' + B\beta'' + C\gamma'' = D'' \quad \ldots\ldots\ldots \quad (3)$$

we shall have

$$\begin{aligned}
C^3\,T &= D b'^2 - 2 D' b b' + D'' b^2 \\
C^3\,U &= -D a' b' + D' (a b' + b a') - D'' a b \\
C^3\,V &= D a'^2 - 2 D' a a' + D'' a^2
\end{aligned}$$

From this we find, after the reckoning has been carried out,

$$C^6 (T V - U^2) = (D D'' - D'^2)(a b' - b a')^2 = (D D'' - D'^2)\, C^2$$

and therefore the formula for the measure of curvature

$$k = \frac{D D'' - D'^2}{(A^2 + B^2 + C^2)^2}$$

11.

By means of the formula just found we are going to establish another, which may be counted among the most productive theorems in the theory of curved surfaces. Let us introduce the following notation :

$$aa + bb + cc = E$$
$$aa' + bb' + cc' = F$$
$$a'a' + b'b' + c'c' = G$$
$$a\alpha + b\beta + c\gamma = m \quad \ldots\ldots\ldots\ldots \quad (4)$$
$$a\alpha' + b\beta' + c\gamma' = m' \quad \ldots\ldots\ldots\ldots \quad (5)$$
$$a\alpha'' + b\beta'' + c\gamma'' = m'' \quad \ldots\ldots\ldots\ldots \quad (6)$$
$$a'\alpha + b'\beta + c'\gamma = n \quad \ldots\ldots\ldots\ldots \quad (7)$$
$$a'\alpha' + b'\beta' + c'\gamma' = n' \quad \ldots\ldots\ldots\ldots \quad (8)$$
$$a'\alpha'' + b'\beta'' + c'\gamma'' = n'' \quad \ldots\ldots\ldots\ldots \quad (9)$$
$$AA + BB + CC = EG - FF = \Delta$$

Eliminemus ex aequationibus 1, 4, 7, quantitates β, γ, quod fit multiplicando illas per $bc' - cb'$, $b'C - c'B$, $cB - bC$, et addendo: ita oritur

$$(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC))\alpha$$
$$= D(bc' - cb') + m(b'C - c'B) + n(cB - bC)$$

quam aequationem facile transformamus in hanc:

$$AD = \alpha\Delta + a(nF - mG) + a'(mF - nE)$$

Simili modo eliminatio quantitatum α, γ vel α, β ex iisdem aequationibus suppeditat

$$BD = \beta\Delta + b(nF - mG) + b'(mF - nE)$$
$$CD = \gamma\Delta + c(nF - mG) + c'(mF - nE)$$

Multiplicando has tres aequationes per α'', β'', γ'' et addendo obtinemus

$$DD'' = (\alpha\beta'' + \beta\beta'' + \gamma\gamma'')\Delta + m''(nF - mG) + n''(mF - nE) \quad \ldots\ldots \quad (10)$$

Si perinde tractamus aequationes 2, 5, 8, prodit

$$AD' = \alpha'\Delta + a(n'F - m'G) + a'(m'F - n'E)$$
$$BD' = \beta'\Delta + b(n'F - m'G) + b'(m'F - n'E)$$
$$CD' = \gamma'\Delta + c(n'F - m'G) + c'(m'F - n'E)$$

quibus aequationibus per α', β', γ' multiplicatis, additio suppeditat:

$$D'D' = (\alpha'\alpha' + \beta'\beta' + \gamma'\gamma')\Delta + m'(n'F - m'G) + n'(m'F - n'E)$$

General Investigations of Curved Surfaces

$$a^2+b^2+c^2=E$$
$$aa'+bb'+cc'=F$$
$$a'^2+b'^2+c'^2=G$$
$$a\alpha+b\beta+c\gamma=m \quad \ldots\ldots \quad (4)$$
$$a\alpha'+b\beta'+c\gamma'=m' \quad \ldots\ldots \quad (5)$$
$$a\alpha''+b\beta''+c\gamma''=m'' \quad \ldots\ldots \quad (6)$$
$$a'\alpha+b'\beta+c'\gamma=n \quad \ldots\ldots \quad (7)$$
$$a'\alpha'+b'\beta'+c'\gamma'=n' \quad \ldots\ldots \quad (8)$$
$$a'\alpha''+b'\beta''+c'\gamma''=n'' \quad \ldots\ldots \quad (9)$$
$$A^2+B^2+C^2=EG-F^2=\Delta$$

Let us eliminate from the equations 1, 4, 7 the quantities β, γ, which is done by multiplying them by $bc'-cb'$, $b'C-c'B$, $cB-bC$ respectively and adding. In this way we obtain

$$(A(bc'-cb')+a(b'C-c'B)+a'(cB-bC))\,\alpha$$
$$=D(bc'-cb')+m(b'C-c'B)+n(cB-bC)$$

an equation which is easily transformed into

$$AD=\alpha\Delta+a(nF-mG)+a'(mF-nE)$$

Likewise the elimination of α, γ or α, β from the same equations gives

$$BD=\beta\Delta+b(nF-mG)+b'(mF-nE)$$
$$CD=\gamma\Delta+c(nF-mG)+c'(mF-nE)$$

Multiplying these three equations by α'', β'', γ'' respectively and adding, we obtain

$$DD''=(\alpha\alpha''+\beta\beta''+\gamma\gamma'')\Delta+m''(nF-mG)+n''(mF-nE) \quad \ldots \quad (10)$$

If we treat the equations 2, 5, 8 in the same way, we obtain

$$AD'=\alpha'\Delta+a(n'F-m'G)+a'(m'F-n'E)$$
$$BD'=\beta'\Delta+b(n'F-m'G)+b'(m'F-n'E)$$
$$CD'=\gamma'\Delta+c(n'F-m'G)+c'(m'F-n'E)$$

and after these equations are multiplied by α', β', γ' respectively, addition gives

$$D'^2=(\alpha'^2+\beta'^2+\gamma'^2)\Delta+m'(n'F-m'G)+n'(m'F-n'E)$$

Disquisitiones Generales circa Superficies Curvas

Combinatio huius aequationis cum aequatione (10) producit

$$DD''-D'D' = (\alpha\alpha''+6 6''+\gamma\gamma''-\alpha'\alpha'-6'6'-\gamma'\gamma')\Delta$$
$$+E(n'n'-nn'')+F(nm''-2m'n'+mn'')+G(m'm'-mm'')$$

Iam patet esse

$$\frac{dE}{dp}=2m,\quad \frac{dE}{dq}=2m',\quad \frac{dF}{dp}=m'+n,\quad \frac{dF}{dq}=m''+n',\quad \frac{dG}{dp}=2n',\quad \frac{dG}{dq}=2n''$$

sive

$$m=\tfrac{1}{2}\frac{dE}{dp},\qquad m'=\tfrac{1}{2}\frac{dE}{dq},\qquad m''=\frac{dF}{dq}-\tfrac{1}{2}\frac{dG}{dp}$$
$$n=\frac{dF}{dp}-\tfrac{1}{2}\frac{dE}{dq},\qquad n'=\tfrac{1}{2}\frac{dG}{dp},\qquad n''=\tfrac{1}{2}\frac{dG}{dq}$$

Porro facile confirmatur, haberi

$$\alpha\alpha''+6 6''+\gamma\gamma''-\alpha'\alpha'-6'6'-\gamma'\gamma' = \frac{dn}{dq}-\frac{dn'}{dp}=\frac{dm''}{dp}-\frac{dm'}{dq}$$
$$=-\tfrac{1}{2}\cdot\frac{ddE}{dq^2}+\frac{ddF}{dp.dq}-\tfrac{1}{2}\cdot\frac{ddG}{dp^2}$$

Quodsi iam has expressiones diversas in formula pro mensura curvaturae in fine art. praec. eruta substituimus, pervenimus ad formulam sequentem, e solis quantitatibus F, F, G atque earum quotientibus differentialibus primi et secundi ordinis concinnatam:

$$4(EG-FF)^2k = E\left(\frac{dE}{dq}\cdot\frac{dG}{dq}-2\frac{dF}{dp}\cdot\frac{dG}{dq}+\left(\frac{dG}{dp}\right)^2\right)$$
$$+F\left(\frac{dE}{dp}\cdot\frac{dG}{dq}-\frac{dE}{dq}\cdot\frac{dG}{dp}-2\frac{dE}{dq}\cdot\frac{dF}{dq}+4\frac{dF}{dp}\cdot\frac{dF}{dq}-2\frac{dF}{dp}\cdot\frac{dG}{dp}\right)$$
$$+G\left(\frac{dE}{dp}\cdot\frac{dG}{dp}-2\cdot\frac{dE}{dp}\cdot\frac{dF}{dq}+\left(\frac{dE}{dq}\right)^2\right)$$
$$-2(EG-FF)\left(\frac{ddE}{dq^2}-2\frac{ddF}{dp.dq}+\frac{ddG}{dp^2}\right)$$

12.

Quum indefinite habeatur

$$dx^2+dy^2+dz^2 = Edp^2+2Fdp.dq+Gdq^2$$

patet, $\sqrt{(Edp^2+2Fdp.dq+Gdq^2)}$ esse expressionem generalem elementi linearis in superficie curva. Docet itaque analysis in art. praec. explicata, ad inveniendam mensuram curvaturae haud opus esse formulis finitis, quae coordina-

A combination of this equation with equation (10) gives

$$DD''-D'^2=(\alpha\alpha''+\beta\beta''+\gamma\gamma''-\alpha'^2-\beta'^2-\gamma'^2)\Delta$$
$$+E(n'^2-nn'')+F(nm''-2m'n'+mn'')+G(m'^2-mm'')$$

It is clear that we have

$$\frac{\partial E}{\partial p}=2m,\quad \frac{\partial E}{\partial q}=2m',\quad \frac{\partial F}{\partial p}=m'+n,\quad \frac{\partial F}{\partial q}=m''+n',\quad \frac{\partial G}{\partial p}=2n',\quad \frac{\partial G}{\partial q}=2n'',$$

or

$$m=\tfrac{1}{2}\frac{\partial E}{\partial p},\qquad m'=\tfrac{1}{2}\frac{\partial E}{\partial q},\quad m''=\frac{\partial F}{\partial q}-\tfrac{1}{2}\frac{\partial G}{\partial p}$$
$$n=\frac{\partial F}{\partial p}-\tfrac{1}{2}\frac{\partial E}{\partial q},\quad n'=\tfrac{1}{2}\frac{\partial G}{\partial p},\quad n''=\tfrac{1}{2}\frac{\partial G}{\partial q}$$

Moreover, it is easily shown that we shall have

$$\alpha\alpha''+\beta\beta''+\gamma\gamma''-\alpha'^2-\beta'^2-\gamma'^2=\frac{\partial n}{\partial q}-\frac{\partial n'}{\partial p}=\frac{\partial m''}{\partial p}-\frac{\partial m'}{\partial q}$$
$$=-\tfrac{1}{2}\cdot\frac{\partial^2 E}{\partial q^2}+\frac{\partial^2 F}{\partial p\,.\,\partial q}-\tfrac{1}{2}\cdot\frac{\partial^2 G}{\partial p^2}$$

If we substitute these different expressions in the formula for the measure of curvature derived at the end of the preceding article, we obtain the following formula, which involves only the quantities E, F, G and their differential quotients of the first and second orders:

$$4(EG-F^2)^2k=E\left(\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial q}-2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial q}+\left(\frac{\partial G}{\partial p}\right)^2\right)$$
$$+F\left(\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial q}-\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial p}-2\frac{\partial E}{\partial q}\cdot\frac{\partial F}{\partial q}+4\frac{\partial E}{\partial p}\cdot\frac{\partial F}{\partial q}-2\frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial p}\right)$$
$$+G\left(\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial p}-2\frac{\partial E}{\partial p}\cdot\frac{\partial F}{\partial q}+\left(\frac{\partial E}{\partial q}\right)^2\right)-2(EG-F^2)\left(\frac{\partial^2 E}{\partial q^2}-2\frac{\partial^2 F}{\partial p\,.\,\partial q}+\frac{\partial^2 G}{\partial p^2}\right)$$

12.

Since we always have

$$dx^2+dy^2+dz^2=E\,dp^2+2F\,dp\,.\,dq+G\,dq^2,$$

it is clear that

$$\sqrt{(E\,dp^2+2F\,dp\,.\,dq+G\,dq^2)}$$

is the general expression for the linear element on the curved surface. The analysis developed in the preceding article thus shows us that for finding the measure of curvature there is no need of finite formulæ, which express the coordinates x, y, z as tas x, y, z tamquam functiones indeterminatarum p, q exhibeant, sed sufficere expressionem generalem pro magnitudine cuiusvis elementi linearis. Progrediamur ad aliquot applicationes huius gravissimi theorematis.

Supponamus, superficiem nostram curvam explicari posse in aliam superficiem, curvam seu planam, ita ut cuivis puncto prioris superficiei per coordinatas x, y, z determinato respondeat punctum determinatum superficiei posterioris, cuius coordinatae sint x', y', z'. Manifesto itaque x', y', z' quoque considerari possunt tamquam functiones indeterminatarum p, q, unde pro elemento $\surd(\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2)$ prodibit expressio talis

$$\surd(E'\mathrm{d}p^2 + 2F'\mathrm{d}p \,.\, \mathrm{d}q + G'\mathrm{d}q^2)$$

denotantibus etiam E', F', G' functiones ipsarum p, q. At per ipsam notionem *explicationis* superficiei in superficiem patet, elementa in utraque superficie correspondentia necessario aequalia esse, adeoque identice fieri

$$E = E', \quad F = F', \quad G = G'$$

Formula itaque art. praec. sponte perducit ad egregium

THEOREMA. *Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet.*

Manifesto quoque *quaevis pars finita superficiei curvae post explicationem in aliam superficiem eandem curvaturam integram retinebit.*

Casum specialem, ad quem geometrae hactenus investigationes suas restrinxerunt, sistunt superficies in planum explicabiles. Theoria nostra sponte docet, talium superficierum mensuram curvaturae in quovis puncto fieri $= 0$, quocirca, si earum indoles secundum modum tertium exprimitur, ubique erit

$$\frac{\mathrm{dd}z}{\mathrm{d}x^2} \cdot \frac{\mathrm{dd}z}{\mathrm{d}y^2} - \left(\frac{\mathrm{dd}z}{\mathrm{d}x \,.\, \mathrm{d}y}\right)^2 = 0$$

quod criterium, dudum quidem notum, plerumque nostro saltem iudicio haud eo rigore qui desiderari posset demonstratur.

13.

Quae in art. praec. exposuimus, cohaerent cum modo peculiari superficies considerandi, summopere digno, qui a geometris diligenter excolatur. Scilicet quatenus superficies consideratur non tamquam limes solidi, sed tamquam soli-

General Investigations
of
Curved Surfaces

functions of the indeterminates p, q; but that the general expression for the magnitude of any linear element is sufficient. Let us proceed to some applications of this very important theorem.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminates p, q, and therefore for the element $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$ we shall have an expression of the form

$$\sqrt{(E'\,dp^2 + 2\,F'\,dp\,.\,dq + G'\,dq^2)}$$

where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'.$$

Thus the formula of the preceding article leads of itself to the remarkable

THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

Also it is evident that *any finite part whatever of the curved surface will retain the same integral curvature after development upon another surface.*

Surfaces developable upon a plane constitute the particular case to which geometers have heretofore restricted their attention. Our theory shows at once that the measure of curvature at every point of such surfaces is equal to zero. Consequently, if the nature of these surfaces is defined according to the third method, we shall have at every point

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,.\,\partial y}\right)^2 = 0$$

a criterion which, though indeed known a short time ago, has not, at least to our knowledge, commonly been demonstrated with as much rigor as is desirable.

13.

What we have explained in the preceding article is connected with a particular method of studying surfaces, a very worthy method which may be thoroughly developed by geometers. When a surface is regarded, not as the boundary of a solid, but

Disquisitiones Generales circa Superficies Curvas

dum, cuius dimensio una pro evanescente habetur, flexile quidem, sed non extensibile, qualitates superficiei partim a forma pendent, in quam illa reducta concipitur, partim absolutae sunt, atque invariatae manent, in quamcunque formam illa flectatur. Ad has posteriores, quarum investigatio campum geometriae novum fertilemque aperit, referendae sunt mensura curvaturae atque curvatura integra eo sensu, quo hae expressiones a nobis accipiuntur; porro huc pertinet doctrina de lineis brevissimis, pluraque alia, de quibus in posterum agere nobis reservamus. In hoc considerationis modo superficies plana atque superficies in planum explicabilis, e. g. cylindrica, conica etc. tamquam essentialiter identicae spectantur, modusque genuinus indolem superficiei ita consideratae generaliter exprimendi semper innititur formulae $\sqrt{(E\,dp^2 + 2F\,dp\,.\,dq + G\,dq^2)}$, quae nexum elementi cum duabus indeterminatis p, q sistit. Sed antequam hoc argumentum ulterius prosequamur, principia theoriae linearum brevissimarum in superficie curva data praemittere oportet.

14.

Indoles lineae curvae in spatio generaliter ita datur, ut coordinatae x, y, z singulis illius punctis respondentes exhibeantur in forma functionum unius variabilis, quam per w denotabimus. Longitudo talis lineae a puncto initiali arbitrario usque ad punctum, cuius coordinatae sunt x, y, z, exprimitur per integrale

$$\int dw\,.\sqrt{\left(\left(\frac{dx}{dw}\right)^2 + \left(\frac{dy}{dw}\right)^2 + \left(\frac{dz}{dw}\right)^2\right)}$$

Si supponimus, situm lineae curvae variationem infinite parvam pati, ita ut coordinatae singulorum punctorum accipiant variationes δx, δy, δz, variatio totius longitudinis invenitur

$$= \int \frac{dx\,.\,d\delta x + dy\,.\,d\delta y + dz\,.\,d\delta z}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$

quam expressionem in hanc formam transmutamus:

$$\frac{dx\,.\,\delta x + dy\,.\,\delta y + dz\,.\,\delta z}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$
$$- \int \left(\delta x\,.\,d\frac{dx}{\sqrt{(dx^2 + dy^2 + dz^2)}} + \delta y\,.\,d\frac{dy}{\sqrt{(dx^2 + dy^2 + dz^2)}} + \delta z\,.\,d\frac{dz}{\sqrt{(dx^2 + dy^2 + dz^2)}}\right)$$

In casu eo, ubi linea est brevissima inter puncta sua extrema, constat, ea, quae hic sub signo integrali sunt, evanescere debere. Quatenus linea esse debet in su-

General Investigations of Curved Surfaces

as a flexible, though not extensible solid, one dimension of which is supposed to vanish, then the properties of the surface depend in part upon the form to which we can suppose it reduced, and in part are absolute and remain invariable, whatever may be the form into which the surface is bent. To these latter properties, the study of which opens to geometry a new and fertile field, belong the measure of curvature and the integral curvature, in the sense which we have given to these expressions. To these belong also the theory of shortest lines, and a great part of what we reserve to be treated later. From this point of view, a plane surface and a surface developable on a plane, *e. g.*, cylindrical surfaces, conical surfaces, etc., are to be regarded as essentially identical; and the generic method of defining in a general manner the nature of the surfaces thus considered is always based upon the formula

$$\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)},$$

which connects the linear element with the two indeterminates p, q. But before following this study further, we must introduce the principles of the theory of shortest lines on a given curved surface.

14.

The nature of a curved line in space is generally given in such a way that the coordinates x, y, z corresponding to the different points of it are given in the form of functions of a single variable, which we shall call w. The length of such a line from an arbitrary initial point to the point whose coordinates are x, y, z, is expressed by the integral

$$\int dw\,.\sqrt{\left(\left(\frac{dx}{dw}\right)^2 + \left(\frac{dy}{dw}\right)^2 + \left(\frac{dz}{dw}\right)^2\right)}$$

If we suppose that the position of the line undergoes an infinitely small variation, so that the coordinates of the different points receive the variations δx, δy, δz, the variation of the whole length becomes

$$\int \frac{dx\,.\,d\,\delta x + dy\,.\,d\,\delta y + dz\,.\,d\,\delta z}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$

which expression we can change into the form

$$\frac{dx\,.\,\delta x + dy\,.\,\delta y + dz\,.\,\delta z}{\sqrt{(dx^2 + dy^2 + dz^2)}}$$

$$-\int\left(\delta x\,.\,d\frac{dx}{\sqrt{(dx^2+dy^2+dz^2)}} + \delta y\,.\,d\frac{dy}{\sqrt{(dx^2+dy^2+dz^2)}} + \delta z\,.\,d\frac{dz}{\sqrt{(dx^2+dy^2+dz^2)}}\right)$$

We know that, in case the line is to be the shortest between its end points, all that stands under the integral sign must vanish. Since the line must lie on the given perficie data, cuius indoles exprimitur per aequationem $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0$, etiam variationes $\delta x, \delta y, \delta z$ satisfacere debent aequationi $P\delta x + Q\delta y + R\delta z = 0$, unde per principia nota facile colligitur, differentialia

$$\mathrm{d}\frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)}},\quad \mathrm{d}\frac{\mathrm{d}y}{\sqrt{(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)}},\quad \mathrm{d}\frac{\mathrm{d}z}{\sqrt{(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)}}$$

resp. quantitatibus P, Q, R proportionalia esse debere. Iam sit $\mathrm{d}r$ elementum lineae curvae, λ punctum in superficie sphaerica repraesentans directionem huius elementi, L punctum in superficie sphaerica repraesentans directionem normalis in superficiem curvam; denique sint ξ, η, ζ coordinatae puncti λ, atque X, Y, Z coordinatae puncti L respectu centri sphaerae. Ita erit

$$\mathrm{d}x = \xi\mathrm{d}r,\quad \mathrm{d}y = \eta\mathrm{d}r,\quad \mathrm{d}z = \zeta\mathrm{d}r$$

unde colligimus, differentialia illa fieri $\mathrm{d}\xi$, $\mathrm{d}\eta$, $\mathrm{d}\zeta$. Et quum quantitates P, Q, R proportionales sint ipsis X, Y, Z, character lineae brevissimae consistit in aequationibus

$$\frac{\mathrm{d}\xi}{X} = \frac{\mathrm{d}\eta}{Y} = \frac{\mathrm{d}\zeta}{Z}$$

Ceterum facile perspicitur, $\sqrt{(\mathrm{d}\xi^2 + \mathrm{d}\eta^2 + \mathrm{d}\zeta^2)}$ aequari arculo in superficie sphaerica, qui mensurat angulum inter directiones tangentium in initio et fine elementi $\mathrm{d}r$, adeoque esse $= \frac{\mathrm{d}r}{\rho}$, si ρ denotet radium curvaturae in hoc loco curvae brevissimae; ita fiet

$$\rho\mathrm{d}\xi = X\mathrm{d}r,\quad \rho\mathrm{d}\eta = Y\mathrm{d}r,\quad \rho\mathrm{d}\zeta = Z\mathrm{d}r$$

15.

Supponamus, in superficie curva a puncto dato A proficisci innumeras curvas brevissimas, quas inter se distinguemus per angulum, quem constituit singularum elementum primum cum elemento primo unius ex his lineis pro prima assumtae: sit φ ille angulus, vel generalius functio illius anguli, nec non r longitudo talis lineae brevissimae a puncto A usque ad punctum, cuius coordinatae sunt x, y, z. Quum itaque valoribus determinatis variabilium r, φ respondeant puncta determinata superficiei, coordinatae x, y, z considerari possunt tamquam functiones ipsarum r, φ. Notationes λ, L, ξ, η, ζ, X, Y, Z in eadem significasurface, whose nature is defined by the equation

$$P\,dx + Q\,dy + R\,dz = 0,$$

the variations δx, δy, δz also must satisfy the equation

$$P\,\delta x + Q\,\delta y + R\,\delta z = 0,$$

and from this it follows at once, according to well-known rules, that the differentials

$$d\frac{dx}{\sqrt{(dx^2+dy^2+dz^2)}},\quad d\frac{dy}{\sqrt{(dx^2+dy^2+dz^2)}},\quad d\frac{dz}{\sqrt{(dx^2+dy^2+dz^2)}}$$

must be proportional to the quantities P, Q, R respectively. Let dr be the element of the curved line; λ the point on the sphere representing the direction of this element; L the point on the sphere representing the direction of the normal to the curved surface; finally, let ξ, η, ζ be the coordinates of the point λ, and X, Y, Z be those of the point L with reference to the centre of the sphere. We shall then have

$$dx = \xi\,dr,\quad dy = \eta\,dr,\quad dz = \zeta\,dr$$

from which we see that the above differentials become $d\xi$, $d\eta$, $d\zeta$. And since the quantities P, Q, R are proportional to X, Y, Z, the character of shortest lines is expressed by the equations

$$\frac{d\xi}{X} = \frac{d\eta}{Y} = \frac{d\zeta}{Z}$$

Moreover, it is easily seen that

$$\sqrt{(d\xi^2 + d\eta^2 + d\zeta^2)}$$

is equal to the small arc on the sphere which measures the angle between the directions of the tangents at the beginning and at the end of the element dr, and is thus equal to $\frac{dr}{\rho}$, if ρ denotes the radius of curvature of the shortest line at this point. Thus we shall have

$$\rho\,d\xi = X\,dr,\quad \rho\,d\eta = Y\,dr,\quad \rho\,d\zeta = Z\,dr$$

15.

Suppose that an infinite number of shortest lines go out from a given point A on the curved surface, and suppose that we distinguish these lines from one another by the angle that the first element of each of them makes with the first element of one of them which we take for the first. Let ϕ be that angle, or, more generally, a function of that angle, and r the length of such a shortest line from the point A to the point whose coordinates are x, y, z. Since to definite values of the variables r, ϕ there correspond definite points of the surface, the coordinates x, y, z can be regarded as functions of r, ϕ. We shall retain for the notation λ, L, ξ, η, ζ, X, Y, Z the same tione retinebimus, in qua in art. praec. acceptae fuerunt, modo indefinite ad punctum indefinitum cuiuslibet linearum brevissimarum referantur.

Lineae brevissimae omnes, quae sunt aequalis longitudinis r, terminabuntur ad aliam lineam, cuius longitudinem ab initio arbitrario numeratam denotamus per v. Considerari poterit itaque v tamquam functio indeterminatarum r, φ, et si per λ' designamus punctum in superficie sphaerica respondens directioni elementi $\mathrm{d}v$, nec non per ξ', η', ζ' coordinatas huius puncti respectu centri sphaerae, habebimus:

$$\frac{\mathrm{d}x}{\mathrm{d}\varphi} = \xi'.\frac{\mathrm{d}v}{\mathrm{d}\varphi}, \quad \frac{\mathrm{d}y}{\mathrm{d}\varphi} = \eta'.\frac{\mathrm{d}v}{\mathrm{d}\varphi}, \quad \frac{\mathrm{d}z}{\mathrm{d}\varphi} = \zeta'.\frac{\mathrm{d}v}{\mathrm{d}\varphi}$$

Hinc et ex

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \xi, \quad \frac{\mathrm{d}y}{\mathrm{d}r} = \eta, \quad \frac{\mathrm{d}z}{\mathrm{d}r} = \zeta$$

sequitur

$$\frac{\mathrm{d}x}{\mathrm{d}r}\cdot\frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{d}y}{\mathrm{d}r}\cdot\frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{d}z}{\mathrm{d}r}\cdot\frac{\mathrm{d}z}{\mathrm{d}\varphi} = (\xi\xi' + \eta\eta' + \zeta\zeta')\cdot\frac{\mathrm{d}v}{\mathrm{d}\varphi} = \cos\lambda\lambda'.\frac{\mathrm{d}v}{\mathrm{d}\varphi}$$

Membrum primum huius aequationis, quod etiam erit functio ipsarum r, φ, per S denotamus; cuius differentiatio secundum r suppeditat:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}r} &= \frac{\mathrm{dd}x}{\mathrm{d}r^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{dd}y}{\mathrm{d}r^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{dd}z}{\mathrm{d}r^2}\cdot\frac{\mathrm{d}z}{\mathrm{d}\varphi} + \tfrac{1}{2}\cdot\frac{\mathrm{d}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right)}{\mathrm{d}\varphi} \\ &= \frac{\mathrm{d}\xi}{\mathrm{d}r}\cdot\frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{d}\eta}{\mathrm{d}r}\cdot\frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{d}\zeta}{\mathrm{d}r}\cdot\frac{\mathrm{d}z}{\mathrm{d}\varphi} + \tfrac{1}{2}\cdot\frac{\mathrm{d}(\xi\xi + \eta\eta + \zeta\zeta)}{\mathrm{d}\varphi}\end{aligned}$$

Sed $\xi\xi + \eta\eta + \zeta\zeta = 1$, adeoque ipsius differentiale $= 0$; et per art. praec. habemus, si etiam hic ρ denotat radium curvaturae in linea r,

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{X}{\rho}, \quad \frac{\mathrm{d}\eta}{\mathrm{d}r} = \frac{Y}{\rho}, \quad \frac{\mathrm{d}\zeta}{\mathrm{d}r} = \frac{Z}{\rho}$$

Ita obtinemus

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{1}{\rho}.(X\xi' + Y\eta' + Z\zeta').\frac{\mathrm{d}v}{\mathrm{d}\varphi} = \frac{1}{\rho}.\cos L\lambda'.\frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$$

quoniam manifesto λ' iacet in circulo maximo, cuius polus L. Hinc itaque concludimus, S independentem esse ab r et proin functionem solius φ. At pro $r = 0$ manifesto fit $v = 0$, et proin etiam $\frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$, nec non $S = 0$, independenter a φ. Necessario itaque generaliter esse debebit $S = 0$, adeoque $\cos\lambda\lambda' = 0$, i. e. $\lambda\lambda' = 90^0$. Hinc colligimus meaning as in the preceding article, this notation referring to any point whatever on any one of the shortest lines.

All the shortest lines that are of the same length r will end on another line whose length, measured from an arbitrary initial point, we shall denote by v. Thus v can be regarded as a function of the indeterminates r, ϕ, and if λ' denotes the point on the sphere corresponding to the direction of the element dv, and also ξ', η', ζ' denote the coordinates of this point with reference to the centre of the sphere, we shall have

$$\frac{\partial x}{\partial \phi}=\xi'\cdot\frac{\partial v}{\partial \phi},\quad \frac{\partial y}{\partial \phi}=\eta'\cdot\frac{\partial v}{\partial \phi},\quad \frac{\partial z}{\partial \phi}=\zeta'\cdot\frac{\partial v}{\partial \phi}$$

From these equations and from the equations

$$\frac{\partial x}{\partial r}=\xi,\quad \frac{\partial y}{\partial r}=\eta,\quad \frac{\partial z}{\partial r}=\zeta$$

we have

$$\frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial y}{\partial r}\cdot\frac{\partial y}{\partial \phi}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \phi}=(\xi\xi'+\eta\eta'+\zeta\zeta')\cdot\frac{\partial v}{\partial \phi}=\cos\lambda\lambda'\cdot\frac{\partial v}{\partial \phi}$$

Let S denote the first member of this equation, which will also be a function of r, ϕ. Differentiation of S with respect to r gives

$$\frac{\partial S}{\partial r}=\frac{\partial^2 x}{\partial r^2}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial^2 y}{\partial r^2}\cdot\frac{\partial y}{\partial \phi}+\frac{\partial^2 z}{\partial r^2}\cdot\frac{\partial z}{\partial \phi}+\tfrac{1}{2}\cdot\frac{\partial\left(\left(\frac{\partial x}{\partial r}\right)^2+\left(\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial z}{\partial r}\right)^2\right)}{\partial \phi}$$
$$=\frac{\partial \xi}{\partial r}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial \eta}{\partial r}\cdot\frac{\partial y}{\partial \phi}+\frac{\partial \zeta}{\partial r}\cdot\frac{\partial z}{\partial \phi}+\tfrac{1}{2}\cdot\frac{\partial(\xi^2+\eta^2+\zeta^2)}{\partial \phi}$$

But

$$\xi^2+\eta^2+\zeta^2=1,$$

and therefore its differential is equal to zero; and by the preceding article we have, if ρ denotes the radius of curvature of the line r,

$$\frac{\partial \xi}{\partial r}=\frac{X}{\rho},\quad \frac{\partial \eta}{\partial r}=\frac{Y}{\rho},\quad \frac{\partial \zeta}{\partial r}=\frac{Z}{\rho}$$

Thus we have

$$\frac{\partial S}{\partial r}=\frac{1}{\rho}\cdot(X\xi'+Y\eta'+Z\zeta')\cdot\frac{\partial v}{\partial \phi}=\frac{1}{\rho}\cdot\cos L\lambda'\cdot\frac{\partial v}{\partial \phi}=0$$

since λ' evidently lies on the great circle whose pole is L. From this we see that S is independent of r, and is, therefore, a function of ϕ alone. But for $r=0$ we evidently have $v=0$, consequently $\frac{\partial v}{\partial \phi}=0$, and $S=0$ independently of ϕ. Thus, in general, we have necessarily $S=0$, and so $\cos\lambda\lambda'=0$, *i. e.*, $\lambda\lambda'=90°$. From this follows the

Disquisitiones Generales circa Superficies Curvas

Theorema. *Ductis in superficie curva ab eodem puncto initiali innumeris lineis brevissimis aequalis longitudinis, linea earum extremitates iungens ad illas singulas erit normalis.*

Operae pretium esse duximus, hoc theorema e proprietate fundamentali linearum brevissimarum deducere: ceterum eius veritas etiam absque calculo per sequens ratiocinium intelligi potest. Sint AB, AB' duae lineae brevissimae eiusdem longitudinis, angulum infinite parvum ad A includentes, supponamusque, alterutrum angulorum elementi BB' cum lineis BA, $B'A$ differre quantitate finita ab angulo recto, unde per legem continuitatis alter maior alter minor erit angulo recto. Supponamus, angulum ad B esse $= 90^0 - \omega$, capiamusque in linea BA punctum C, ita ut sit $BC = BB'.\operatorname{cosec}\omega$: hinc quum triangulum infinite parvum $BB'C$ tamquam planum tractare liceat, erit $CB' = BC.\cos\omega$, et proin

$$AC + CB' = AC + BC.\cos\omega = AB - BC.(1 - \cos\omega) = AB' - BC(1 - \cos\omega)$$

i.e. transitus a puncto A ad B' per punctum C brevior linea brevissima. Q.E.A.

16.

Theoremati art. praec. associamus aliud, quod ita enunciamus. *Si in superficie curva concipitur linea qualiscunque, a cuius punctis singulis proficiscantur sub angulis rectis et versus eandem plagam innumerae lineae brevissimae aequalis longitudinis, curva, quae earum extremitates alteras iungit, illas singulas sub angulis rectis secabit.* Ad demonstrationem nihil in analysi praecedente mutandum est, nisi quod φ designare debet longitudinem curvae *datae* inde a puncto arbitrario numeratam, aut si mavis functionem huius longitudinis; ita omnia ratiocinia etiamnum valebunt, ea modificatione, quod veritas aequationis $S = 0$ pro $r = 0$ nunc iam in ipsa hypothesi implicatur. Ceterum hoc alterum theorema generalius est praecedente, quod adeo in illo comprehendi censeri potest, dum pro linea data adoptamus circulum infinite parvum circa centrum A descriptum. Denique monemus, hic quoque considerationes geometricas analyseos vice fungi posse, quibus tamen, quum satis obviae sint, hic non immoramur.

17.

Revertimur ad formulam $\sqrt{(E\,dp^2 + 2F\,dp.dq + G\,dq^2)}$, quae indefinite

General Investigations
of
Curved Surfaces

THEOREM. *If on a curved surface an infinite number of shortest lines of equal length be drawn from the same initial point, the lines joining their extremities will be normal to each of the lines.*

We have thought it worth while to deduce this theorem from the fundamental property of shortest lines; but the truth of the theorem can be made apparent without any calculation by means of the following reasoning. Let AB, AB' be two shortest lines of the same length including at A an infinitely small angle, and let us suppose that one of the angles made by the element BB' with the lines BA, $B'A$ differs from a right angle by a finite quantity. Then, by the law of continuity, one will be greater and the other less than a right angle. Suppose the angle at B is equal to $90° - \omega$, and take on the line AB a point C, such that

$$BC = BB' \,.\, \operatorname{cosec} \omega.$$

Then, since the infinitely small triangle $BB'C$ may be regarded as plane, we shall have

$$CB' = BC \,.\, \cos \omega,$$

and consequently

$$AC + CB' = AC + BC \,.\, \cos\omega = AB - BC \,.\, (1 - \cos\omega) = AB' - BC \,.\, (1 - \cos\omega),$$

i. e., the path from A to B' through the point C is shorter than the shortest line, *Q. E. A.*

16.

With the theorem of the preceding article we associate another, which we state as follows: *If on a curved surface we imagine any line whatever, from the different points of which are drawn at right angles and toward the same side an infinite number of shortest lines of the same length, the curve which joins their other extremities will cut each of the lines at right angles.* For the demonstration of this theorem no change need be made in the preceding analysis, except that ϕ must denote the length of the *given* curve measured from an arbitrary point; or rather, a function of this length. Thus all of the reasoning will hold here also, with this modification, that $S = 0$ for $r = 0$ is now implied in the hypothesis itself. Moreover, this theorem is more general than the preceding one, for we can regard it as including the first one if we take for the given line the infinitely small circle described about the centre A. Finally, we may say that here also geometric considerations may take the place of the analysis, which, however, we shall not take the time to consider here, since they are sufficiently obvious.

17.

We return to the formula

$$\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)},$$

Disquisitiones Generales circa Superficies Curvas

magnitudinem elementi linearis in superficie curva exprimit, atque ante omnia significationem geometricam coëfficientium E, F, G examinamus. Iam in art. 5 monuimus, in superficie curva concipi posse duo systemata linearum, alterum, in quibus singulis sola p sit variabilis, q constans; alterum, in quibus sola q variabilis, p constans. Quodlibet punctum superficiei considerari potest tamquam intersectio lineae primi systematis cum linea secundi: tuncque elementum lineae primae huic puncto adiacens et variationi $\mathrm{d}p$ respondens erit $= \sqrt{E}.\mathrm{d}p$, nec non elementum lineae secundae respondens variationi $\mathrm{d}q$ erit $= \sqrt{G}.\mathrm{d}q$; denique denotando per ω angulum inter haec elementa, facile perspicitur, fieri $\cos\omega = \frac{F}{\sqrt{EG}}$. Area autem elementi parallelogrammatici in superficie curva inter duas lineas primi systematis, quibus respondent q, $q+\mathrm{d}q$, atque duas lineas systematis secundi, quibus respondent p, $p+\mathrm{d}p$, erit $\sqrt{(EG-FF)}\mathrm{d}p.\mathrm{d}q$.

Linea quaecunque in superficie curva ad neutrum illorum systematum pertinens, oritur, dum p et q concipiuntur esse functiones unius variabilis novae, vel altera illarum functio alterius. Sit s longitudo talis curvae ab initio arbitrario numerata et versus directionem utramvis pro positiva habita. Denotemus per θ angulum, quem efficit elementum $\mathrm{d}s = \sqrt{(E\mathrm{d}p^2 + 2F\mathrm{d}p.\mathrm{d}q + G\mathrm{d}q^2)}$ cum linea primi systematis per initium elementi ducta, et quidem ne ulla ambiguitas remaneat, hunc angulum semper ab eo ramo illius lineae, in quo valores ipsius p crescunt, inchoari, et versus eam plagam positive accipi supponemus, versus quam valores ipsius q crescunt. His ita intellectis facile perspicitur haberi

$$\cos\theta.\mathrm{d}s = \sqrt{E}.\mathrm{d}p + \sqrt{G}.\cos\omega.\mathrm{d}q = \frac{E\mathrm{d}p + F\mathrm{d}q}{\sqrt{E}}$$
$$\sin\theta.\mathrm{d}s = \sqrt{G}.\sin\omega.\mathrm{d}q = \frac{\sqrt{(EG-FF)}.\mathrm{d}q}{\sqrt{E}}$$

18.

Investigabimus nunc, quaenam sit conditio, ut haec linea sit brevissima. Quum ipsius longitudo s expressa sit per integrale

$$s = \int\sqrt{(E\mathrm{d}p^2 + 2F\mathrm{d}p.\mathrm{d}q + G\mathrm{d}q^2)}$$

conditio minimi requirit, ut variatio huius integralis a mutatione infinite parva tractus lineae oriunda fiat $= 0$. Calculus ad propositum nostrum in hoc casu commodius absolvitur, si p tamquam functionem ipsius q consideramus. Quo which expresses generally the magnitude of a linear element on the curved surface, and investigate, first of all, the geometric meaning of the coefficients E, F, G. We have already said in Art. 5 that two systems of lines may be supposed to lie on the curved surface, p being variable, q constant along each of the lines of the one system; and q variable, p constant along each of the lines of the other system. Any point whatever on the surface can be regarded as the intersection of a line of the first system with a line of the second; and then the element of the first line adjacent to this point and corresponding to a variation dp will be equal to $\sqrt{E}\,.\,dp$, and the element of the second line corresponding to the variation dq will be equal to $\sqrt{G}\,.\,dq$. Finally, denoting by ω the angle between these elements, it is easily seen that we shall have

$$\cos\omega = \frac{F}{\sqrt{EG}}.$$

Furthermore, the area of the surface element in the form of a parallelogram between the two lines of the first system, to which correspond q, $q+dq$, and the two lines of the second system, to which correspond p, $p+dp$, will be

$$\sqrt{(EG-F^2)}\,dp\,.\,dq.$$

Any line whatever on the curved surface belonging to neither of the two systems is determined when p and q are supposed to be functions of a new variable, or one of them is supposed to be a function of the other. Let s be the length of such a curve, measured from an arbitrary initial point, and in either direction chosen as positive. Let θ denote the angle which the element

$$ds = \sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)}$$

makes with the line of the first system drawn through the initial point of the element, and, in order that no ambiguity may arise, let us suppose that this angle is measured from that branch of the first line on which the values of p increase, and is taken as positive toward that side toward which the values of q increase. These conventions being made, it is easily seen that

$$\cos\theta\,.\,ds = \sqrt{E}\,.\,dp + \sqrt{G}\,.\cos\omega\,.\,dq = \frac{E\,dp + F\,dq}{\sqrt{E}}$$

$$\sin\theta\,.\,ds = \sqrt{G}\,.\sin\omega\,.\,dq = \frac{\sqrt{(EG-F^2)}\,.\,dq}{\sqrt{E}}$$

18.

We shall now investigate the condition that this line be a shortest line. Since its length s is expressed by the integral

$$s = \int\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)}$$

the condition for a minimum requires that the variation of this integral arising from an infinitely small change in the position become equal to zero. The calculation, for our purpose, is more simply made in this case, if we regard p as a function of q.

pacto, si variatio per characteristicam δ denotatur, habemus

$$\delta s = \int \frac{(\frac{dE}{dp}.dp^2 + \frac{2dF}{dp}.dp.dq + \frac{dG}{dp}dq^2)\delta p + (2E\,dp + 2F\,dq)d\delta p}{2ds}$$

$$= \frac{E\,dp + F\,dq}{ds}.\delta p + \int \delta p.\{\frac{\frac{dE}{dp}.dp^2 + \frac{2dF}{dp}.dp.dq + \frac{dG}{dp}.dq^2}{2ds} - d.\frac{E\,dp + F\,dq}{ds}\}$$

constatque, quae hic sunt sub signo integrali, independenter a δp evanescere debere. Fit itaque

$$\frac{dE}{dp}.dp^2 + \frac{2dF}{dp}.dp.dq + \frac{dG}{dp}.dq^2 = 2ds.d.\frac{E\,dp + F\,dq}{ds}$$

$$= 2ds.d.\sqrt{E}.\cos\theta = \frac{ds.dE.\cos\theta}{\sqrt{E}} - 2ds.d\theta.\sqrt{E}.\sin\theta$$

$$= \frac{(E\,dp + F\,dq)dE}{E} - \sqrt{(EG - FF)}.dq.d\theta$$

$$= (\frac{E\,dp + F\,dq}{E}).(\frac{dE}{dp}.dp + \frac{dE}{dq}.dq) - 2\sqrt{(EG - FF)}.dq.d\theta$$

Hinc itaque nanciscimur aequationem conditionalem pro linea brevissima sequentem:

$$\sqrt{(EG - FF)}.d\theta = \tfrac{1}{2}\frac{F}{E}.\frac{dE}{dp}.dp + \tfrac{1}{2}.\frac{F}{E}.\frac{dE}{dq}.dq + \tfrac{1}{2}.\frac{dE}{dq}.dp - \frac{dF}{dp}.dp - \tfrac{1}{2}.\frac{dG}{dp}.dq$$

quam etiam ita scribere licet

$$\sqrt{(EG - FF)}.d\theta = \tfrac{1}{2}\frac{F}{E}.dE + \tfrac{1}{2}.\frac{dE}{dq}.dp - \frac{dF}{dp}.dp - \tfrac{1}{2}.\frac{dG}{dp}.dq$$

Ceterum adiumento aequationis

$$\operatorname{cotg}\theta = \frac{E}{\sqrt{(EG - FF)}}.\frac{dp}{dq} + \frac{F}{\sqrt{(EG - FF)}}$$

ex illa aequatione angulus θ eliminari, atque sic aequatio differentio-differentialis inter p et q evolvi potest, quae tamen magis complicata et ad applicationes minus utilis evaderet, quam praecedens.

19.

Formulae generales, quas pro mensura curvaturae et pro variatione directionis lineae brevissimae in artt. 11, 18 eruimus, multo simpliciores fiunt, si quantitates p, q ita sunt electae, ut lineae primi systematis lineas secundi systematis ubique orthogonaliter secent, i. e. ut generaliter habeatur $\omega = 90^0$, sive $F = 0$. Tunc scilicet fit, pro mensura curvaturae, When this is done, if the variation is denoted by the characteristic δ, we have

$$\delta s = \int \frac{\left(\frac{\partial E}{\partial p} \cdot dp^2 + 2\frac{\partial F}{\partial p} \cdot dp \cdot dq + \frac{\partial G}{\partial p} \cdot dq^2\right)\delta p + (2\,E\,dp + 2\,F\,dq)\,d\,\delta p}{2\,ds}$$

$$= \frac{E\,dp + F\,dq}{ds} \cdot \delta p +$$

$$+ \int \delta p \left(\frac{\frac{\partial E}{\partial p} \cdot dp^2 + 2\frac{\partial F}{\partial p} \cdot dp \cdot dq + \frac{\partial G}{\partial p} \cdot dq^2}{2\,ds} - d \cdot \frac{E\,dp + F\,dq}{ds} \right)$$

and we know that what is included under the integral sign must vanish independently of δp. Thus we have

$$\frac{\partial E}{\partial p} \cdot dp^2 + 2\frac{\partial F}{\partial p} \cdot dp \cdot dq + \frac{\partial G}{\partial p} \cdot dq^2 = 2\,ds \cdot d \cdot \frac{E\,dp + F\,dq}{ds}$$

$$= 2\,ds \cdot d \cdot \sqrt{E} \cdot \cos\theta$$

$$= \frac{ds \cdot dE \cdot \cos\theta}{\sqrt{E}} - 2\,ds \cdot d\theta \cdot \sqrt{E} \cdot \sin\theta$$

$$= \frac{(E\,dp + F\,dq)\,dE}{E} - \sqrt{(EG - F^2)} \cdot dp \cdot d\theta$$

$$= \left(\frac{E\,dp + F\,dq}{E}\right) \cdot \left(\frac{\partial E}{\partial p} \cdot dp + \frac{\partial E}{\partial q} \cdot dq\right) - 2\sqrt{(EG - F^2)} \cdot dq \cdot d\theta$$

This gives the following conditional equation for a shortest line:

$$\sqrt{(EG - F^2)} \cdot d\theta = \frac{1}{2} \cdot \frac{F}{E} \cdot \frac{\partial E}{\partial p} \cdot dp + \frac{1}{2} \cdot \frac{F}{E} \cdot \frac{\partial E}{\partial q} \cdot dq + \frac{1}{2} \cdot \frac{\partial E}{\partial q} \cdot dp$$
$$- \frac{\partial F}{\partial p} \cdot dp - \frac{1}{2} \cdot \frac{\partial G}{\partial p} \cdot dq$$

which can also be written

$$\sqrt{(EG - F^2)} \cdot d\theta = \frac{1}{2} \cdot \frac{F}{E} \cdot dE + \frac{1}{2} \cdot \frac{\partial E}{\partial q} \cdot dp - \frac{\partial F}{\partial p} \cdot dp - \frac{1}{2} \cdot \frac{\partial G}{\partial p} \cdot dq$$

From this equation, by means of the equation

$$\cot\theta = \frac{E}{\sqrt{(EG - F^2)}} \cdot \frac{dp}{dq} + \frac{F}{\sqrt{(EG - F^2)}}$$

it is also possible to eliminate the angle θ, and to derive a differential equation of the second order between p and q, which, however, would become more complicated and less useful for applications than the preceding.

19.

The general formulæ, which we have derived in Arts. 11, 18 for the measure of curvature and the variation in the direction of a shortest line, become much simpler if the quantities p, q are so chosen that the lines of the first system cut everywhere orthogonally the lines of the second system; *i. e.*, in such a way that we have generally $\omega = 90°$, or $F = 0$. Then the formula for the measure of curvature becomes

$$4EEGGk = E.\frac{dE}{dq}.\frac{dG}{dq} + E(\frac{dG}{dp})^2 + G.\frac{dE}{dp}.\frac{dG}{dp} + G(\frac{dE}{dq})^2 - 2EG(\frac{ddE}{dq^2} + \frac{ddG}{dp^2})$$

et pro variatione anguli θ

$$\sqrt{EG}.d\theta = \tfrac{1}{2}.\frac{dE}{dq}.dp - \tfrac{1}{2}.\frac{dG}{dp}.dq$$

Inter varios casus, in quibus haec conditio orthogonalitatis valet, primarium locum tenet is, ubi lineae omnes alterutrius systematis, e. g. primi, sunt lineae brevissimae. Hic itaque pro valore constante ipsius q, angulus θ fit $= 0$, unde aequatio pro variatione anguli θ modo tradita docet, fieri debere $\frac{dE}{dq} = 0$, sive coëfficientem E a q independentem, i. e. E esse debet vel constans vel functio solius p. Simplicissimum erit, pro p adoptare longitudinem ipsam cuiusque lineae primi systematis, et quidem, quoties omnes lineae primi systematis in uno puncto concurrunt, ab hoc puncto numeratam, vel, si communis intersectio non adest, a qualibet linea secundi systematis. Quibus ita intellectis patet, p et q iam eadem denotare, quae in artt. 15, 16 per r et φ expresseramus, atque fieri $E = 1$. Ita duae formulae praecedentes iam transeunt in has:

$$4GGk = (\frac{dG}{dp})^2 - 2G\frac{ddG}{dp^2}$$
$$\sqrt{G}.d\theta = -\tfrac{1}{2}.\frac{dG}{dp}.dq$$

vel statuendo $\sqrt{G} = m$,

$$k = -\frac{1}{m}.\frac{ddm}{dp^2}, \qquad d\theta = -\frac{dm}{dp}.dq$$

Generaliter loquendo m erit functio ipsarum p, q atque $m\,dq$ expressio elementi cuiusvis lineae secundi systematis. In casu speciali autem, ubi omnes lineae p ab eodem puncto proficiscuntur, manifesto pro $p = 0$ esse debet $m = 0$; porro si in hoc casu pro q adoptamus angulum ipsum, quem elementum primum cuiusvis lineae primi systematis facit cum elemento alicuius ex ipsis ad arbitrium electae, quum pro valore infinite parvo ipsius p, elementum lineae secundi systematis (quae considerari potest tamquam circulus radio p descriptus), sit $= p\,dq$, erit pro valore infinite parvo ipsius $p, m = p$, adeoque, pro $p = 0$ simul $m = 0$ et $\frac{dm}{dp} = 1$.

20.

Immoremur adhuc eidem suppositioni, puta p designare indefinite longitudinem lineae brevissimae a puncto determinato A ad punctum quodlibet super-

$$4\,E^2\,G^2\,k = E\cdot\frac{\partial E}{\partial q}\cdot\frac{\partial G}{\partial q} + E\left(\frac{\partial G}{\partial p}\right)^2 + G\cdot\frac{\partial E}{\partial p}\cdot\frac{\partial G}{\partial p} + G\left(\frac{\partial E}{\partial q}\right)^2 - 2\,E\,G\left(\frac{\partial^2 E}{\partial q^2} + \frac{\partial^2 G}{\partial p^2}\right),$$

and for the variation of the angle θ

$$\sqrt{E\,G}\,.\,d\,\theta = \frac{1}{2}\cdot\frac{\partial E}{\partial q}\cdot dp - \frac{1}{2}\cdot\frac{\partial G}{\partial p}\cdot dq$$

Among the various cases in which we have this condition of orthogonality, the most important is that in which all the lines of one of the two systems, *e. g.*, the first, are shortest lines. Here for a constant value of q the angle θ becomes equal to zero, and therefore the equation for the variation of θ just given shows that we must have $\frac{\partial E}{\partial q} = 0$, or that the coefficient E must be independent of q; *i. e.*, E must be either a constant or a function of p alone. It will be simplest to take for p the length of each line of the first system, which length, when all the lines of the first system meet in a point, is to be measured from this point, or, if there is no common intersection, from any line whatever of the second system. Having made these conventions, it is evident that p and q denote now the same quantities that were expressed in Arts. 15, 16 by r and ϕ, and that $E = 1$. Thus the two preceding formulæ become:

$$4\,G^2\,k = \left(\frac{\partial G}{\partial p}\right)^2 - 2\,G\frac{\partial^2 G}{\partial p^2}$$

$$\sqrt{G}\,.\,d\,\theta = -\frac{1}{2}\cdot\frac{\partial G}{\partial p}\cdot dq$$

or, setting $\sqrt{G} = m$,

$$k = -\frac{1}{m}\cdot\frac{\partial^2 m}{\partial p^2},\quad d\,\theta = -\frac{\partial m}{\partial p}\cdot dq$$

Generally speaking, m will be a function of p, q, and $m\,dq$ the expression for the element of any line whatever of the second system. But in the particular case where all the lines p go out from the same point, evidently we must have $m = 0$ for $p = 0$. Furthermore, in the case under discussion we will take for q the angle itself which the first element of any line whatever of the first system makes with the element of any one of the lines chosen arbitrarily. Then, since for an infinitely small value of p the element of a line of the second system (which can be regarded as a circle described with radius p) is equal to $p\,dq$, we shall have for an infinitely small value of p, $m = p$, and consequently, for $p = 0$, $m = 0$ at the same time, and $\frac{\partial m}{\partial p} = 1$.

20.

We pause to investigate the case in which we suppose that p denotes in a general manner the length of the shortest line drawn from a fixed point A to any other

Disquisitiones Generales circa Superficies Curvas

ficiei ductum, atque q angulum, quem primum elementum huius lineae efficit cum elemento primo alicuius lineae brevissimae ex A proficiscentis datae. Sit B punctum determinatum in hac linea pro qua $q = 0$, atque C aliud punctum determinatum superficiei, pro quo valorem ipsius q simpliciter per A designabimus. Supponamus, puncta B, C per lineam brevissimam iuncta, cuius partes, inde a puncto B numeratas, indefinite ut in art. 18 per s denotabimus, nec non perinde ut illic, per θ angulum, quem quodvis elementum $\mathrm{d}s$ facit cum elemento $\mathrm{d}p$: denique sint θ^0, θ' valores anguli θ in punctis B, C. Habemus itaque in superficie curva triangulum lineis brevissimis inclusum, eiusque anguli ad B et C, per has ipsas literas simpliciter designandi aequales erunt ille complemento anguli θ^0 ad 180^0, hic ipsi angulo θ'. Sed quum analysin nostram inspicienti facile pateat, omnes angulos non per gradus sed per numeros expressos concipi, ita ut angulus $57^0 17' 45''$, cui respondet arcus radio aequalis, pro unitate habeatur, statuere oportet, denotando per 2π peripheriam circuli

$$\theta^0 = \pi - B, \qquad \theta' = C$$

Inquiramus nunc in curvaturam integram huius trianguli, quae fit $= \int k \,\mathrm{d}\sigma$, denotante $\mathrm{d}\sigma$ elementum superficiale trianguli; quare quum hoc elementum exprimatur per $m\,\mathrm{d}p \,.\, \mathrm{d}q$, eruere oportet integrale $\int\int k m \,\mathrm{d}p \,.\, \mathrm{d}q$ supra totam trianguli superficiem. Incipiamus ab integratione secundum p, quae propter $k = -\frac{1}{m} \cdot \frac{\mathrm{d}\mathrm{d}m}{\mathrm{d}p^2}$, suppeditat $\mathrm{d}q.(\mathrm{Const.} - \frac{\mathrm{d}m}{\mathrm{d}p})$ pro curvatura integra areae iacentis inter lineas primi systematis, quibus respondent valores indeterminatae secundae q, $q + \mathrm{d}q$: quum haec curvatura pro $p = 0$ evanescere debeat, quantitas constans per integrationem introducta aequalis esse debet valori ipsius $\frac{\mathrm{d}m}{\mathrm{d}p}$ pro $p = 0$, i. e. unitati. Habemus itaque $\mathrm{d}q(1 - \frac{\mathrm{d}m}{\mathrm{d}p})$, ubi pro $\frac{\mathrm{d}m}{\mathrm{d}p}$ accipere oportet valorem respondentem fini illius areae in linea CB. In hac linea vero fit per art. praec. $\frac{\mathrm{d}m}{\mathrm{d}p} . \mathrm{d}q = -\mathrm{d}\theta$, unde expressio nostra mutatur in $\mathrm{d}q + \mathrm{d}\theta$. Accedente iam integratione altera a $q = 0$ usque ad $q = A$ extendenda, obtinemus curvaturam integram trianguli $= A + \theta' - \theta^0 = A + B + C - \pi$.

Curvatura integra aequalis est areae eius partis superficiei sphaericae, quae respondet triangulo, signo positivo vel negativo affectae, prout superficies curva, in qua triangulum iacet, est concavo-concava vel concavo-convexa: pro unitate areae accipiendum est quadratum, cuius latus est unitas (radius sphaerae), quo pacto superficies tota sphaerae fit $= 4\pi$. Est itaque pars superficiei sphaericae point whatever of the surface, and q the angle that the first element of this line makes with the first element of another given shortest line going out from A. Let B be a definite point in the latter line, for which $q = 0$, and C another definite point of the surface, at which we denote the value of q simply by A. Let us suppose the points B, C joined by a shortest line, the parts of which, measured from B, we denote in a general way, as in Art. 18, by s; and, as in the same article, let us denote by θ the angle which any element ds makes with the element dp; finally, let us denote by θ°, θ' the values of the angle θ at the points B, C. We have thus on the curved surface a triangle formed by shortest lines. The angles of this triangle at B and C we shall denote simply by the same letters, and B will be equal to $180^\circ - \theta$, C to θ' itself. But, since it is easily seen from our analysis that all the angles are supposed to be expressed, not in degrees, but by numbers, in such a way that the angle $57^\circ 17' 45''$, to which corresponds an arc equal to the radius, is taken for the unit, we must set

$$\theta^\circ = \pi - B, \quad \theta' = C$$

where 2π denotes the circumference of the sphere. Let us now examine the integral curvature of this triangle, which is equal to

$$\int k\, d\sigma,$$

$d\sigma$ denoting a surface element of the triangle. Wherefore, since this element is expressed by $m\, dp \,.\, dq$, we must extend the integral

$$\iint m\, dp \,.\, dq$$

over the whole surface of the triangle. Let us begin by integration with respect to p, which, because $k = -\frac{1}{m} \cdot \frac{\partial^2 m}{\partial p^2}$, gives $dq \,.\, \left(\text{const.} - \frac{\partial m}{\partial p}\right)$, for the integral curvature of the area lying between the lines of the first system, to which correspond the values q, $q + dq$ of the second indeterminate. Since this integral curvature must vanish for $p = 0$, the constant introduced by integration must be equal to the value of $\frac{\partial m}{\partial q}$ for $p = 0$, *i. e.*, equal to unity. Thus we have

$$dq\left(1 - \frac{\partial m}{\partial p}\right),$$

where for $\frac{\partial m}{\partial p}$ must be taken the value corresponding to the end of this area on the line CB. But on this line we have, by the preceding article,

$$\frac{\partial m}{\partial q} \,.\, dq = -d\,\theta,$$

whence our expression is changed into $dq + d\theta$. Now by a second integration, taken from $q = 0$ to $q = A$, we obtain for the integral curvature

$$A + \theta' - \theta^\circ,$$

or

$$A + B + C - \pi.$$

The integral curvature is equal to the area of that part of the sphere which corresponds to the triangle, taken with the positive or negative sign according as the curved surface on which the triangle lies is concavo-concave or concavo-convex. For unit area will be taken the square whose side is equal to unity (the radius of the sphere), and then the whole surface of the sphere becomes equal to 4π. Thus the triangulo respondens ad sphaerae superficiem integram ut $\pm(A+B+C-\pi)$ ad 4π. Hoc theorema, quod ni fallimur ad elegantissima in theoria superficierum curvarum referendum esse videtur, etiam sequenti modo enuntiari potest:

Excessus summae angulorum trianguli a lineis brevissimis in superficie curva concavo-concava formati ultra 180^0, *vel defectus summae angulorum trianguli a lineis brevissimis in superficie curva concavo-convexa formati a* 180^0 *mensuratur per aream partis superficiei sphaericae, quae illi triangulo per directiones normalium respondet, si superficies integra* 720 *gradibus aequiparatur.*

Generalius in quovis polygono n laterum, quae singula formantur per lineas brevissimas, excessus summae angulorum supra $2n-4$ rectos, vel defectus a $2n-4$ rectis (pro indole curvaturae superficiei), aequatur areae polygoni respondentis in superficie sphaerica, dum tota superficies sphaerae 720 gradibus aequiparatur, uti per discerptionem polygoni in triangula e theoremate praecedenti sponte demanat.

21.

Restituamus characteribus p, q, E, F, G, ω significationes generales, quibus supra accepti fuerant, supponamusque, indolem superficiei curvae praeterea alio simili modo per duas alias variabiles p', q' determinari, ubi elementum lineare indefinitum exprimatur per

$$\sqrt{(E'\mathrm{d}p'^2+2F'\mathrm{d}p'.\mathrm{d}q'+G'\mathrm{d}q'^2)}$$

Ita cuivis puncto superficiei per valores determinatos variabilium p, q definito respondebunt valores determinati variabilium p', q', quocirca hae erunt functiones ipsarum p, q, e quarum differentiatione prodire supponemus

$$\mathrm{d}p' = \alpha\,\mathrm{d}p+\beta\,\mathrm{d}q$$
$$\mathrm{d}q' = \gamma\,\mathrm{d}p+\delta\,\mathrm{d}q$$

Iam proponimus nobis investigare significationem geometricam horum coëfficientium α, β, γ, δ.

Quatuor itaque nunc systemata linearum in superficie curva concipi possunt, pro quibus resp. q, p, q', p' sint constantes. Si per punctum determinatum, cui respondent variabilium valores p, q, p', q', quatuor lineas ad singula illa systemata pertinentes ductas supponimus, harum elementa, variationibus positivis part of the surface of the sphere corresponding to the triangle is to the whole surface of the sphere as $\pm\ (A+B+C-\pi)$ is to 4π. This theorem, which, if we mistake not, ought to be counted among the most elegant in the theory of curved surfaces, may also be stated as follows :

The excess over 180° of the sum of the angles of a triangle formed by shortest lines on a concavo-concave curved surface, or the deficit from 180° of the sum of the angles of a triangle formed by shortest lines on a concavo-convex curved surface, is measured by the area of the part of the sphere which corresponds, through the directions of the normals, to that triangle, if the whole surface of the sphere is set equal to 720 degrees.

More generally, in any polygon whatever of n sides, each formed by a shortest line, the excess of the sum of the angles over $(2n-4)$ right angles, or the deficit from $(2n-4)$ right angles (according to the nature of the curved surface), is equal to the area of the corresponding polygon on the sphere, if the whole surface of the sphere is set equal to 720 degrees. This follows at once from the preceding theorem by dividing the polygon into triangles.

21.

Let us again give to the symbols p, q, E, F, G, ω the general meanings which were given to them above, and let us further suppose that the nature of the curved surface is defined in a similar way by two other variables, p', q', in which case the general linear element is expressed by

$$\sqrt{(E'\,dp'^2+2\,F'\,dp'.\,dq'+G'\,dq'^2)}$$

Thus to any point whatever lying on the surface and defined by definite values of the variables p, q will correspond definite values of the variables p', q', which will therefore be functions of p, q. Let us suppose we obtain by differentiating them

$$dp'=\alpha\,dp+\beta\,dq$$
$$dq'=\gamma\,dp+\delta\,dq$$

We shall now investigate the geometric meaning of the coefficients α, β, γ, δ.

Now *four* systems of lines may thus be supposed to lie upon the curved surface, for which p, q, p', q' respectively are constants. If through the definite point to which correspond the values p, q, p', q' of the variables we suppose the four lines belonging to these different systems to be drawn, the elements of these lines, corres-

dp, dq, dp', dq', respondentes erunt

$$\sqrt{E}.dp, \quad \sqrt{G}.dq, \quad \sqrt{E'}.dq', \quad \sqrt{G'}.dq'$$

Angulos, quos horum elèmentorum directiones faciunt cum directione fixa arbitraria, denotabimus per M, N, M', N', numerando eo sensu, quo iacet secunda respectu primae, ita ut $\sin(N-M)$ fiat quantitas positiva: eodem sensu iacere supponemus (quod licet) quartam respectu tertiae, ita ut etiam $\sin(N'-M')$ sit quantitas positiva. His ita intellectis, si consideramus punctum aliud, a priore infinite parum distans, cui respondeant valores variabilium

$$p+dp, \quad q+dq, \quad p'+dp', \quad q'+dq'$$

levi attentione adhibita cognoscemus, fieri generaliter, i. e. independenter a valoribus variationum dp, dq, dp', dq',

$$\sqrt{E}.dp.\sin M+\sqrt{G}.dq.\sin N=\sqrt{E'}.dp'.\sin M'+\sqrt{G'}.dq'.\sin N'$$

quum utraque expressio nihil aliud sit, nisi distantia puncti novi a linea, a qua anguli directionum incipiunt. Sed habemus, per notationem iam supra introductam $N-M=\omega$, et per analogiam statuemus $N'-M'=\omega'$, nec non insuper $N-M'=\psi$. Ita aequatio modo inventa exhiberi potest in forma sequenti

$$\begin{aligned}\sqrt{E}.dp.\sin(M'-\omega+\psi)+\sqrt{G}.dq.\sin(M'+\psi)\\ =\sqrt{E'}.dp'.\sin M'+\sqrt{G'}.dq'.\sin(M'+\omega')\end{aligned}$$

vel ita

$$\begin{aligned}\sqrt{E}.dp.\sin(N'-\omega-\omega'+\psi)+\sqrt{G}.dq.\sin(N'-\omega'+\psi)\\ =\sqrt{E'}.dp'.\sin(N'-\omega')+\sqrt{G'}.dq'.\sin N'\end{aligned}$$

Et quum aequatio manifesto independens esse debeat a directione initiali, hanc ad lubitum accipere licet. Statuendo itaque in forma secunda $N'=0$, vel in prima $M'=0$, obtinemus aequationes sequentes:

$$\begin{aligned}\sqrt{E'}.\sin\omega'.dp'&=\sqrt{E}.\sin(\omega+\omega'-\psi).dp+\sqrt{G}.\sin(\omega'-\psi).dq\\ \sqrt{G'}.\sin\omega'.dq'&=\sqrt{E}.\sin(\psi-\omega).dp+\sqrt{G}.\sin\psi.dq\end{aligned}$$

quae aequationes quum identicae esse debeant cum his

$$\begin{aligned}dp'&=\alpha dp+\beta dq\\ dq'&=\gamma dp+\delta dq\end{aligned}$$

General Investigations of Curved Surfaces

ponding to the positive increments dp, dq, dp', dq', will be

$$\sqrt{E}\,.\,dp, \quad \sqrt{G}\,.\,dq, \quad \sqrt{E'}\,.\,dp', \quad \sqrt{G'}\,.\,dq'.$$

The angles which the directions of these elements make with an arbitrary fixed direction we shall denote by M, N, M', N', measuring them in the sense in which the second is placed with respect to the first, so that $\sin(N-M)$ is positive. Let us suppose (which is permissible) that the fourth is placed in the same sense with respect to the third, so that $\sin(N'-M')$ also is positive. Having made these conventions, if we consider another point at an infinitely small distance from the first point, and to which correspond the values $p+dp$, $q+dq$, $p'+dp'$, $q'+dq'$ of the variables, we see without much difficulty that we shall have generally, *i. e.*, independently of the values of the increments dp, dq, dp', dq',

$$\sqrt{E}\,.\,dp\,.\,\sin M + \sqrt{G}\,.\,dq\,.\,\sin N = \sqrt{E'}\,.\,dp'\,.\,\sin M' + \sqrt{G'}\,.\,dq'\,.\,\sin N'$$

since each of these expressions is merely the distance of the new point from the line from which the angles of the directions begin. But we have, by the notation introduced above,

$$N-M=\omega.$$

In like manner we set

$$N'-M'=\omega',$$

and also

$$N-M'=\psi.$$

Then the equation just found can be thrown into the following form:

$$\sqrt{E}\,.\,dp\,.\,\sin(M'-\omega+\psi)+\sqrt{G}\,.\,dq\,.\,\sin(M'+\psi)$$
$$=\sqrt{E'}\,.\,dp'\,.\,\sin M'+\sqrt{G'}\,.\,dq'\,.\,\sin(M'+\omega')$$

or

$$\sqrt{E}\,.\,dp\,.\,\sin(N'-\omega-\omega'+\psi)+\sqrt{G}\,.\,dq\,.\,\sin(N'-\omega'+\psi)$$
$$=\sqrt{E'}\,.\,dp'\,.\,\sin(N'-\omega')+\sqrt{G'}\,.\,dq'\,.\,\sin N'$$

And since the equation evidently must be independent of the initial direction, this direction can be chosen arbitrarily. Then, setting in the second formula $N'=0$, or in the first $M'=0$, we obtain the following equations:

$$\sqrt{E'}\,.\,\sin\omega'\,.\,dp'=\sqrt{E}\,.\,\sin(\omega+\omega'-\psi)\,.\,dp+\sqrt{G}\,.\,\sin(\omega'-\psi)\,.\,dq$$
$$\sqrt{G'}\,.\,\sin\omega'\,.\,dq'=\sqrt{E}\,.\,\sin(\psi-\omega)\,.\,dp+\sqrt{G}\,.\,\sin\psi\,.\,dq$$

and these equations, since they must be identical with

$$dp'=\alpha\,dp+\beta\,dq$$
$$dq'=\gamma\,dp+\delta\,dq$$

suppeditabunt determinationem coëfficientium α, β, γ, δ. Erit scilicet

$$\alpha = \sqrt{\frac{E}{E'}} \cdot \frac{\sin(\omega + \omega' - \psi)}{\sin \omega'}, \qquad \beta = \sqrt{\frac{G}{E'}} \cdot \frac{\sin(\omega' - \psi)}{\sin \omega'}$$

$$\gamma = \sqrt{\frac{E}{G'}} \cdot \frac{\sin(\psi - \omega)}{\sin \omega'}, \qquad \delta = \sqrt{\frac{G}{G'}} \cdot \frac{\sin \psi}{\sin \omega'}$$

Adiungi debent aequationes

$$\cos \omega = \frac{F}{\sqrt{EG}}, \quad \cos \omega' = \frac{F'}{\sqrt{E'G'}}, \quad \sin \omega = \sqrt{\frac{EG - FF}{EG}}, \quad \sin \omega' = \sqrt{\frac{E'G' - F'F'}{E'G'}}$$

unde quatuor aequationes ita quoque exhiberi possunt

$$\alpha \sqrt{(E'G' - F'F')} = \sqrt{EG'} \,.\, \sin(\omega + \omega' - \psi)$$
$$\beta \sqrt{(E'G' - F'F')} = \sqrt{GG'} \,.\, \sin(\omega' - \psi)$$
$$\gamma \sqrt{(E'G' - F'F')} = \sqrt{EE'} \,.\, \sin(\psi - \omega)$$
$$\delta \sqrt{(E'G' - F'F')} = \sqrt{GE'} \,.\, \sin \psi$$

Quum per substitutiones $dp' = \alpha \, dp + \beta \, dq$, $dq' = \gamma \, dp + \delta \, dq$ trinomium $E' dp'^2 + 2F' dp' . dq' + G' dq'^2$ transire debeat in $E dp^2 + 2F dp . dq + G dq^2$, facile obtinemus

$$EG - FF = (E'G' - F'F')(\alpha\delta - \beta\gamma)^2$$

et quum vice versa trinomium posterius rursus transire debeat in prius per substitutionem

$$(\alpha\delta - \beta\gamma) dp = \delta \, dp' - \beta \, dq', \qquad (\alpha\delta - \beta\gamma) \, dq = -\gamma \, dp' + \alpha \, dq'$$

invenimus

$$E\delta\delta - 2F\gamma\delta + G\gamma\gamma = \frac{EG - FF}{E'G' - F'F'} \cdot E'$$
$$E\beta\delta - F(\alpha\delta + \beta\gamma) + G\alpha\gamma = -\frac{EG - FF}{E'G' - F'F'} \cdot F'$$
$$E\beta\beta - 2F\alpha\beta + G\alpha\alpha = \frac{EG - FF}{E'G' - F'F'} \cdot G'$$

22.

A disquisitione generali art. praec. descendimus ad applicationem latissime patentem, ubi, dum p et q etiam significatione generalissima accipiuntur, pro p', q', adoptamus quantitates in art. 15 per r, φ denotatas, quibus characteribus determine the coefficients α, β, γ, δ. We shall have

$$\alpha = \sqrt{\frac{E}{E'}} \cdot \frac{\sin(\omega + \omega' - \psi)}{\sin \omega'}, \quad \beta = \sqrt{\frac{G}{E'}} \cdot \frac{\sin(\omega' - \psi)}{\sin \omega'}$$

$$\gamma = \sqrt{\frac{E}{G'}} \cdot \frac{\sin(\psi - \omega)}{\sin \omega'}, \quad \delta = \sqrt{\frac{G}{G'}} \cdot \frac{\sin \psi}{\sin \omega'}$$

These four equations, taken in connection with the equations

$$\cos \omega = \frac{F}{\sqrt{EG}}, \quad \cos \omega' = \frac{F'}{\sqrt{E'G'}},$$

$$\sin \omega = \sqrt{\frac{EG - F^2}{EG}}, \quad \sin \omega' = \sqrt{\frac{E'G' - F'^2}{E'G'}},$$

may be written

$$\alpha \sqrt{(E'G' - F'^2)} = \sqrt{EG'} \, . \sin(\omega + \omega' - \psi)$$
$$\beta \sqrt{(E'G' - F'^2)} = \sqrt{GG'} \, . \sin(\omega' - \psi)$$
$$\gamma \sqrt{(E'G' - F'^2)} = \sqrt{EE'} \, . \sin(\psi - \omega)$$
$$\delta \sqrt{(E'G' - F'^2)} = \sqrt{GE'} \, . \sin \psi$$

Since by the substitutions

$$dp' = \alpha \, dp + \beta \, dq,$$
$$dq' = \gamma \, dp + \delta \, dq$$

the trinomial

$$E' dp'^2 + 2F' dp' . dq' + G' dq'^2$$

is transformed into

$$E dp^2 + 2F dp . dq + G dq^2,$$

we easily obtain

$$EG - F^2 = (E'G' - F'^2)(\alpha\delta - \beta\gamma)^2$$

and since, *vice versa*, the latter trinomial must be transformed into the former by the substitution

$$(\alpha\delta - \beta\gamma) dp = \delta \, dp' - \beta \, dq', \quad (\alpha\delta - \beta\gamma) dq = -\gamma \, dp' + \alpha \, dq',$$

we find

$$E\delta^2 - 2F\gamma\delta + G\gamma^2 = \frac{EG - F^2}{E'G' - F'^2} \cdot E'$$

$$-E\beta\delta + F(\alpha\delta + \beta\gamma) - G\alpha\gamma = \frac{EG - F^2}{E'G' - F'^2} \cdot F'$$

$$E\beta^2 - 2F\alpha\beta + G\alpha^2 = \frac{EG - F^2}{E'G' - F'^2} \cdot G'$$

22.

From the general discussion of the preceding article we proceed to the very extended application in which, while keeping for p, q their most general meaning, we take for p', q' the quantities denoted in Art. 15 by r, ϕ. We shall use r, ϕ here

Disquisitiones Generales circa Superficies Curvas

etiam hic utemur, scilicet ut pro quovis puncto superficiei r sit distantia minima a puncto determinato, atque φ angulus in hoc puncto inter elementum primum ipsius r atque directionem fixam. Ita habemus $E'=1$, $F'=0$, $\omega'=90^0$: statuemus insuper $\sqrt{G'}=m$, ita ut elementum lineare quodcunque fiat $=\sqrt{(\mathrm{d}r^2+mm\,\mathrm{d}\varphi^2)}$. Hinc quatuor aequationes in art. praec. pro α, β, γ, δ, erutae, suppeditant:

$$\sqrt{E}.\cos(\omega-\psi)=\frac{\mathrm{d}r}{\mathrm{d}p} \quad \ldots\ldots \quad (1)$$

$$\sqrt{G}.\cos\psi=\frac{\mathrm{d}r}{\mathrm{d}q} \quad \ldots\ldots \quad (2)$$

$$\sqrt{E}.\sin(\psi-\omega)=m.\frac{\mathrm{d}\varphi}{\mathrm{d}p} \quad \ldots\ldots \quad (3)$$

$$\sqrt{G}.\sin\psi=m.\frac{\mathrm{d}\varphi}{\mathrm{d}q} \quad \ldots\ldots \quad (4)$$

Ultima et penultima vero has

$$EG-FF=E\left(\frac{\mathrm{d}r}{\mathrm{d}q}\right)^2-2F.\frac{\mathrm{d}r}{\mathrm{d}p}.\frac{\mathrm{d}r}{\mathrm{d}q}+G\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 \quad \ldots\ldots \quad (5)$$

$$\left(E.\frac{\mathrm{d}r}{\mathrm{d}q}-F.\frac{\mathrm{d}r}{\mathrm{d}p}\right).\frac{\mathrm{d}\varphi}{\mathrm{d}q}=\left(F.\frac{\mathrm{d}r}{\mathrm{d}q}-G.\frac{\mathrm{d}r}{\mathrm{d}p}\right).\frac{\mathrm{d}\varphi}{\mathrm{d}p} \quad \ldots\ldots \quad (6)$$

Ex his aequationibus petenda est determinatio quantitatum r, φ, ψ et (si opus videatur) m, per p et q: scilicet integratio aequationis (5) dabit r, qua inventa integratio aequationis (6) dabit φ, atque alterutra aequationum (1), (2) ipsam ψ: denique m habebitur per alterutram aequationum (3), (4).

Integratio generalis aequationum (5), (6) necessario duas functiones arbitrarias introducere debet, quae quid sibi velint facile intelligemus, si perpendimus, illas aequationes ad casum eum quem hic consideramus non limitari, sed perinde valere, si r et φ accipiantur in significatione generaliore art. 16, ita ut sit r longitudo lineae brevissimae ad lineam arbitrariam determinatam normaliter ductae, atque φ functio arbitraria longitudinis eius partis lineae, quae inter lineam brevissimam indefinitam et punctum arbitrarium determinatum intercipitur. Solutio itaque generalis haec omnia indefinite amplecti debet, functionesque arbitrariae tunc demum in definitas abibunt, quando linea illa arbitraria atque functio partium, quam φ exhibere debet, praescriptae sunt. In casu nostro circulus infinite parvus adoptari potest, centrum in eo puncto habens, a quo distantiae r numerantur, et φ denotabit partes huius circuli ipsas per radium divisas, also in such a way that, for any point whatever on the surface, r will be the shortest distance from a fixed point, and ϕ the angle at this point between the first element of r and a fixed direction. We have thus

$$E' = 1, \quad F' = 0, \quad \omega' = 90^\circ.$$

Let us set also

$$\sqrt{G'} = m,$$

so that any linear element whatever becomes equal to

$$\sqrt{(dr^2 + m^2\, d\phi^2)}.$$

Consequently, the four equations deduced in the preceding article for α, β, γ, δ give

$$\sqrt{E} \,.\, \cos(\omega - \psi) = \frac{\partial r}{\partial p} \qquad (1)$$

$$\sqrt{G} \,.\, \cos\psi = \frac{\partial r}{\partial q} \qquad (2)$$

$$\sqrt{E} \,.\, \sin(\psi - \omega) = m \,.\, \frac{\partial \phi}{\partial p} \qquad (3)$$

$$\sqrt{G} \,.\, \sin\psi = m \,.\, \frac{\partial \phi}{\partial q} \qquad (4)$$

But the last and the next to the last equations of the preceding article give

$$EG - F^2 = E\left(\frac{\partial r}{\partial q}\right)^2 - 2F \cdot \frac{\partial r}{\partial p} \cdot \frac{\partial r}{\partial q} + G\left(\frac{\partial r}{\partial p}\right)^2 \qquad (5)$$

$$\left(E \cdot \frac{\partial r}{\partial q} - F \cdot \frac{\partial r}{\partial p}\right) \cdot \frac{\partial \phi}{\partial q} = \left(F \cdot \frac{\partial r}{\partial q} - G \cdot \frac{\partial r}{\partial p}\right) \cdot \frac{\partial \phi}{\partial p} \qquad (6)$$

From these equations must be determined the quantities r, ϕ, ψ and (if need be) m, as functions of p and q. Indeed, integration of equation (5) will give r; r being found, integration of equation (6) will give ϕ; and one or other of equations (1), (2) will give ψ itself. Finally, m is obtained from one or other of equations (3), (4).

The general integration of equations (5), (6) must necessarily introduce two arbitrary functions. We shall easily understand what their meaning is, if we remember that these equations are not limited to the case we are here considering, but are equally valid if r and ϕ are taken in the more general sense of Art. 16, so that r is the length of the shortest line drawn normal to a fixed but arbitrary line, and ϕ is an arbitrary function of the length of that part of the fixed line which is intercepted between any shortest line and an arbitrary fixed point. The general solution must embrace all this in a general way, and the arbitrary functions must go over into definite functions only when the arbitrary line and the arbitrary functions of its parts, which ϕ must represent, are themselves defined. In our case an infinitely small circle may be taken, having its centre at the point from which the distances r are measured, and ϕ will denote the parts themselves of this circle, divided by the unde facile colligitur, aequationes (5), (6) pro casu nostro complete sufficere, dummodo ea, quae indefinita relinquunt, ei conditioni accommodentur, ut r et φ pro puncto illo initiali atque punctis ab eo infinite parum distantibus quadrent.

Ceterum quod attinet ad integrationem ipsam aequationum (5), (6), constat, eam reduci posse ad integrationem aequationum differentialium vulgarium, quae tamen plerumque tam intricatae evadunt, ut parum lucri inde redundet. Contra evolutio in series, quae ad usus practicos, quoties de partibus superficiei modicis agitur, abunde sufficiunt, nullis difficultatibus obnoxia est, atque sic formulae allatae fontem uberem aperiunt, ad multa problemata gravissima solvenda. Hoc vero loco exemplum unicum ad methodi indolem monstrandam evolvemus.

23.

Considerabimus casum eum, ubi omnes lineae, pro quibus p constans est, sunt lineae brevissimae orthogonaliter secantes lineam, pro qua $\varphi = 0$, et quam tamquam lineam abscissarum contemplari possumus. Sit A punctum, pro quo $r = 0$, D punctum indefinitum in linea abscissarum, $AD = p$, B punctum indefinitum in linea brevissima ipsi AD in D normali, atque $BD = q$, ita ut p considerari possit tamquam abscissa, q tamquam ordinata puncti B; abscissas positivas assumimus in eo ramo lineae abscissarum, cui respondet $\varphi = 0$, dum r semper tamquam quantitatem positivam spectamus; ordinatas positivas statuimus in plaga ea, ubi φ numeratur inter 0 et 180^0.

Per theorema art. 16 habebimus $\omega = 90^0$, $F = 0$, nec non $G = 1$; statuemus insuper $\sqrt{E} = n$. Erit itaque n functio ipsarum p, q, et quidem talis, quae pro $q = 0$ fieri debet $= 1$. Applicatio formulae in art. 18 allatae ad casum nostrum docet, in *quavis* linea brevissima esse debere $d\theta = -\frac{dn}{dq} \cdot dp$, denotante θ angulum inter elementum huius lineae atque elementum lineae, pro qua q constans: iam quum linea abscissarum ipsa sit brevissima, atque pro ea ubique $\theta = 0$, patet, pro $q = 0$ ubique fieri debere $\frac{dn}{dq} = 0$. Hinc igitur colligimus, si n in seriem secundum potestates ipsius q progredientem evolvatur, hanc habere debere formam sequentem

$$n = 1 + fqq + gq^3 + hq^4 + \text{ etc.}$$

ubi f, g, h etc. erunt functiones ipsius p, et quidem statuemus radius. Whence it is easily seen that the equations (5), (6) are quite sufficient for our case, provided that the functions which they leave undefined satisfy the condition which r and ϕ satisfy for the initial point and for points at an infinitely small distance from this point.

Moreover, in regard to the integration itself of the equations (5), (6), we know that it can be reduced to the integration of ordinary differential equations, which, however, often happen to be so complicated that there is little to be gained by the reduction. On the contrary, the development in series, which are abundantly sufficient for practical requirements, when only a finite portion of the surface is under consideration, presents no difficulty; and the formulæ thus derived open a fruitful source for the solution of many important problems. But here we shall develop only a single example in order to show the nature of the method.

23.

We shall now consider the case where all the lines for which p is constant are shortest lines cutting orthogonally the line for which $\phi = 0$, which line we can regard as the axis of abscissas. Let A be the point for which $r = 0$, D any point whatever on the axis of abscissas, $AD = p$, B any point whatever on the shortest line normal to AD at D, and $BD = q$, so that p can be regarded as the abscissa, q the ordinate of the point B. The abscissas we assume positive on the branch of the axis of abscissas to which $\phi = 0$ corresponds, while we always regard r as positive. We take the ordinates positive in the region in which ϕ is measured between 0 and 180°.

By the theorem of Art. 16 we shall have

$$\omega = 90°, \quad F = 0, \quad G = 1,$$

and we shall set also

$$\sqrt{E} = n.$$

Thus n will be a function of p, q, such that for $q = 0$ it must become equal to unity. The application of the formula of Art. 18 to our case shows that on any shortest line *whatever* we must have

$$d\theta = \frac{\partial n}{\partial q} \cdot dp,$$

where θ denotes the angle between the element of this line and the element of the line for which q is constant. Now since the axis of abscissas is itself a shortest line, and since, for it, we have everywhere $\theta = 0$, we see that for $q = 0$ we must have everywhere

$$\frac{\partial n}{\partial q} = 0.$$

Therefore we conclude that, if n is developed into a series in ascending powers of q, this series must have the following form:

$$n = 1 + fq^2 + gq^3 + hq^4 + \text{etc.}$$

where f, g, h, etc., will be functions of p, and we set

Disquisitiones Generales circa Superficies Curvas

$$f = f^0 + f'p + f''pp + \text{etc.}$$
$$g = g^0 + g'p + g''pp + \text{etc.}$$
$$h = h^0 + h'p + h''pp + \text{etc.}$$

etc. sive

$$\begin{aligned} n = 1 &+ f^0 qq + f'pqq + f''ppqq + \text{etc.} \\ &+ g^0 q^3 + g'pq^3 + \text{etc.} \\ &+ h^0 q^4 + \text{etc. etc.} \end{aligned}$$

24.

Aequationes art. 22 in casu nostro suppeditant

$$n \sin\psi = \frac{dr}{dp}, \quad \cos\psi = \frac{dr}{dq}, \quad -n\cos\psi = m \cdot \frac{d\varphi}{dp}, \quad \sin\psi = m \cdot \frac{d\varphi}{dq}$$
$$nn = nn\left(\frac{dr}{dq}\right)^2 + \left(\frac{dr}{dp}\right)^2, \quad nn \cdot \frac{dr}{dq} \cdot \frac{d\varphi}{dq} + \frac{dr}{dp} \cdot \frac{d\varphi}{dp} = 0$$

Adiumento harum aequationum, quarum quinta et sexta iam in reliquis continentur, series evolvi poterunt pro r, φ, ψ, m, vel pro quibuslibet functionibus harum quantitatum, e quibus eas, quae imprimis attentione sunt dignae, hic sistemus.

Quum pro valoribus infinite parvis ipsarum p, q fieri debeat $rr = pp + qq$, series pro rr incipiet a terminis $pp + qq$: terminos altiorum ordinum obtinemus per methodum coëfficientium indeterminatorum *) adiumento aequationis

$$\left(\frac{1}{n} \cdot \frac{drr}{dp}\right)^2 + \left(\frac{drr}{dq}\right)^2 = 4rr$$

scilicet

[1]
$$\begin{aligned} rr = pp &+ \tfrac{2}{3} f^0 ppqq + \tfrac{1}{2} f' p^3 qq + \left(\tfrac{2}{5} f'' - \tfrac{4}{45} f^0 f^0\right) p^4 qq \quad \text{etc.} \\ + qq &+ \tfrac{1}{2} g^0 ppq^3 + \tfrac{2}{5} g' p^3 q^3 \\ &+ \left(\tfrac{2}{5} h^0 - \tfrac{7}{45} f^0 f^0\right) ppq^4 \end{aligned}$$

Dein habemus, ducente formula $r \sin\psi = \frac{1}{2n} \cdot \frac{drr}{dp}$,

[2]
$$\begin{aligned} r\sin\psi = p &- \tfrac{1}{3} f^0 pqq - \tfrac{1}{4} f' ppqq - \left(\tfrac{1}{5} f'' + \tfrac{8}{45} f^0 f^0\right) p^3 qq \quad \text{etc.} \\ &- \tfrac{1}{2} g^0 pq^3 - \tfrac{2}{5} g' ppq^3 \\ &- \left(\tfrac{3}{5} h^0 - \tfrac{8}{45} f^0 f^0\right) pq^4 \end{aligned}$$

*) Calculum, qui per nonnulla artificia paullulum contrahi potest, hic adscribere superfluum duximus.

$$f = f^\circ + f' p + f'' p^2 + \text{etc.}$$
$$g = g^\circ + g' p + g'' p^2 + \text{etc.}$$
$$h = h^\circ + h' p + h'' p^2 + \text{etc.}$$

or

$$\begin{aligned} n = 1 + f^\circ q^2 + f' p q^2 + f'' p^2 q^2 &+ \text{etc.} \\ + g^\circ q^3 + g' p q^3 &+ \text{etc.} \\ + h^\circ q^4 &+ \text{etc. etc.} \end{aligned}$$

24.

The equations of Art. 22 give, in our case,

$$n \sin\psi = \frac{\partial r}{\partial p}, \quad \cos\psi = \frac{\partial r}{\partial q}, \quad -n\cos\psi = m \,.\, \frac{\partial \phi}{\partial p}, \quad \sin\psi = m \,.\, \frac{\partial \phi}{\partial q},$$

$$n^2 = n^2 \left(\frac{\partial r}{\partial q}\right)^2 + \left(\frac{\partial r}{\partial p}\right)^2, \quad n^2 \cdot \frac{\partial r}{\partial q} \cdot \frac{\partial \phi}{\partial q} + \frac{\partial r}{\partial p} \cdot \frac{\partial \phi}{\partial p} = 0$$

By the aid of these equations, the fifth and sixth of which are contained in the others, series can be developed for r, ϕ, ψ, m, or for any functions whatever of these quantities. We are going to establish here those series that are especially worthy of attention.

Since for infinitely small values of p, q we must have

$$r^2 = p^2 + q^2,$$

the series for r^2 will begin with the terms $p^2 + q^2$. We obtain the terms of higher order by the method of undetermined coefficients,* by means of the equation

$$\left(\frac{1}{n} \cdot \frac{\partial (r^2)}{\partial p}\right)^2 + \left(\frac{\partial (r^2)}{\partial q}\right)^2 = 4\, r^2$$

Thus we have

[1]
$$\begin{aligned} r^2 = p^2 &+ \tfrac{2}{3} f^\circ p^2 q^2 + \tfrac{1}{2} f' p^3 q^2 + (\tfrac{2}{5} f'' - \tfrac{4}{45} f^{\circ 2}) p^4 q^2 \quad \text{etc.} \\ + q^2 &\qquad\qquad + \tfrac{1}{2} g^\circ p^2 q^3 + \tfrac{2}{5} g' p^3 q^3 \\ &\qquad\qquad\qquad + (\tfrac{2}{5} h^\circ - \tfrac{7}{45} f^{\circ 2}) p^2 q^4 \end{aligned}$$

Then we have, from the formula

$$r \sin\psi = \frac{1}{2n} \cdot \frac{\partial (r^2)}{\partial p},$$

[2]
$$\begin{aligned} r \sin\psi = p &- \tfrac{1}{3} f^\circ p q^2 - \tfrac{1}{4} f' p^2 q^2 - (\tfrac{1}{5} f'' + \tfrac{8}{45} f^{\circ 2}) p^3 q^2 \quad \text{etc.} \\ &- \tfrac{1}{2} g^\circ p q^3 - \tfrac{2}{5} g' p^2 q^3 \\ &\qquad - (\tfrac{3}{5} h^\circ - \tfrac{8}{45} f^{\circ 2}) p q^4 \end{aligned}$$

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* We have thought it useless to give the calculation here, which can be somewhat abridged by certain artifices.

Disquisitiones Generales circa Superficies Curvas

nec non per formulam $r\cos\psi = \frac{1}{2}\cdot\frac{\mathrm{d}rr}{\mathrm{d}q}$

$$[3] \qquad r\cos\psi = q + \tfrac{2}{3}f^0ppq + \tfrac{1}{2}f'p^3q + (\tfrac{2}{5}f'' - \tfrac{4}{45}f^0f^0)p^4q \quad \text{etc.}$$
$$+ \tfrac{3}{4}g^0ppqq + \tfrac{3}{5}g'p^3qq$$
$$+ (\tfrac{1}{5}h^0 - \tfrac{14}{45}f^0f^0)ppq^3$$

Hinc simul innotescit angulus ψ. Perinde ad computum anguli φ concinnius evolvuntur series pro $r\cos\varphi$ atque $r\sin\varphi$, quibus inserviunt aequationes differentiales partiales

$$\frac{\mathrm{d}\,.\,r\cos\varphi}{\mathrm{d}p} = n\cos\varphi\,.\,\sin\psi - r\sin\varphi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}p}$$
$$\frac{\mathrm{d}\,.\,r\cos\varphi}{\mathrm{d}q} = \cos\varphi\,.\cos\psi - r\sin\varphi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}q}$$
$$\frac{\mathrm{d}\,.\,r\sin\varphi}{\mathrm{d}p} = n\sin\varphi\,.\,\sin\psi + r\cos\varphi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}p}$$
$$\frac{\mathrm{d}\,.\,r\sin\varphi}{\mathrm{d}q} = \sin\varphi\,.\,\cos\psi + r\cos\varphi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}q}$$
$$n\cos\psi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}q} + \sin\psi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}p} = 0$$

quarum combinatio suppeditat

$$\frac{r\sin\psi}{n}\,.\,\frac{\mathrm{d}\,.\,r\cos\varphi}{\mathrm{d}p} + r\cos\psi\,.\,\frac{\mathrm{d}\,.\,r\cos\varphi}{\mathrm{d}q} = r\cos\varphi$$
$$\frac{r\sin\psi}{n}\,.\,\frac{\mathrm{d}\,.\,r\sin\varphi}{\mathrm{d}p} + r\cos\psi\,.\,\frac{\mathrm{d}\,.\,r\sin\varphi}{\mathrm{d}q} = r\sin\varphi$$

Hinc facile evolvuntur series pro $r\cos\varphi$, $r\sin\varphi$, quarum termini primi manifesto esse debent p et q, puta

$$[4] \qquad r\cos\varphi = p + \tfrac{2}{3}f^0pqq + \tfrac{5}{12}f'ppqq + (\tfrac{3}{10}f'' - \tfrac{8}{45}f^0f^0)p^3qq \quad \text{etc.}$$
$$+ \tfrac{1}{2}g^0pq^3 + \tfrac{7}{20}g'ppq^3$$
$$+ (\tfrac{2}{5}h^0 - \tfrac{7}{45}f^0f^0)pq^4$$
$$[5] \qquad r\sin\varphi = q - \tfrac{1}{3}f^0ppq - \tfrac{1}{6}f'p^3q - (\tfrac{1}{10}f'' - \tfrac{7}{90}f^0f^0)p^4q \quad \text{etc.}$$
$$- \tfrac{1}{4}g^0ppqq - \tfrac{3}{20}g'p^3qq$$
$$- (\tfrac{1}{5}h^0 + \tfrac{13}{90}f^0f^0)ppq^3$$

E combinatione aequationum [2], [3], [4], [5] derivari posset series pro $rr\cos(\psi+\varphi)$, atque hinc, dividendo per seriem [1], series pro $\cos(\psi+\varphi)$, a qua ad seriem pro ipso angulo $\psi+\varphi$ descendere liceret. Elegantius tamen eadem obtinetur se-and from the formula

$$r\cos\psi = \tfrac{1}{2}\frac{\partial(r^2)}{\partial q}$$

[3] $$r\cos\psi = q + \tfrac{2}{3}f^\circ p^2 q + \tfrac{1}{2}f' p^3 q + (\tfrac{2}{5}f'' - \tfrac{4}{45}f^{\circ 2})p^4 q \quad \text{etc.}$$
$$+ \tfrac{3}{4}g^\circ p^2 q^2 + \tfrac{3}{5}g' p^3 q^2$$
$$+ (\tfrac{4}{5}h^\circ - \tfrac{14}{45}f^{\circ 2})p^2 q^3$$

These formulæ give the angle ψ. In like manner, for the calculation of the angle ϕ, series for $r\cos\phi$ and $r\sin\phi$ are very elegantly developed by means of the partial differential equations

$$\frac{\partial\,.\,r\cos\phi}{\partial p} = n\cos\phi\,.\,\sin\psi - r\sin\phi\,.\,\frac{\partial\phi}{\partial p}$$

$$\frac{\partial\,.\,r\cos\phi}{\partial q} = \cos\phi\,.\,\cos\psi - r\sin\phi\,.\,\frac{\partial\phi}{\partial q}$$

$$\frac{\partial\,.\,r\sin\phi}{\partial p} = n\sin\phi\,.\,\sin\psi + r\cos\phi\,.\,\frac{\partial\phi}{\partial p}$$

$$\frac{\partial\,.\,r\sin\phi}{\partial q} = \sin\phi\,.\,\cos\psi + r\cos\phi\,.\,\frac{\partial\phi}{\partial q}$$

$$n\cos\psi\,.\,\frac{\partial\phi}{\partial q} + \sin\psi\,.\,\frac{\partial\phi}{\partial p} = 0$$

A combination of these equations gives

$$\frac{r\sin\psi}{n}\,.\,\frac{\partial\,.\,r\cos\phi}{\partial p} + r\cos\psi\,.\,\frac{\partial\,.\,r\cos\phi}{\partial q} = r\cos\phi$$

$$\frac{r\sin\psi}{n}\,.\,\frac{\partial\,.\,r\sin\phi}{\partial p} + r\cos\psi\,.\,\frac{\partial\,.\,r\sin\phi}{\partial q} = r\sin\phi$$

From these two equations series for $r\cos\phi$, $r\sin\phi$ are easily developed, whose first terms must evidently be p, q respectively. The series are

[4] $$r\cos\phi = p + \tfrac{2}{3}f^\circ p q^2 + \tfrac{5}{12}f' p^2 q^2 + (\tfrac{3}{10}f'' - \tfrac{8}{45}f^{\circ 2})p^3 q^2 \quad \text{etc.}$$
$$+ \tfrac{1}{2}g^\circ p q^3 + \tfrac{7}{20}g' p^2 q^3$$
$$+ (\tfrac{2}{5}h^\circ - \tfrac{7}{45}f^{\circ 2})p q^4$$

[5] $$r\sin\phi = q - \tfrac{1}{3}f^\circ p^2 q - \tfrac{1}{6}f' p^3 q - (\tfrac{1}{10}f'' - \tfrac{7}{90}f^{\circ 2})p^4 q \quad \text{etc.}$$
$$- \tfrac{1}{4}g^\circ p^2 q^2 - \tfrac{3}{20}g' p^3 q^2$$
$$- (\tfrac{1}{5}h^\circ + \tfrac{13}{90}f^{\circ 2})p^2 q^3$$

From a combination of equations [2], [3], [4], [5] a series for $r^2\cos(\psi+\phi)$, may be derived, and from this, dividing by the series [1], a series for $\cos(\psi+\phi)$, from quenti modo. Differentiando aequationem primam et secundam ex iis, quae initio huius art. allatae sunt, obtinemus

$$\sin\psi\,.\,\frac{\mathrm{d}n}{\mathrm{d}q}+n\cos\psi\,.\,\frac{\mathrm{d}\psi}{\mathrm{d}q}+\sin\psi\,.\,\frac{\mathrm{d}\psi}{\mathrm{d}p}=0$$

qua combinata cum hac

$$n\cos\psi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}q}+\sin\psi\,.\,\frac{\mathrm{d}\varphi}{\mathrm{d}p}=0$$

prodit

$$\frac{r\sin\psi}{n}\,.\,\frac{\mathrm{d}n}{\mathrm{d}q}+\frac{r\sin\psi}{n}\,.\,\frac{\mathrm{d}(\psi+\varphi)}{\mathrm{d}p}+r\cos\psi\,.\,\frac{\mathrm{d}(\psi+\varphi)}{\mathrm{d}q}=0$$

Ex hac aequatione adiumento methodi coëfficientium indeterminatorum facile eliciemus seriem pro $\psi+\varphi$, si perpendimus, ipsius terminum primum esse debere $\frac{1}{2}\pi$, radio pro unitate accepto, atque denotante 2π peripheriam circuli,

$$[6]\qquad \begin{aligned}\psi+\varphi=\tfrac{1}{2}\pi&-f^0pq-\tfrac{2}{3}f'ppq-(\tfrac{1}{2}f''-\tfrac{1}{6}f^0f^0)p^3q \quad\text{etc.}\\ &-g^0pqq\;-\tfrac{3}{4}g'ppqq\\ &-(h^0-\tfrac{1}{3}f^0f^0)pq^3\end{aligned}$$

Operae pretium videtur, etiam aream trianguli ABD in seriem evolvere. Huic evolutioni inservit aequatio conditionalis sequens, quae e considerationibus geometricis satis obviis facile derivatur, et in qua S aream quaesitam denotat:

$$\frac{r\sin\psi}{n}\,.\,\frac{\mathrm{d}S}{\mathrm{d}p}+r\cos\psi\,.\,\frac{\mathrm{d}S}{\mathrm{d}q}=\frac{r\sin\psi}{n}\,.\int n\,\mathrm{d}q$$

integratione a $q=0$ incepta. Hinc scilicet obtinemus per methodum coëfficientium indeterminatorum

$$[7]\qquad \begin{aligned}S=\tfrac{1}{2}pq&-\tfrac{1}{12}f^0p^3q-\tfrac{1}{20}f'p^4q\;-(\tfrac{1}{30}f''-\tfrac{1}{60}f^0f^0)p^5q \quad\text{etc.}\\ &-\tfrac{1}{12}f^0pq^3-\tfrac{3}{40}g^0p^3qq\;-\tfrac{1}{20}g'p^4qq\\ &-\tfrac{7}{120}f'ppq^3-(\tfrac{1}{15}h^0+\tfrac{2}{45}f''+\tfrac{1}{60}f^0f^0)p^3q^3\\ &-\tfrac{1}{10}g^0pq^4\;-\tfrac{3}{40}g'ppq^4\\ &-(\tfrac{1}{10}h^0-\tfrac{1}{30}f^0f^0)pq^5\end{aligned}$$

25.

A formulis art. praec., quae referuntur ad triangulum a lineis brevissimis formatum rectangulum, progredimur ad generalia. Sit C aliud punctum in ea-which may be found a series for the angle $\psi + \phi$ itself. However, the same series can be obtained more elegantly in the following manner. By differentiating the first and second of the equations introduced at the beginning of this article, we obtain

$$\sin\psi \cdot \frac{\partial n}{\partial q} + n\cos\psi \cdot \frac{\partial \psi}{\partial q} + \sin\psi \cdot \frac{\partial \psi}{\partial p} = 0$$

and this combined with the equation

$$n\cos\psi \cdot \frac{\partial \phi}{\partial q} + \sin\psi \cdot \frac{\partial \phi}{\partial p} = 0$$

gives

$$\frac{r\sin\psi}{n} \cdot \frac{\partial n}{\partial q} + \frac{r\sin\psi}{n} \cdot \frac{\partial(\psi+\phi)}{\partial p} + r\cos\psi \cdot \frac{\partial(\psi+\phi)}{\partial q} = 0$$

From this equation, by aid of the method of undetermined coefficients, we can easily derive the series for $\psi + \phi$, if we observe that its first term must be $\frac{1}{2}\pi$, the radius being taken equal to unity and 2π denoting the circumference of the circle,

$$[6] \qquad \begin{aligned} \psi + \phi = \tfrac{1}{2}\pi - f^\circ pq - \tfrac{2}{3}f'p^2q - (\tfrac{1}{2}f'' - \tfrac{1}{6}f^{\circ 2})p^3q \quad \text{etc.} \\ - g^\circ pq^2 - \tfrac{3}{4}g'p^2q^2 \\ - (h^\circ - \tfrac{1}{3}f^{\circ 2})pq^3 \end{aligned}$$

It seems worth while also to develop the area of the triangle ABD into a series. For this development we may use the following conditional equation, which is easily derived from sufficiently obvious geometric considerations, and in which S denotes the required area:

$$\frac{r\sin\psi}{n} \cdot \frac{\partial S}{\partial p} + r\cos\psi \cdot \frac{\partial S}{\partial q} = \frac{r\sin\psi}{n} \cdot \int n\, dq$$

the integration beginning with $q = 0$. From this equation we obtain, by the method of undetermined coefficients,

$$[7] \qquad \begin{aligned} S = \tfrac{1}{2}pq - \tfrac{1}{12}f^\circ p^3q - \tfrac{1}{20}f'p^4q - (\tfrac{1}{30}f'' - \tfrac{1}{60}f^{\circ 2})p^5q \quad \text{etc.} \\ - \tfrac{1}{12}f^\circ pq^3 - \tfrac{3}{40}g^\circ p^3q^2 - \tfrac{1}{20}g'p^4q^2 \\ - \tfrac{7}{120}f'p^2q^3 - (\tfrac{1}{15}h^\circ + \tfrac{2}{45}f'' + \tfrac{1}{60}f^{\circ 2})p^3q^3 \\ - \tfrac{1}{10}g^\circ pq^4 - \tfrac{3}{40}g'p^2q^4 \\ - (\tfrac{1}{10}h^\circ - \tfrac{1}{30}f^{\circ 2})pq^5 \end{aligned}$$

25.

From the formulæ of the preceding article, which refer to a right triangle formed by shortest lines, we proceed to the general case. Let C be another point on the dem linea brevissima DB, pro quo, manente p, characteres q', r', φ', ψ', S' eadem designent, quae q, r, φ, ψ, S pro puncto B. Ita oritur triangulum inter puncta A, B, C, cuius angulos per A, B, C, latera opposita per a, b, c, aream per σ denotamus; mensuram curvaturae in punctis A, B, C resp. per α, β, γ exprimemus. Supponendo itaque (quod licet), quantitates p, q, $q-q'$ esse positivas, habemus

$$A = \varphi - \varphi', \quad B = \psi, \quad C = \pi - \psi', \quad a = q - q', \quad b = r', \quad c = r, \quad \sigma = S - S'$$

Ante omnia aream σ per seriem exprimemus. Mutando in [7] singulas quantitates ad B relatas in eas, quae ad C referuntur, prodit formula pro S', unde, usque ad quantitates sexti ordinis obtinemus

$$\begin{aligned}\sigma = \tfrac{1}{2}p(q-q')\{1 &- \tfrac{1}{6}f^0(pp+qq+qq'+q'q') \\ &- \tfrac{1}{60}f'p(6pp+7qq+7qq'+7q'q') \\ &- \tfrac{1}{20}g^0(q+q')(3pp+4qq+4qq'+4q'q')\}\end{aligned}$$

Haec formula, adiumento seriei [2] puta

$$c\sin B = p(1 - \tfrac{1}{3}f^0qq - \tfrac{1}{4}f'pqq - \tfrac{1}{2}g^0q^3 - \text{ etc.})$$

transit in sequentem

$$\begin{aligned}\sigma = \tfrac{1}{2}ac\sin B\{1 &- \tfrac{1}{6}f^0(pp-qq+qq'+q'q') \\ &- \tfrac{1}{60}f'p(6pp-8qq+7qq'+7q'q') \\ &- \tfrac{1}{20}g^0(3ppq+3ppq'-6q^3+4qqq'+4qq'q'+4q'^3)\}\end{aligned}$$

Mensura curvaturae pro quovis superficiei puncto fit (per art. 19, ubi m, p, q erant quae hic sunt n, q, p)

$$= -\frac{1}{n}\cdot\frac{\mathrm{d}\mathrm{d}n}{\mathrm{d}q^2} = -\frac{2f+6gq+12hqq+\text{ etc.}}{1+fqq+\text{ etc.}} = -2f-6gq-(12h-2ff)qq - \text{ etc.}$$

Hinc fit, quatenus p, q ad punctum B referuntur,

$$\beta = -2f^0 - 2f'p - 6g^0q - 2f''pp - 6g'pq - (12h^0 - 2f^0f^0)qq - \text{ etc.}$$

nec non

$$\begin{aligned}\gamma &= -2f^0 - 2f'p - 6g^0q' - 2f''pp - 6g'pq' - (12h^0 - 2f^0f^0)q'q' - \text{ etc.} \\ \alpha &= -2f^0.\end{aligned}$$

General Investigations of Curved Surfaces

same shortest line DB, for which point p remains the same as for the point B, and q', r', ϕ', ψ', S' have the same meanings as q, r, ϕ, ψ, S have for the point B. There will thus be a triangle between the points A, B, C, whose angles we denote by A, B, C, the sides opposite these angles by a, b, c, and the area by σ. We represent the measure of curvature at the points A, B, C by α, β, γ respectively. And then supposing (which is permissible) that the quantities p, q, $q-q'$ are positive, we shall have

$$A = \phi - \phi', \quad B = \psi, \quad C = \pi - \psi',$$
$$a = q - q', \quad b = r', \quad c = r, \quad \sigma = S - S'.$$

We shall first express the area σ by a series. By changing in [7] each of the quantities that refer to B into those that refer to C, we obtain a formula for S'. Whence we have, exact to quantities of the sixth order,

$$\begin{aligned}\sigma = \tfrac{1}{2} p(q-q')\big(1 &- \tfrac{1}{6} f^\circ (p^2 + q^2 + qq' + q'^2) \\ &- \tfrac{1}{60} f'p(6p^2 + 7q^2 + 7qq' + 7q'^2) \\ &- \tfrac{1}{20} g^\circ (q+q')(3p^2 + 4q^2 + 4q'^2)\big)\end{aligned}$$

This formula, by aid of series [2], namely,

$$c \sin B = p\left(1 - \tfrac{1}{3} f^\circ q^2 - \tfrac{1}{4} f'pq^2 - \tfrac{1}{2} g^\circ q^3 - \text{etc.}\right)$$

can be changed into the following:

$$\begin{aligned}\sigma = \tfrac{1}{2} ac \sin B\big(1 &- \tfrac{1}{6} f^\circ (p^2 - q^2 + qq' + q'^2) \\ &- \tfrac{1}{60} f'p(6p^2 - 8q^2 + 7qq' + 7q'^2) \\ &- \tfrac{1}{20} g^\circ (3p^2 q + 3p^2 q' - 6p^3 + 4q^2 q' + 4qq'^2 + 4q'^3)\big)\end{aligned}$$

The measure of curvature for any point whatever of the surface becomes (by Art. 19, where m, p, q were what n, q, p are here)

$$\begin{aligned}k &= -\frac{1}{n}\cdot\frac{\partial^2 n}{\partial q^2} = -\frac{2f + 6gq + 12hq^2 + \text{etc.}}{1 + fq^2 + \text{etc.}} \\ &= -2f - 6gq - (12h - 2f^2)q^2 - \text{etc.}\end{aligned}$$

Therefore we have, when p, q refer to the point B,

$$\beta = -2f^\circ - 2f'p - 6g^\circ q - 2f''p^2 - 6g'pq - (12h^\circ - 2f^{\circ 2})q^2 - \text{etc.}$$

Also

$$\gamma = -2f^\circ - 2f'p - 6g^\circ q' - 2f''p^2 - 6g'pq' - (12h^\circ - 2f^{\circ 2})q'^2 - \text{etc.}$$
$$\alpha = -2f^\circ$$

Introducendo has mensuras curvaturae in serie pro σ, obtinemus expressionem sequentem, usque ad quantitates sexti ordinis (excl.) exactam:

$$\begin{aligned}\sigma = \tfrac{1}{2} ac\sin B\{1 &+ \tfrac{1}{120}\alpha(4pp - 2qq + 3qq' + 3q'q')\\ &+ \tfrac{1}{120}\beta(3pp - 6qq + 6qq' + 3q'q')\\ &+ \tfrac{1}{120}\gamma(3pp - 2qq + qq' + 4q'q')\}\end{aligned}$$

Praecisio eadem manebit, si pro p, q, q' substituimus $c\sin B$, $c\cos B$, $c\cos B - a$, quo pacto prodit

[8]
$$\begin{aligned}\sigma = \tfrac{1}{2} ac\sin B\{1 &+ \tfrac{1}{120}\alpha(3aa + 4cc - 9ac\cos B)\\ &+ \tfrac{1}{120}\beta(3aa + 3cc - 12ac\cos B)\\ &+ \tfrac{1}{120}\gamma(4aa + 3cc - 9ac\cos B)\}\end{aligned}$$

Quum ex hac aequatione omnia, quae ad lineam AD normaliter ad BC ductam referuntur, evanuerint, etiam puncta A, B, C cum correlatis inter se permutare licebit, quapropter erit eadem praecisione

[9]
$$\begin{aligned}\sigma = \tfrac{1}{2} bc\sin A\{1 &+ \tfrac{1}{120}\alpha(3bb + 3cc - 12bc\cos A)\\ &+ \tfrac{1}{120}\beta(3bb + 4cc - 9bc\cos A)\\ &+ \tfrac{1}{120}\gamma(4bb + 3cc - 9bc\cos A)\}\end{aligned}$$

[10]
$$\begin{aligned}\sigma = \tfrac{1}{2} ab\sin C\{1 &+ \tfrac{1}{120}\alpha(3aa + 4bb - 9ab\cos C)\\ &+ \tfrac{1}{120}\beta(4aa + 3bb - 9ab\cos C)\\ &+ \tfrac{1}{120}\gamma(3aa + 3bb - 12ab\cos C)\}\end{aligned}$$

26.

Magnam utilitatem affert consideratio trianguli plani rectilinei, cuius latera aequalia sunt ipsis a, b, c; anguli illius trianguli, quos per A^*, B^*, C^* designabimus, different ab angulis trianguli in superficie curva, puta ab A, B, C, quantitatibus secundi ordinis, operaeque pretium erit, has differentias accurate evolvere. Calculorum autem prolixiorum quam difficiliorum, primaria momenta apposuisse sufficiet.

Mutando in formulis [1], [4], [5], quantitates, quae referuntur ad B, in eas, quae referuntur ad C, nanciscemur formulas pro $r'r'$, $r'\cos\varphi'$, $r'\sin\varphi'$. Tunc evolutio expressionis $rr + r'r' - (q - q')^2 - 2r\cos\varphi . r'\cos\varphi' - 2r\sin\varphi . r'\sin\varphi'$, quae fit $= bb + cc - aa - 2bc\cos A = 2bc(\cos A^* - \cos A)$, combinata cum evolutione

Introducing these measures of curvature into the expression for σ, we obtain the following expression, exact to quantities of the sixth order (exclusive):

$$\begin{aligned}\sigma = \tfrac{1}{2}\, ac \sin B \big(1 &+ \tfrac{1}{120}\, \alpha\, (4p^2 - 2q^2 + 3qq' + 3q'^2) \\ &+ \tfrac{1}{120}\, \beta\, (3p^2 - 6q^2 + 6qq' + 3q'^2) \\ &+ \tfrac{1}{120}\, \gamma\, (3p^2 - 2q^2 + qq' + 4q'^2)\big)\end{aligned}$$

The same precision will remain, if for p, q, q' we substitute $c \sin B$, $c \cos B$, $c \cos B - a$. This gives

[8]
$$\begin{aligned}\sigma = \tfrac{1}{2}\, ac \sin B \big(1 &+ \tfrac{1}{120}\, \alpha\, (3a^2 + 4c^2 - 9ac \cos B) \\ &+ \tfrac{1}{120}\, \beta\, (3a^2 + 3c^2 - 12ac \cos B) \\ &+ \tfrac{1}{120}\, \gamma\, (4a^2 + 3c^2 - 9ac \cos B)\big)\end{aligned}$$

Since all expressions which refer to the line AD drawn normal to BC have disappeared from this equation, we may permute among themselves the points A, B, C and the expressions that refer to them. Therefore we shall have, with the same precision,

[9]
$$\begin{aligned}\sigma = \tfrac{1}{2}\, bc \sin A \big(1 &+ \tfrac{1}{120}\, \alpha\, (3b^2 + 3c^2 - 12bc \cos A) \\ &+ \tfrac{1}{120}\, \beta\, (3b^2 + 4c^2 - 9bc \cos A) \\ &+ \tfrac{1}{120}\, \gamma\, (4b^2 + 3c^2 - 9bc \cos A)\big)\end{aligned}$$

[10]
$$\begin{aligned}\sigma = \tfrac{1}{2}\, ab \sin C \big(1 &+ \tfrac{1}{120}\, \alpha\, (3a^2 + 4b^2 - 9ab \cos C) \\ &+ \tfrac{1}{120}\, \beta\, (4a^2 + 3b^2 - 9ab \cos C) \\ &+ \tfrac{1}{120}\, \gamma\, (3a^2 + 3b^2 - 12ab \cos C)\big)\end{aligned}$$

26.

The consideration of the rectilinear triangle whose sides are equal to a, b, c is of great advantage. The angles of this triangle, which we shall denote by A^*, B^*, C^*, differ from the angles of the triangle on the curved surface, namely, from A, B, C, by quantities of the second order; and it will be worth while to develop these differences accurately. However, it will be sufficient to show the first steps in these more tedious than difficult calculations.

Replacing in formulæ [1], [4], [5] the quantities that refer to B by those that refer to C, we get formulæ for r'^2, $r' \cos \phi'$, $r' \sin \phi'$. Then the development of the expression

$$\begin{aligned}r^2 + r'^2 - (q - q')^2 - 2r \cos \phi \,.\, r' \cos \phi' - 2r \sin \phi \,.\, r' \sin \phi' \\ = b^2 + c^2 - a^2 - 2bc \cos A \\ = 2bc\,(\cos A^* - \cos A),\end{aligned}$$

expressionis $r\sin\varphi \,.\, r'\cos\varphi' - r\cos\varphi \,.\, r'\sin\varphi'$, quae fit $= bc\sin A$, suppeditat formulam sequentem

$$\begin{aligned}\cos A^* - \cos A = -(q-q')p\sin A\,\{&\tfrac{1}{3}f^0 + \tfrac{1}{6}f'p + \tfrac{1}{4}g^0(q+q')\\ &+ (\tfrac{1}{10}f'' - \tfrac{1}{45}f^0f^0)\,pp + \tfrac{3}{20}g'p(q+q')\\ &+ (\tfrac{1}{5}h^0 - \tfrac{7}{90}f^0f^0)(qq+qq'+q'q') + \text{etc.}\}\end{aligned}$$

Hinc fit porro, usque ad quantitates quinti ordinis

$$\begin{aligned}A^* - A = -(q-q')p\,\{&\tfrac{1}{3}f^0 + \tfrac{1}{6}f'p + \tfrac{1}{4}g^0(q+q') + \tfrac{1}{10}f''pp\\ &+ \tfrac{3}{20}g'p(q+q') + \tfrac{1}{5}h^0(qq+qq'+q'q')\\ &- \tfrac{1}{90}f^0f^0\,(7pp+7qq+12qq'+7q'q')\}\end{aligned}$$

Combinando hanc formulam cum hac

$$2\sigma = ap\big(1 - \tfrac{1}{6}f^0(pp+qq+qq'+q'q' - \text{etc.})\big)$$

atque cum valoribus quantitatum α, β, γ in art. praec. allatis, obtinemus usque ad quantitates quinti ordinis

$$\begin{aligned}[11]\quad A^* = A - \sigma\,\{&\tfrac{1}{6}\alpha + \tfrac{1}{12}\beta + \tfrac{1}{12}\gamma + \tfrac{2}{15}f''pp + \tfrac{1}{5}g'p(q+q')\\ &+ \tfrac{1}{5}h^0(3qq - 2qq' + 3q'q')\\ &+ \tfrac{1}{90}f^0f^0(4pp - 11qq + 14qq' - 11q'q')\}\end{aligned}$$

Per operationes prorsus similes evolvimus

$$\begin{aligned}[12]\quad B^* = B - \sigma\,\{&\tfrac{1}{12}\alpha + \tfrac{1}{6}\beta + \tfrac{1}{12}\gamma + \tfrac{1}{10}f''pp + \tfrac{1}{10}g'p(2q+q')\\ &+ \tfrac{1}{5}h^0(4qq - 4qq' + 3q'q')\\ &- \tfrac{1}{90}f^0f^0(2pp + 8qq - 8qq' + 11q'q')\}\\ [13]\quad C^* = C - \sigma\,\{&\tfrac{1}{12}\alpha + \tfrac{1}{12}\beta + \tfrac{1}{6}\gamma + \tfrac{1}{10}f''pp + \tfrac{1}{10}g'p(q+2q')\\ &+ \tfrac{1}{5}h^0(3qq - 4qq' + 4q'q')\\ &- \tfrac{1}{90}f^0f^0(2pp + 11qq - 8qq' + 8q'q')\}\end{aligned}$$

Hinc simul deducimus, quum summa $A^* + B^* + C^*$ duobus rectis aequalis sit, excessum summae $A + B + C$ supra duos angulos rectos, puta

$$\begin{aligned}[14]\quad A + B + C = \pi + \sigma\{&\tfrac{1}{3}\alpha + \tfrac{1}{3}\beta + \tfrac{1}{3}\gamma + \tfrac{1}{3}f''pp + \tfrac{1}{2}g'p(q+q')\\ &+ (2h^0 - \tfrac{1}{3}f^0f^0)(qq - qq' + q'q')\}\end{aligned}$$

Haec ultima aequatio etiam formulae [6] superstrui potuisset.

combined with the development of the expression

$$r \sin \phi \,.\, r' \cos \phi' - r \cos \phi \,.\, r' \sin \phi' = bc \sin A,$$

gives the following formula:

$$\begin{aligned}\cos A^* - \cos A = -(q-q')\,p \sin A \,\bigl(&\tfrac{1}{3} f^\circ + \tfrac{1}{6} f' p + \tfrac{1}{4} g^\circ (q+q') \\ &+ (\tfrac{1}{10} f'' - \tfrac{1}{45} f^{\circ 2})\, p^2 + \tfrac{3}{20} g' p (q+q') \\ &+ (\tfrac{1}{5} h^\circ - \tfrac{7}{90} f^{\circ 2})(q^2 + qq' + q'^2) + \text{etc.}\bigr)\end{aligned}$$

From this we have, to quantities of the fifth order,

$$\begin{aligned}A^* - A = +(q-q')\,p\,\bigl(&\tfrac{1}{3} f^\circ + \tfrac{1}{6} f' p + \tfrac{1}{4} g^\circ (q+q') + \tfrac{1}{10} f'' p^2 \\ &+ \tfrac{3}{20} g' p (q+q') + \tfrac{1}{5} h^\circ (q^2 + qq' + q'^2) \\ &- \tfrac{1}{90} f^{\circ 2} (7p^2 + 7q^2 + 12qq' + 7q'^2)\bigr)\end{aligned}$$

Combining this formula with

$$2\sigma = ap\bigl(1 - \tfrac{1}{6} f^\circ (p^2 + q^2 + qq' + q'^2) - \text{etc.}\bigr)$$

and with the values of the quantities α, β, γ found in the preceding article, we obtain, to quantities of the fifth order,

[11]
$$\begin{aligned}A^* = A - \sigma\bigl(&\tfrac{1}{6}\alpha + \tfrac{1}{12}\beta + \tfrac{1}{12}\gamma + \tfrac{2}{15} f'' p^2 + \tfrac{1}{5} g' p (q+q') \\ &+ \tfrac{1}{5} h^\circ (3q^2 - 2qq' + 3q'^2) \\ &+ \tfrac{1}{90} f^{\circ 2} (4p^2 - 11q^2 + 14qq' - 11q'^2)\bigr)\end{aligned}$$

By precisely similar operations we derive

[12]
$$\begin{aligned}B^* = B - \sigma\bigl(&\tfrac{1}{12}\alpha + \tfrac{1}{6}\beta + \tfrac{1}{12}\gamma + \tfrac{1}{10} f'' p^2 + \tfrac{1}{10} g' p (2q+q') \\ &+ \tfrac{1}{5} h^\circ (4q^2 - 4qq' + 3q'^2) \\ &- \tfrac{1}{90} f^{\circ 2} (2p^2 + 8q^2 - 8qq' + 11q'^2)\bigr)\end{aligned}$$

[13]
$$\begin{aligned}C^* = C - \sigma\bigl(&\tfrac{1}{12}\alpha + \tfrac{1}{12}\beta + \tfrac{1}{6}\gamma + \tfrac{1}{10} f'' p^2 + \tfrac{1}{10} g' p (q+2q') \\ &+ \tfrac{1}{5} h^\circ (3q^2 - 4qq' + 4q'^2) \\ &- \tfrac{1}{90} f^{\circ 2} (2p^2 + 11q^2 - 8qq' + 8q'^2)\bigr)\end{aligned}$$

From these formulæ we deduce, since the sum $A^* + B^* + C^*$ is equal to two right angles, the excess of the sum $A + B + C$ over two right angles, namely,

[14]
$$\begin{aligned}A + B + C = \pi + \sigma\bigl(&\tfrac{1}{3}\alpha + \tfrac{1}{3}\beta + \tfrac{1}{3}\gamma + \tfrac{1}{3} f'' p^2 + \tfrac{1}{2} g' p (q+q') \\ &+ (2h^\circ - \tfrac{1}{3} f^{\circ 2})(q^2 - qq' + q'^2)\bigr)\end{aligned}$$

This last equation could also have been derived from formula [6].

Disquisitiones Generales circa Superficies Curvas

27.

Si superficies curva est sphaera, cuius radius $= R$, erit

$$\alpha = \beta = \gamma = -2f^0 = \frac{1}{RR};\quad f'' = 0,\quad g' = 0,\quad 6h^0 - f^0 f^0 = 0 \text{ sive } k^0 = \frac{1}{24R^4}$$

Hinc formula [14] fit

$$A + B + C = \pi + \frac{\sigma}{RR}$$

quae praecisione absoluta gaudet; formulae 11—13 autem suppeditant

$$A^* = A - \frac{\sigma}{3RR} - \frac{\sigma}{180R^4}(2pp - qq + 4qq' - q'q')$$
$$B^* = B - \frac{\sigma}{3RR} + \frac{\sigma}{180R^4}(pp - 2qq + 2qq' + q'q')$$
$$C^* = C - \frac{\sigma}{3RR} + \frac{\sigma}{180R^4}(pp + qq + 2qq' - 2q'q')$$

sive aeque exacte

$$A^* = A - \frac{\sigma}{3RR} - \frac{\sigma}{180R^4}(bb + cc - 2aa)$$
$$B^* = B - \frac{\sigma}{3RR} - \frac{\sigma}{180R^4}(aa + cc - 2bb)$$
$$C^* = C - \frac{\sigma}{3RR} - \frac{\sigma}{180R^4}(aa + bb - 2cc)$$

Neglectis quantitatibus quarti ordinis, prodit hinc theorema notum a clar. LEGENDRE primo propositum.

28.

Formulae nostrae generales, reiectis terminis quarti ordinis, persimplices evadunt, scilicet

$$A^* = A - \tfrac{1}{12}\sigma(2\alpha + \beta + \gamma)$$
$$B^* = B - \tfrac{1}{12}\sigma(\alpha + 2\beta + \gamma)$$
$$C^* = C - \tfrac{1}{12}\sigma(\alpha + \beta + 2\gamma)$$

Angulis itaque A, B, C in superficie non sphaerica reductiones inaequales applicandae sunt, ut mutatorum sinus lateribus oppositis fiant proportionales. Inaequalitas generaliter loquendo erit tertii ordinis, at si superficies parum a sphaera discrepat, illa ad ordinem altiorem referenda erit: in triangulis vel maximis in superficie telluris, quorum quidem angulos dimetiri licet, differentia semper pro

General Investigations of Curved Surfaces

27.

If the curved surface is a sphere of radius R, we shall have

$$\alpha = \beta = \gamma = -2f^\circ = \frac{1}{R^2};\ f'' = 0, \quad g' = 0, \quad 6h^\circ - f^{\circ 2} = 0,$$

or

$$h^\circ = \frac{1}{24R^4}.$$

Consequently, formula [14] becomes

$$A + B + C = \pi + \frac{\sigma}{R^2},$$

which is absolutely exact. But formulæ [11], [12], [13] give

$$A^* = A - \frac{\sigma}{3R^2} - \frac{\sigma}{180R^4}(2p^2 - q^2 + 4qq' - q'^2)$$

$$B^* = B - \frac{\sigma}{3R^2} + \frac{\sigma}{180R^4}(p^2 - 2q^2 + 2qq' + q'^2)$$

$$C^* = C - \frac{\sigma}{3R^2} + \frac{\sigma}{180R^4}(p^2 + q^2 + 2qq' - 2q'^2)$$

or, with equal exactness,

$$A^* = A - \frac{\sigma}{3R^2} - \frac{\sigma}{180R^4}(b^2 + c^2 - 2a^2)$$

$$B^* = B - \frac{\sigma}{3R^2} - \frac{\sigma}{180R^4}(a^2 + c^2 - 2b^2)$$

$$C^* = C - \frac{\sigma}{3R^2} - \frac{\sigma}{180R^4}(a^2 + b^2 - 2c^2)$$

Neglecting quantities of the fourth order, we obtain from the above the well-known theorem first established by the illustrious Legendre.

28.

Our general formulæ, if we neglect terms of the fourth order, become extremely simple, namely:

$$A^* = A - \tfrac{1}{12}\sigma(2\alpha + \beta + \gamma)$$
$$B^* = B - \tfrac{1}{12}\sigma(\alpha + 2\beta + \gamma)$$
$$C^* = C - \tfrac{1}{12}\sigma(\alpha + \beta + 2\gamma)$$

Thus to the angles A, B, C on a non-spherical surface, unequal reductions must be applied, so that the sines of the changed angles become proportional to the sides opposite. The inequality, generally speaking, will be of the third order; but if the surface differs little from a sphere, the inequality will be of a higher order. Even in the greatest triangles on the earth's surface, whose angles it is possible to measure, insensibili haberi potest. Ita e.g. in triangulo maximo inter ea, quae annis praecedentibus dimensi sumus, puta inter puncta Hohehagen, Brocken, Inselsberg, ubi excessus summae angulorum fuit $= 14''85348$, calculus sequentes reductiones angulis applicandas prodidit:

Hohehagen	$-4''95113$
Brocken	-4.95104
Inselsberg	-4.95131

29.

Coronidis caussa adhuc comparationem areae trianguli in superficie curva cum area trianguli rectilinei, cuius latera sunt a, b, c, adiiciemus. Aream posteriorem denotabimus per σ^*, quae fit $= \frac{1}{2} bc \sin A^* = \frac{1}{2} ac \sin B^* = \frac{1}{2} ab \sin C^*$

Habemus, usque ad quantitates ordinis quarti

$$\sin A^* = \sin A - \tfrac{1}{12}\sigma \cos A \,.\, (2\alpha + \beta + \gamma)$$

sive aeque exacte

$$\sin A = \sin A^* . (1 + \tfrac{1}{24} bc \cos A \,.\, (2\alpha + \beta + \gamma))$$

Substituto hoc valore in formula [9], erit usque ad quantitates sexti ordinis

$$\sigma = \tfrac{1}{2} bc \sin A^* . \{1 + \tfrac{1}{120}\alpha(3bb + 3cc - 2bc\cos A) + \tfrac{1}{120}\beta(3bb + 4cc - 4bc\cos A) + \tfrac{1}{120}\gamma(4bb + 3cc - 4bc\cos A)\}$$

sive aeque exacte

$$\sigma = \sigma^* \{1 + \tfrac{1}{120}\alpha(aa + 2bb + 2cc) + \tfrac{1}{120}\beta(2aa + bb + 2cc) + \tfrac{1}{120}\gamma(2aa + 2bb + cc)\}$$

Pro superficie sphaerica haec formula sequentem induit formam

$$\sigma = \sigma^* (1 + \tfrac{1}{24}\alpha(aa + bb + cc))$$

cuius loco etiam sequentem salva eadem praecisione adoptari posse facile confirmatur

$$\sigma = \sigma^* \sqrt{\frac{\sin A \,.\, \sin B \,.\, \sin C}{\sin A^* . \sin B^* . \sin C^*}}$$

Si eadem formula triangulis in superficie curva non sphaerica applicatur, error generaliter loquendo erit quinti ordinis, sed insensibilis in omnibus triangulis qualia in superficie telluris dimetiri licet.

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the difference can always be regarded as insensible. Thus, *e. g.*, in the greatest of the triangles which we have measured in recent years, namely, that between the points Hohehagen, Brocken, Inselberg, where the excess of the sum of the angles was 14″.85348, the calculation gave the following reductions to be applied to the angles:

—Hohehagen	—4″.95113
Brocken	—4″.95104
Inselberg	—4″.95131.

29.

We shall conclude this study by comparing the area of a triangle on a curved surface with the area of the rectilinear triangle whose sides are a, b, c. We shall denote the area of the latter by σ^*; hence

$$\sigma^* = \tfrac{1}{2}\, b c \sin A^* = \tfrac{1}{2}\, a c \sin B^* = \tfrac{1}{2}\, a b \sin C^*$$

We have, to quantities of the fourth order,

$$\sin A^* = \sin A - \tfrac{1}{12}\,\sigma \cos A \,.\, (2\,\alpha + \beta + \gamma)$$

or, with equal exactness,

$$\sin A = \sin A^* \,.\, (1 + \tfrac{1}{24}\, b c \cos A \,.\, (2\,\alpha + \beta + \gamma))$$

Substituting this value in formula [9], we shall have, to quantities of the sixth order,

$$\begin{aligned}\sigma = \tfrac{1}{2}\, b c \sin A^* \,.\, (1 &+ \tfrac{1}{120}\,\alpha\,(3\,b^2 + 3\,c^2 - 2\,b c \cos A)\\ &+ \tfrac{1}{120}\,\beta\,(3\,b^2 + 4\,c^2 - 4\,b c \cos A)\\ &+ \tfrac{1}{120}\,\gamma\,(4\,b^2 + 3\,c^2 - 4\,b c \cos A)),\end{aligned}$$

or, with equal exactness,

$$\sigma = \sigma^* (1 + \tfrac{1}{120}\,\alpha\,(a^2 + 2\,b^2 + 2\,c^2) + \tfrac{1}{120}\,\beta\,(2\,a^2 + b^2 + 2\,c^2) + \tfrac{1}{120}\,\gamma\,(2\,a^2 + 2\,b^2 + c^2))$$

For the sphere this formula goes over into the following form:

$$\sigma = \sigma^* (1 + \tfrac{1}{24}\,\alpha\,(a^2 + b^2 + c^2)).$$

It is easily verified that, with the same precision, the following formula may be taken instead of the above:

$$\sigma = \sigma^* \sqrt{\frac{\sin A \,.\, \sin B \,.\, \sin C}{\sin A^* \,.\, \sin B^* \,.\, \sin C^*}}$$

If this formula is applied to triangles on non-spherical curved surfaces, the error, generally speaking, will be of the fifth order, but will be insensible in all triangles such as may be measured on the earth's surface.

ANZEIGE

Göttingische gelehrte Anzeigen. 1827 November 5.

Am 8. October überreichte Hr. Hofr. GAUSS der Königl. Societät eine Vorlesung:

Disquisitiones generales circa superficies curvas.

Obgleich die Geometer sich viel mit allgemeinen Untersuchungen über die krummen Flächen beschäftigt haben, und ihre Resultate einen bedeutenden Theil des Gebiets der höhern Geometrie ausmachen, so ist doch dieser Gegenstand noch so weit davon entfernt, erschöpft zu sein, dass man vielmehr behaupten kann, es sei bisher nur erst ein kleiner Theil eines höchst fruchtbaren Feldes angebauet. Der Verf. hat schon vor einigen Jahren durch die Auflösung der Aufgabe, alle Darstellungen einer gegebenen Fläche auf einer andern zu finden, bei welchen die kleinsten Theile ähnlich bleiben, dieser Lehre eine neue Seite abzugewinnen gesucht: der Zweck der gegenwärtigen Abhandlung ist, abermals andere neue Gesichtspunkte zu eröffnen, und einen Theil der neuen Wahrheiten, die dadurch zugänglich werden, zu entwickeln. Wir werden davon hier anzeigen, was ohne zu grosse Weitläuftigkeit verständlich gemacht werden kann, müssen aber dabei im Voraus bemerken, dass sowohl die neuen Begriffsbildungen, als die Theoreme, wenn die grösste Allgemeinheit umfasst werden soll, zum Theil noch einiger Beschränkungen oder nähern Bestimmungen bedürfen, welche hier übergangen werden müssen.

GAUSS'S ABSTRACT OF THE DISQUISITIONES GENERALES CIRCA SUPERFICIES CURVAS, PRESENTED TO THE ROYAL SOCIETY OF GÖTTINGEN.

GÖTTINGISCHE GELEHRTE ANZEIGEN. No. 177. PAGES 1761–1768. 1827. NOVEMBER 5.

On the 8th of October, Hofrath Gauss presented to the Royal Society a paper:

Disquisitiones generales circa superficies curvas.

Although geometers have given much attention to general investigations of curved surfaces and their results cover a significant portion of the domain of higher geometry, this subject is still so far from being exhausted, that it can well be said that, up to this time, but a small portion of an exceedingly fruitful field has been cultivated. Through the solution of the problem, to find all representations of a given surface upon another in which the smallest elements remain unchanged, the author sought some years ago to give a new phase to this study. The purpose of the present discussion is further to open up other new points of view and to develop some of the new truths which thus become accessible. We shall here give an account of those things which can be made intelligible in a few words. But we wish to remark at the outset that the new theorems as well as the presentations of new ideas, if the greatest generality is to be attained, are still partly in need of some limitations or closer determinations, which must be omitted here.

Anzeige

342

Bei Untersuchungen, wo eine Mannigfaltigkeit von Richtungen gerader Linien im Raume ins Spiel kommt, ist es vortheilhaft, diese Richtungen durch diejenigen Punkte auf der Oberfläche einer festen Kugel zu bezeichnen, welche die Endpunkte der mit jenen parallel gezogenen Radien sind: Mittelpunkt und Halbmesser dieser *Hülfskugel* sind hierbei ganz willkürlich; für letztern mag die Lineareinheit gewählt werden. Diess Verfahren kommt im Grunde mit demjenigen überein, welches in der Astronomie in stetem Gebrauch ist, wo man alle Richtungen auf eine fingirte Himmelskugel von unendlich grosem Halbmesser bezieht. Die sphärische Trigonometrie, und einige andere Lehrsätze, welchen der Verf. noch einen neuen von häufiger Anwendbarkeit beigefügt hat. dienen dann zur Auflösung der Aufgaben, welche die Vergleichung der verschiedenen vorkommenden Richtungen darbieten kann.

Wenn man die Richtung der an jedem Punkt einer krummen Fläche auf diese errichteten Normale durch den nach dem angedeuteten Verfahren entsprechenden Punkt der Kugelfläche bezeichnet, also jedem Punkt der krummen Fläche in dieser Beziehung einen Punkt der Oberfläche der Hülfskugel entsprechen lässt, so wird, allgemein zu reden, jeder Linie auf der krummen Fläche eine Linie auf der Oberfläche der Hülfskugel, und jedem Flächenstück von jener ein Flächenstück von dieser entsprechen. Je geringer die Abweichung jenes Stücks von der Ebene ist, desto kleiner wird der entsprechende Theil der Kugelfläche sein, und es ist mithin ein sehr natürlicher Gedanke zum Maassstabe der Totalkrümmung, welche einem Stück der krummen Fläche beizulegen ist, den Inhalt des entsprechenden Stücks der Kugelfläche zu gebrauchen. Der Verf. nennt daher diesen Inhalt die *ganze Krümmung* des entsprechenden Stücks der krummen Fläche. Ausser der Grösse kommt aber zugleich noch die *Lage* der Theile in Betracht, die, ganz abgesehen von dem Grössenverhältniss, in den beiden Stücken entweder eine ähnliche, oder eine verkehrte sein kann: diese beiden Fälle werden durch das der Totalkrümmung vorzusetzende positive oder negative Zeichen unterschieden werden können. Diese Unterscheidung hat jedoch nur insofern eine bestimmte Bedeutung, als die Figuren auf bestimmten Seiten der beiden Flächen gedacht werden: der Verf. nimmt sie bei der Kugelfläche auf der äussern und bei der krummen Fläche auf derjenigen Seite, wo man sich die Normale errichtet denkt, und es folgt dann, dass das positive Zeichen bei convex-convexen oder concav-concaven Flächen (die nicht wesentlich verschieden sind), und das nega-

Abstract

In researches in which an infinity of directions of straight lines in space is concerned, it is advantageous to represent these directions by means of those points upon a fixed sphere, which are the end points of the radii drawn parallel to the lines. The centre and the radius of this *auxiliary sphere* are here quite arbitrary. The radius may be taken equal to unity. This procedure agrees fundamentally with that which is constantly employed in astronomy, where all directions are referred to a fictitious celestial sphere of infinite radius. Spherical trigonometry and certain other theorems, to which the author has added a new one of frequent application, then serve for the solution of the problems which the comparison of the various directions involved can present.

If we represent the direction of the normal at each point of the curved surface by the corresponding point of the sphere, determined as above indicated, namely, in this way, to every point on the surface, let a point on the sphere correspond; then, generally speaking, to every line on the curved surface will correspond a line on the sphere, and to every part of the former surface will correspond a part of the latter. The less this part differs from a plane, the smaller will be the corresponding part on the sphere. It is, therefore, a very natural idea to use as the measure of the total curvature, which is to be assigned to a part of the curved surface, the area of the corresponding part of the sphere. For this reason the author calls this area the *integral curvature* of the corresponding part of the curved surface. Besides the magnitude of the part, there is also at the same time its *position* to be considered. And this position may be in the two parts similar or inverse, quite independently of the relation of their magnitudes. The two cases can be distinguished by the positive or negative sign of the total curvature. This distinction has, however, a definite meaning only when the figures are regarded as upon definite sides of the two surfaces. The author regards the figure in the case of the sphere on the outside, and in the case of the curved surface on that side upon which we consider the normals erected. It follows then that the positive sign is taken in the case of convexo-convex or concavo-concave surfaces (which are not essentially different), and the negative in the case of concavo-convex

Anzeige

tive bei concav-convexen Statt hat. Wenn das in Rede stehende Stück der krummen Fläche in dieser Beziehung aus Theilen ungleicher Art besteht, so werden noch nähere Bestimmungen nothwendig, die hier übergangen werden müssen.

Die Vergleichung des Inhalts zweier einander correspondirender Stücke der krummen Fläche und der Oberfläche der Hülfskugel führt nun (auf dieselbe Art wie z. B. aus der Vergleichung von Volumen und Masse der Begriff von Dichtigkeit hervorgeht) zu einem neuen Begriffe. Der Verf. nennt nämlich *Krümmungsmaass* in einem Punkt der krummen Fläche den Werth des Bruches, dessen Nenner der Inhalt eines unendlich kleinen Stücks der krummen Fläche in diesem Punkt, und der Zähler der Inhalt des entsprechenden Stücks der Fläche der Hülfskugel, oder die ganze Krümmung jenes Elements ist. Man sieht, dass, in dem Sinn des Verf., ganze Krümmung und Krümmungsmaass bei krummen Flächen dem analog ist, was bei krummen Linien resp. Amplitudo und schlechthin Krümmung genannt wird; er fand Bedenken, die letztern mehr durch Gewohnheit als wegen Angemessenheit recipirten Ausdrücke auf die krummen Flächen zu übertragen. Uebrigens liegt weniger an den Benennungen selbst, als daran, dass ihre Einführung durch prägnante Sätze gerechtfertigt wird.

Die Auflösung der Aufgabe, das Krümmungsmaass in jedem Punkt einer krummen Fläche zu finden, erscheint in verschiedener Gestalt, nach Maassgabe der Art, wie die Natur der krummen Fläche gegeben ist. Die einfachste Art ist, indem die Punkte im Raum allgemein durch drei rechtwinklige Coordinaten x, y, z unterschieden werden, eine Coordinate als Function der beiden andern darzustellen: dabei erhält man den einfachsten Ausdruck für das Krümmungsmaass. Zugleich ergibt sich aber ein merkwürdiger Zusammenhang zwischen diesem Krümmungsmaass und den Krümmungen derjenigen Curven, die durch den Schnitt der krummen Fläche mit Ebenen senkrecht auf dieselbe, hervorgehen. Bekanntlich hat Euler zuerst gezeigt, dass zwei dieser schneidenden Ebenen, die einander gleichfalls unter einem rechten Winkel schneiden, die Eigenschaft haben, dass in der einen der grösste, in der andern der kleinste Krümmungshalbmesser Statt findet, oder richtiger, dass in ihnen die beiden äussersten Krümmungen vorkommen. Hier ergibt sich nun aus dem erwähnten Ausdruck für das Krümmungsmaass, dass dieses einem Bruche gleich wird, dessen Zähler die Einheit, der Nenner das Product der beiden äussersten Krümmungshalbmesser wird. — Weniger einfach wird der Ausdruck für das Krümmungsmaas, wenn

Abstract

surfaces. If the part of the curved surface in question consists of parts of these different sorts, still closer definition is necessary, which must be omitted here.

The comparison of the areas of two corresponding parts of the curved surface and of the sphere leads now (in the same manner as, *e. g.*, from the comparison of volume and mass springs the idea of density) to a new idea. The author designates as *measure of curvature* at a point of the curved surface the value of the fraction whose denominator is the area of the infinitely small part of the curved surface at this point and whose numerator is the area of the corresponding part of the surface of the auxiliary sphere, or the integral curvature of that element. It is clear that, according to the idea of the author, integral curvature and measure of curvature in the case of curved surfaces are analogous to what, in the case of curved lines, are called respectively amplitude and curvature simply. He hesitates to apply to curved surfaces the latter expressions, which have been accepted more from custom than on account of fitness. Moreover, less depends upon the choice of words than upon this, that their introduction shall be justified by pregnant theorems.

The solution of the problem, to find the measure of curvature at any point of a curved surface, appears in different forms according to the manner in which the nature of the curved surface is given. When the points in space, in general, are distinguished by three rectangular coordinates, the simplest method is to express one coordinate as a function of the other two. In this way we obtain the simplest expression for the measure of curvature. But, at the same time, there arises a remarkable relation between this measure of curvature and the curvatures of the curves formed by the intersections of the curved surface with planes normal to it. Euler, as is well known, first showed that two of these cutting planes which intersect each other at right angles have this property, that in one is found the greatest and in the other the smallest radius of curvature; or, more correctly, that in them the two extreme curvatures are found. It will follow then from the above mentioned expression for the measure of curvature that this will be equal to a fraction whose numerator is unity and whose denominator is the product of the extreme radii of curvature. The expression for the measure of curvature will be less simple, if the nature of the curved surface is determined by an equation in x, y, z. And it will become still more complex, if the nature of the curved surface is given so that

Anzeige

die Natur der krummen Fläche durch eine Gleichung zwischen x, y, z, bestimmt ist, und noch zusammengesetzter wird jener, wenn die Natur der krummen Fläche dadurch gegeben ist, dass x, y, z in der Gestalt von Functionen zweier neuen veränderlichen Grössen p, q dargestellt sind. Im letzten Fall enthält der Ausdruck funfzehn Elemente, nemlich die partiellen Differentialquotienten der ersten und zweiten Ordnung von x, y, z nach p und q: allein er ist weniger wichtig an sich, als weil er den Übergang zu einem andern bahnt, der zu den merkwürdigsten Sätzen in dieser Lehre gerechnet werden muss. Bei jener Art, die Natur der krummen Fläche darzustellen, hat der allgemeine Ausdruck für irgend ein Linearelement auf derselben,

oder für $\sqrt{(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)}$, die Form $\sqrt{(E\mathrm{d}x^2+2F\mathrm{d}x.\mathrm{d}y+G\mathrm{d}y^2)}$

wo E, F, G wiederum Functionen von p und q werden, der erwähnte neue Ausdruck für das Krümmungsmaass enthält nun bloss diese Grössen, und ihre partiellen Differentialquotienten der ersten und zweiten Ordnung. Man sieht also, dass zur Bestimmung des Krümmungsmaasses bloss die Kenntniss des allgemeinen Ausdrucks eines Linearelements erforderlich ist, ohne dass es der Ausdrücke für die Coordinaten x, y, z selbst bedarf. Eine unmittelbare Folge davon ist der merkwürdige Lehrsatz: Wenn eine krumme Fläche, oder ein Stück derselben auf eine andere Fläche abgewickelt werden kann, so bleibt nach der Abwickelung das Krümmungsmaass in jedem Punkt ungeändert. Als specieller Fall folgt hieraus ferner: In einer krummen Fläche, die in eine Ebene abgewickelt werden kann, ist das Krümmungsmaass überall $= 0$. Man leitet daraus sofort die characteristische Gleichung der in eine Ebene abwickelungsfähigen Flächen ab, nemlich, in so fern z als Function von x und y betrachtet wird,

$$\frac{\mathrm{dd}z}{\mathrm{d}x^2}\cdot\frac{\mathrm{dd}z}{\mathrm{d}y^2}-\left(\frac{\mathrm{dd}z}{\mathrm{d}x.\mathrm{d}y}\right)^2=0$$

eine Gleichung, die zwar längst bekannt, aber nach des Verf. Urtheil bisher nicht mit der erforderlichen Strenge bewiesen war.

Diese Sätze führen dahin, die Theorie der krummen Flächen aus einem neuen Gesichtspunkte zu betrachten, wo sich der Untersuchung ein weites noch ganz unangebautes Feld öffnet. Wenn man die Flächen nicht als Grenzen von Körpern, sondern als Körper, deren eine Dimension verschwindet, und zugleich als biegsam, aber nicht als dehnbar betrachtet, so begreift man, dass zweierlei

Abstract

x, y, z are expressed in the form of functions of two new variables p, q. In this last case the expression involves fifteen elements, namely, the partial differential coefficients of the first and second orders of x, y, z with respect to p and q. But it is less important in itself than for the reason that it facilitates the transition to another expression, which must be classed with the most remarkable theorems of this study. If the nature of the curved surface be expressed by this method, the general expression for any linear element upon it, or for $\sqrt{(dx^2 + dy^2 + dz^2)}$, has the form $\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)}$, where E, F, G are again functions of p and q. The new expression for the measure of curvature mentioned above contains merely these magnitudes and their partial differential coefficients of the first and second order. Therefore we notice that, in order to determine the measure of curvature, it is necessary to know only the general expression for a linear element; the expressions for the coordinates x, y, z are not required. A direct result from this is the remarkable theorem: If a curved surface, or a part of it, can be developed upon another surface, the measure of curvature at every point remains unchanged after the development. In particular, it follows from this further: Upon a curved surface that can be developed upon a plane, the measure of curvature is everywhere equal to zero. From this we derive at once the characteristic equation of surfaces developable upon a plane, namely,

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,.\,\partial y}\right)^2 = 0,$$

when z is regarded as a function of x and y. This equation has been known for some time, but according to the author's judgment it has not been established previously with the necessary rigor.

These theorems lead to the consideration of the theory of curved surfaces from a new point of view, where a wide and still wholly uncultivated field is open to investigation. If we consider surfaces not as boundaries of bodies, but as bodies of which one dimension vanishes, and if at the same time we conceive them as flexible but not extensible, we see that two essentially different relations must be distinguished, namely,

Anzeige

wesentlich verschiedene Relationen zu unterscheiden sind, theils nemlich solche, die eine bestimmte Form der Fläche im Raume voraussetzen, theils solche, welche von den verschiedenen Formen, die die Fläche annehmen kann, unabhängig sind. Die letztern sind es, wovon hier die Rede ist: nach dem, was vorhin bemerkt ist, gehört dazu das Krümmungsmaass; man sieht aber leicht, dass eben dahin die Betrachtung der auf der Fläche construirten Figuren, ihrer Winkel, ihres Flächeninhalts und ihrer Totalkrümmung, die Verbindung der Punkte durch kürzeste Linien u. dgl. gehört. Alle solche Untersuchungen müssen davon ausgehen, dass die Natur der krummen Fläche an sich durch den Ausdruck eines unbestimmten Linearelements in der Form $\surd(E\mathrm{d}p^2 + 2F\mathrm{d}p\,.\,\mathrm{d}q + G\mathrm{d}q^2)$ gegeben ist. Der Verf. hat gegenwärtiger Abhandlung einen Theil seiner seit mehreren Jahren auf diesem Felde angestellten Untersuchuneen einverleibt, indem er sich auf solche einschränkte, die von dem ersten Eintritt nicht zu entfernt liegen und zum Theil als allgemeine Hülfsmittel zu vielfachen weitern Untersuchungen dienen können. Bei unsrer Anzeige müssen wir uns noch mehr beschränken, und uns begnügen, nur einiges als Probe anzuführen. Als solche mögen folgende Lehrsätze dienen.

Wenn auf einer krummen Fläche von Einem Anfangspunkte ein System unendlich vieler kürzester Linien von gleicher Länge ausläuft, so schneidet die durch ihre Endpunkte gehende Linie jede derselben unter rechten Winkeln. Wenn an jedem Punkte einer beliebigen Linie auf einer krummen Fläche kürzeste Linien von gleicher Länge senkrecht gegen jene Linie gezogen sind, so sind diese alle auch senkrecht gegen diejenige Linie, welche ihre andern Endpunkte verbindet. Diese beiden Lehrsätze, wovon der zweite als eine Generalisirung des ersten betrachtet werden kann, werden sowohl analytisch, als durch einfache geometrische Betrachtungen bewiesen. *Der Überschuss der Summe der Winkel eines aus kürzesten Linien gebildeten Dreiecks über zwei Rechte ist der Totalkrümmung des Dreiecks gleich.* Es wird hiebei angenommen, dass für die Winkel derjenige, dem ein dem Halbmesser gleicher Bogen entspricht, $(57^\circ\,17'\,45'')$, und für die ganze Krümmung, als Stück der Fläche der Hülfskugel, der Inhalt eines Quadrats, dessen Seite der Halbmesser der Hülfskugel ist, als Einheit zum Grunde liegt. Offenbar kann man diess wichtige Theorem auch so ausdrücken: der Überschuss der Winkel eines aus kürzesten Linien gebildeten Dreiecks über zwei Rechte verhält sich zu acht Rechten, wie das Stück der Oberfläche der Hülfskuon the one hand, those that presuppose a definite form of the surface in space; on the other hand, those that are independent of the various forms which the surface may assume. This discussion is concerned with the latter. In accordance with what has been said, the measure of curvature belongs to this case. But it is easily seen that the consideration of figures constructed upon the surface, their angles, their areas and their integral curvatures, the joining of the points by means of shortest lines, and the like, also belong to this case. All such investigations must start from this, that the very nature of the curved surface is given by means of the expression of any linear element in the form $\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)}$. The author has embodied in the present treatise a portion of his investigations in this field, made several years ago, while he limits himself to such as are not too remote for an introduction, and may, to some extent, be generally helpful in many further investigations. In our abstract, we must limit ourselves still more, and be content with citing only a few of them as types. The following theorems may serve for this purpose.

If upon a curved surface a system of infinitely many shortest lines of equal lengths be drawn from one initial point, then will the line going through the end points of these shortest lines cut each of them at right angles. If at every point of an arbitrary line on a curved surface shortest lines of equal lengths be drawn at right angles to this line, then will all these shortest lines be perpendicular also to the line which joins their other end points. Both these theorems, of which the latter can be regarded as a generalization of the former, will be demonstrated both analytically and by simple geometrical considerations. *The excess of the sum of the angles of a triangle formed by shortest lines over two right angles is equal to the total curvature of the triangle.* It will be assumed here that that angle (57° 17′ 45″) to which an arc equal to the radius of the sphere corresponds will be taken as the unit for the angles, and that for the unit of total curvature will be taken a part of the spherical surface, the area of which is a square whose side is equal to the radius of the sphere. Evidently we can express this important theorem thus also: the excess over two right angles of the angles of a triangle formed by shortest lines is to eight right angles as the part of the surface of the auxiliary sphere, which corresponds

Anzeige

gel, welches jenem als ganze Krümmung entspricht, zu der ganzen Oberfläche der Hülfskugel. Allgemein wird der Überschuss der Winkel eines Polygons von n Seiten, wenn diese kürzeste Linien sind, über $2n - 4$ Rechte, der ganzen Krümmung des Polygons gleich sein.

Die allgemeinen in der Abhandlung entwickelten Untersuchungen werden am Schluss derselben noch auf die Theorie der durch kürzeste Linien gebildeten Dreiecke angewandt, wovon wir hier nur ein paar Haupttheoreme anführen. Sind a, b, c die Seiten eines solchen Dreiecks (die als Grössen der ersten Ordnung betrachtet werden); A, B, C die gegenüberstehenden Winkel; α, β, γ die Krümmungsmaasse in den Winkelpunkten; σ der Flächeninhalt des Dreiecks, so ist, bis auf Grössen der vierten Ordnung, $\frac{1}{3}(\alpha + \beta + \gamma)\sigma$ der Überschuss der Summe $A + B + C$ über zwei Rechte. Ferner sind, mit derselben Genauigkeit, die Winkel eines ebenen geradlinigen Dreiecks, dessen Seiten a, b, c sind, der Ordnung nach

$$\begin{aligned} &A - \tfrac{1}{12}(2\alpha + \beta + \gamma)\sigma \\ &B - \tfrac{1}{12}(\alpha + 2\beta + \gamma)\sigma \\ &C - \tfrac{1}{12}(\alpha + \beta + 2\gamma)\sigma \end{aligned}$$

Man sieht sogleich, dass das letzte Theorem eine Generalisirung des bekannten von LEGENDRE zuerst aufgestellten ist, nach welchem man, bis auf Grössen der vierten Ordnung, die Winkel des geradlinigen Dreiecks erhält, wenn man die Winkel des sphärischen jeden um den dritten Theil des sphärischen Excesses vermindert. Auf einer nichtsphärischen Fläche muss man also den Winkeln ungleiche Reductionen beifügen, und die Ungleichheit ist allgemein zu reden eine Grösse der dritten Ordnung; wenn jedoch die ganze Fläche nur wenig von der Kugelgestalt abweicht, so involvirt jene noch ausserdem einen Factor von der Ordnung der Abweichung von der Kugelgestalt. Es ist unstreitig für die höhere Geodaesie wichtig, dass man im Stande ist, die Ungleichheiten jener Reductionen zu berechnen, und dadurch die volle Ueberzeugung zu erhalten, dass sie für alle messbaren Dreiecke auf der Oberfläche der Erde als ganz unmerklich zu betrachten sind. So finden sich z. B. in dem grössten Dreiecke der von dem Verf. ausgeführten Triangulirung, dessen grösste Seite fast 15 geographische Meilen lang ist, und in welchem der Ueberschuss der Summe der drei Winkel über zwei Rechte fast 15 Secunden beträgt, die drei Reductionen der Winkel auf die Win-

Abstract

to it as its integral curvature, is to the whole surface of the sphere. In general, the excess over $2n - 4$ right angles of the angles of a polygon of n sides, if these are shortest lines, will be equal to the integral curvature of the polygon.

The general investigations developed in this treatise will, in the conclusion, be applied to the theory of triangles of shortest lines, of which we shall introduce only a couple of important theorems. If a, b, c be the sides of such a triangle (they will be regarded as magnitudes of the first order); A, B, C the angles opposite; α, β, γ the measures of curvature at the angular points; σ the area of the triangle, then, to magnitudes of the fourth order, $\frac{1}{3}(\alpha + \beta + \gamma)\,\sigma$ is the excess of the sum $A + B + C$ over two right angles. Further, with the same degree of exactness, the angles of a plane rectilinear triangle whose sides are a, b, c, are respectively

$$A - \tfrac{1}{12}(2\alpha + \beta + \gamma)\,\sigma$$
$$B - \tfrac{1}{12}(\alpha + 2\beta + \gamma)\,\sigma$$
$$C - \tfrac{1}{12}(\alpha + \beta + 2\gamma)\,\sigma.$$

We see immediately that this last theorem is a generalization of the familiar theorem first established by LEGENDRE. By means of this theorem we obtain the angles of a plane triangle, correct to magnitudes of the fourth order, if we diminish each angle of the corresponding spherical triangle by one-third of the spherical excess. In the case of non-spherical surfaces, we must apply unequal reductions to the angles, and this inequality, generally speaking, is a magnitude of the third order. However, even if the whole surface differs only a little from the spherical form, it will still involve also a factor denoting the degree of the deviation from the spherical form. It is unquestionably important for the higher geodesy that we be able to calculate the inequalities of those reductions and thereby obtain the thorough conviction that, for all measurable triangles on the surface of the earth, they are to be regarded as quite insensible. So it is, for example, in the case of the greatest triangle of the triangulation carried out by the author. The greatest side of this triangle is almost fifteen geographical* miles, and the excess of the sum of its three angles over two right angles amounts almost to fifteen seconds. The three

* This German geographical mile is four minutes of arc at the equator, namely, 7.42 kilometers, and is equal to about 4.6 English statute miles. [Translators.]

Anzeige

kel eines geradlinigen Dreiecks 4″95113, 4″95104, 4″95131. Übrigens hat der Verf. auch die in den obigen Ausdrücken fehlenden Glieder der vierten Ordnung entwickelt, die für die Kugelfläche eine sehr einfache Form erhalten; bei messbaren Dreiecken auf der Oberfläche der Erde sind sie aber ganz unmerklich, und in dem angeführten Beispiel würden sie die erste Reduction nur um zwei Einheiten der fünften Decimale vermindert und die dritte eben so viel vergrössert haben.

reductions of the angles of the plane triangle are 4″.95113, 4″.95104, 4″.95131. Besides, the author also developed the missing terms of the fourth order in the above expressions. Those for the sphere possess a very simple form. However, in the case of measurable triangles upon the earth's surface, they are quite insensible. And in the example here introduced they would have diminished the first reduction by only two units in the fifth decimal place and increased the third by the same amount.

DIFFERENTIAL GEOMETRY - 150 YEARS AFTER

Carl Friedrich Gauss'
Disquisitiones Generales circa Superficies Curvas

by

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Lecture held on April 24, 1977 in Brunswick (GDR)

DIFFERENTIAL GEOMETRY - 150 YEARS AFTER
CARL FRIEDRICH GAUSS'
DISQUISITIONES GENERALES CIRCA SUPERFICIES CURVAS

These days in Brunswick are intended to commemorate the birth of Carl Friedrich Gauss, two hundred years ago on 30 April 1777. For differential geometers, the year 1977 offers a further occasion for a scientific Gauss anniversary :

One hundred and fifty years ago, on 8 October 1827, Gauss presented his essay

(1) "Disquisitiones generales circa superficies curvas"

(in the form of a lecture) to the "Königlichen Gesellschaft der Wissenschaften"(♦) in Göttingen, an event that can without exageration be described as the birth of "Inner (Intrinsic) Differential Geometry".

In his excellent article "Gauss als Geometer"(♦♦), (which is well worth reading), P. Stäckel [24] gives an appreciation of Gauss'work on the differential geometry of curved surfaces, and speaks in this connection (see [24] p. 87 line 18) of the "Gebäude der Disquisitiones generales"(♦♦♦). For many of those not familiar with the original article (written in Latin), this quotation and many similar statements in the subsequent differential-geometric literature have left the impression that, as far as richness of ideas and mastery of calculations are concerned, the essay is similar to a monumental work such as the "Disquisitiones Arithmeticae" of Gauss' younger years. At first sight, however, such an assessment is hardly justified. Compared with the 470 printed pages of the "Disquisitiones Arithmeticae" (see G.W. 1, pp.

(♦) Royal Scientific Society.

(♦♦) "Gauss as a Geometer".

(♦♦♦) "Edifice of the Disquisitiones generales".

1-474), the 40 printed pages of the "Disquisitiones generales" (1) (this volume, pp.2-80) seem decidedly modest. Unlike the case of the "Disquisitiones arithmeticae", one can easily summarize the contents of the "Disquisitiones generales". They contain roughly five essentially new concepts (see below (7), (8), (9), (26), (28)), about ten new theorems (see below (10) and (11), (12), (14), (15), (22), (27), (32), (34), (35), (36)), as well as an explanation of a programme of "Inner Differential Geometry". It is even more impressive since we are led in retrospect to confirm what fundamental influence the "Disquisitiones generales" have had on the development of new ideas, on the objectives as well as on the nature of the results in differential geometry during the last 150 years. Such a review (especially on the occasion of an anniversary!) can easily tend to read more substance and prophetical foresight afterwards into such a "venerable" work than were intended by the author himself, or could, in his time, have been at all conceived of. In order to avoid this danger, we first give a complete report of the contents of the "Disquisitiones generales" (which are divided in 29 articles), following closely the contents of the manuscript but in free translation (i.e. in modern terminology), and we shall quote Gauss word by word at several vital places.

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REPORT ON THE CONTENTS OF THE "DISQUISITIONES GENERALES"

(i) Articles 1 to 3 (this volume, pp.2-8) give elementary preparations, merely summarizing the notational conventions and some (essentially known) results of spherical trigonometry.
In particular : Article 1 fixes the notation of points of the unit sphere

$$(1) \qquad S^2 = \{\underline{a} = (a_1, a_2, a_3) \in \mathbb{E}^3 \mid \langle \underline{a}, \underline{a} \rangle = 1\} \text{ }^{1)}$$

and the canonical basis vectors (1,0,0), (0,1,0), (0,0,1) of $\mathbb{E}^3$ (denoted in Gauss by "(1)", "(2)", "(3)"). In Article 2, angles between lines and lines, planes and planes, and lines and planes in $\mathbb{E}^3$ are defined by means of lengths of segments of great circles and angles between great circles on S^2, and the following identity is proved [1] :

$$(2) \qquad \langle \underline{a} \times \underline{b}, \tilde{\underline{a}} \times \tilde{\underline{b}} \rangle = \langle \underline{a}, \tilde{\underline{a}} \rangle \langle \underline{b}, \tilde{\underline{b}} \rangle - \langle \underline{a}, \tilde{\underline{b}} \rangle \langle \underline{b}, \tilde{\underline{a}} \rangle \quad \text{for } \underline{a}, \underline{b}, \tilde{\underline{a}}, \tilde{\underline{b}} \in S^2 .$$

For $\underline{a} \neq \underline{b}$ and $\tilde{\underline{a}} \neq \tilde{\underline{b}}$, (2) is interpreted as a formula for calculating the angle between the great circle segments $(\underline{a}, \underline{b})$ and $(\tilde{\underline{a}}, \tilde{\underline{b}})$ on S^2. In the draft copy of the "Disquisitiones generales" (see G.W. 8, p. 416, line 5 from below), Gauss remarks concerning (2) : "We add yet another theorem, which to our knowledge has not appeared elsewhere, and which can often be used with profit". (also this volume, p.84 , lines 9/10). [Author's note : (2) is usually called "Lagrange's identity" today. However, the article of Lagrange usually quoted in support of this (see [16]) does not contain formula (2) explicitely, but only shows (see [16], p. 580, line 6 from below and p. 581, line 12) :

$$\langle \underline{a}_1 \times \underline{a}_2, \underline{a}_3 \rangle^2 = \det(\langle \underline{a}_i, \underline{a}_j \rangle)_{i,j=1,2,3} \quad \text{for } \underline{a}_1, \underline{a}_2, \underline{a}_3 \in \mathbb{E}^3 ,$$

---

1) Here and in the sequel we denote, as is usual today, the canonical inner and cross products in $\mathbb{E}^3$ by $\langle .,. \rangle$ and $. \times .$.

and, as a special case ($\underline{a}_3 = \underline{a}_1 \times \underline{a}_2$) :

$$<\underline{a} \times \underline{b}, \underline{a} \times \underline{b}> = <\underline{a},\underline{a}><\underline{b},\underline{b}> - <\underline{a},\underline{b}>^2 \quad \text{for } \underline{a},\underline{b} \in \mathbb{E}^3 .$$

Of course, equation (2) can be derived from this equation by polarization.]

Further, the fundamental identity of spherical trigonometry for a geodesic triangle in S^2 with vertices $\underline{a},\underline{b},\underline{c} \in S^2$, angles α, β, γ and sides a, b, c is derived :

$$(\sin\alpha)(\sin b)(\sin c) = (\sin\beta)(\sin a)(\sin c) = (\sin\gamma)(\sin a)(\sin b) = \pm|<\underline{a},\underline{b}\times\underline{c}>|$$

and the right-hand side (which is independent of the order in which the vertices are taken !) is interpreted as 6 times the volume of the tetrahedron with vertices $\underline{0}$, $\underline{a}$, $\underline{b}$, $\underline{c}$.

Article 3 interprets "smoothness" [2] of a surface M in $\mathbb{E}^3$ at a point $A \in M$ as the existence of a plane through A (the tangent plane T_AM) which contains all limits of straight lines $\overline{AB}$ with $B \in M \setminus \{A\}$ as B tends to A.

(ii) In Articles 4 and 5 (this vol.,p.8-14), normal vectors are calculated and a unit normal field (hence an orientation) is selected, in special representations of some orientable surfaces M in $\mathbb{E}^3$. More precisely, if M is

(3) a level surface of a differentiable function $\varphi : U \to \mathbb{R}$
(where $U \subseteq \mathbb{E}^3$ is open and $d_A\varphi \neq 0$ for $A \in M$)

or

(4) the image of an immersion $f : U \to \mathbb{E}^3$
(where U is open in the (u,v)-plane in $\mathbb{R}^2$)

or

(5) the graph of a function $z(x,y) : U \to \mathbb{R}$ in $\mathbb{E}^3$
(U open in $\mathbb{R}^2$)

then the "outer unit normal field" is defined as being positively proportional to

(6) $$\operatorname{grad} \varphi|_M \quad , \quad f_u \times f_v \quad , \quad (-z_x, -z_y, 1)$$

respectively.

[2] In Gauss "continual curvedness" instead of smoothness.

(iii) Article 6 introduces the following three central (and fundamentally new) concepts of the work :

a) For a surface M in $\mathbb{E}^3$ with continuous unit normal field, a map from M to the unit sphere is defined (this volume, p.16, line 1-4), the famous

(7) Gauss spherical map $\zeta : M \to S^2$.

The idea of introducing this map is explained by the following remark of Gauss in another passage (this vol., p. 84, line 6) : "This procedure is basically the same as the one often used in astronomy, where directions in space are considered to be points of an imaginary sphere with infinite diameter", as well as by the mode of expression, used several times by Gauss, in which the image $\zeta(A) \in S^2$ of a point $A \in M$ under the map (7), is referred to as the "zenith point" of A (see G.W. 8, p. 436, lines 7, 20, 23).

b) Next a preliminary definition is given of the (absolute value of the) "total curvature" ("curvatura totalis" or "curvatura integra", this volume, p. 16, line 11) of a compact subset D of a surface M in $\mathbb{E}^3$, as the area of its spherical image $\zeta(D)$ in S^2 (see (7) and this volume, p. 16, line 10).

c) Then comes the definition of the "measure of curvedness" (="mensura curvaturae", this volume, p.16, line 14) K(A) at a point A of the surface M, which is known today as the

Gaussian curvature K(A) of M at the point $A \in M$ (see b)) :

(8) $$|K(A)| = \lim_{\varepsilon \to 0} \frac{\text{area}(\zeta(D_\varepsilon))}{\text{area}(D_\varepsilon)}$$

(where D_ε is the compact ε-neighborhood of A in M), together with a topological [3] determination of the sign of K(A). If $|K(A)| \neq 0$, then K(A) is defined as positive or negative according to whether (in modern terminology) the differential of ζ at A followed by the translation τ of $\mathbb{E}^3$ which sends $\zeta(A)$ back to A, i.e. the linear mapping $\tau \circ \zeta_*|_A : T_A M \to T_A M$ of the tangent plane of M at A to itself, is orientation preserving or not. This orientation-preserving property is defined topologically by Gauss, on the one hand through the preservation by ζ of the intersection numbers of pairs of transverse curves on M passing through A (this volume, p.16, line 7 from below), and on the other hand through the property of ζ of mapping the winding direction of ∂D_ε

[3] The geometric interpretation of $K(A) > 0$ and $K(A) < 0$ as "hump-type" and "saddle-type" curvedness of M at A is only given much later (see (13) below), and is not part of the definition of K(A) !

around D_ε ("D_ε is to the left of ∂D_ε", see (8)) onto the same winding direction of $\zeta(\partial D_\varepsilon)$ around $\partial(D_\varepsilon)$ (see G.W. 8, p. 425, line 11 from below).

d) With the aid of the Gaussian curvature, the total curvature (with appropriate sign) of an arbitrary compact subset D of an oriented surface M in $\mathbb{E}^3$ is then defined as :

(9) Total curvature (D) (= "curvatura integra" of D) = $\int_D K d\sigma$,

where $d\sigma$ is the surface element of the oriented surface M.

(iv) Articles 7, 8, 9, 10 (this volume, pp.20-32) contain formulas for calculating the Gaussian curvature K of a surface M in $\mathbb{E}^3$, together with the well-known "outer" (= extrinsic) interpretation of (the absolute value and the sign of) K. Precisely, if M is given by (5) (resp. (3), resp. (4)), then Gauss proves that

(10) $$(1+z_x^2+z_y^2)^2 K = z_{xx} z_{yy} - z_{xy}^2 ,$$

resp.

(11) $$(\varphi_x^2+\varphi_y^2+\varphi_z^2)K = (\varphi_{yy}\varphi_{zz}-\varphi_{yz}^2)\varphi_x^2 + (\varphi_{xx}\varphi_{zz}-\varphi_{xz}^2)\varphi_y^2 + (\varphi_{xx}\varphi_{yy}-\varphi_{xy}^2)\varphi_z^2 - 2\left(\begin{vmatrix}\varphi_{xy} & \varphi_{xz}\\ \varphi_{zy} & \varphi_{zz}\end{vmatrix}\varphi_x\varphi_y + \begin{vmatrix}\varphi_{yz} & \varphi_{yx}\\ \varphi_{xz} & \varphi_{xx}\end{vmatrix}\varphi_y\varphi_z + \begin{vmatrix}\varphi_{xz} & \varphi_{xy}\\ \varphi_{yz} & \varphi_{yy}\end{vmatrix}\varphi_x\varphi_z\right) ,$$

resp.

$$\|f_u \times f_v\|^4 K = \langle f_u \times f_v, f_{uu}\rangle \cdot \langle f_u \times f_v, f_{vv}\rangle - \langle f_u \times f_v, f_{uv}\rangle^2 ,$$

(this volume, p.24, line 6, p. 28, line 4 from below, p. 32, line 4 from below). The following result is deduced from (10) (this volume, p.26) :

Theorem .- For each point A of a surface M in $\mathbb{E}^3$,

(12) $$K(A) = \varkappa_1 \cdot \varkappa_2 ,$$

where $\varkappa_1$ and $\varkappa_2$ are the extremal values of the (oriented) curvatures of those plane curves through A obtained as intersections of M with planes containing the normal vector to M at A.

Moreover, $K(A) > 0$ or $K(A) < 0$ according to whether M is curved at A in a "convex-convex" (i.e. "hump-type") or "concave-convex" (i.e. "saddle-type") manner.

(V) Article 11 contains the formula which is the central result of the whole "Disquisitiones generales". If the surface M is given in the form (4), and if

(13) $$E = \langle f_u, f_u \rangle \quad , \quad F = \langle f_u, f_v \rangle \quad , \quad G = \langle f_v, f_v \rangle$$

(this notation was used by Gauss), then we have (this volume, p.36) what is known today as the

Gauss equation :

(14) $$\begin{aligned} 4(EG - F^2)^2 K = {} & E(E_v G_v - 2F_u G_v + G_u^2) + G(E_u G_u - 2E_u F_v + E_v^2) \\ & + F(E_u G_v - E_v G_u - 2E_v F_v + 4F_u F_v - 2F_u G_u) \\ & - 2(EG - F^2)(E_{vv} - 2F_{uv} + G_{uu}) \quad . \end{aligned}$$

The geometric interpretation of (14) is given in Article 12, culminating in the following verbatim formulation (this volume, p. 38) :

"Formula itaque articuli praecendentis sponte perducit ad egregium

($\widetilde{15}$) *THEOREMA* .- Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet."

This is thus the widely known

(15) *Theorema egregium* (Invariance of the Gaussian curvature under isometries) -- If a curved surface in $\mathbb{E}^3$ can be isometrically mapped onto another such surface, then the values of the Gaussian curvature at points corresponding to each other under the isometry are the same.

From this Gauss deduces the following :

(16) An isometry between two curved surfaces in $\mathbb{E}^3$ preserves the total curvature of corresponding compact subsets (see (9)),

and also (this volume, p. 88 , line 12 from below) :

> On a curved surface which can be developed upon a plane, the measure of curvature is everywhere equal to zero. We obtain immediatly from this the characteristic equation for surfaces developable upon a plane, namely,
>
> (17) $$z_{xx} z_{yy} - z_{xy}^2 = 0$$
>
> where z is regarded as a function of x and y (see (5), (10)). This equation has been known for some time, but in the author's opinion has not up to now been proved with the necessary rigour.

[Note.- In contrast with the terminology often used later (particularly by W. Blaschke), Gauss himself calls statement (15) alone the "Theorema egregium", and not equation (14). He merely remarks (see (15)) that formula (14) "leads spontaneously to the Theorema egregium".

The Theorema egregium (15) is also often quoted as a theorem about the "Biegungsinvariantz"(♦) of the Gaussian curvature, a term that was probably first introduced by J. Weingarten (see [26], p. 182, line 18) in 1883. Since there are surfaces in $\mathbb{E}^3$ which are isometric but which cannot be "bent" (i.e. transformed by a continuous family of isometric immersions in $\mathbb{E}^3$) one into the other, the contents of the Theorema egregium proves to be aptly described by the title "Invariance under isometries of the Gaussian curvature". This interpretation also corresponds more closely with the intentions of Gauss. The following variant of (16), given by Gauss elsewhere (prior to the publication of the Disquisitiones generales, in a collection of notes, see G.W. 8, p. 372), indicates almost surely that Gauss had the invariance under isometries and not only under "bendings"(♦♦) in mind. According to this formulation, the total curvature of a "figure" of a curved surface is the same, regardless of the form the surface "assumes" in space. (Note here also the analogous formulation in (18)).]

(vi) In Article 13, Gauss sketches his now famous programme for the "Inner (= intrinsic) Differential Geometry of Surfaces" (as it is called today). Here we quote Gauss himself, from his German résumé ("Selbstanzeige ..."

(♦) Literally, bending invariance.

(♦♦) Literally "bending" means continuous deformation of the surface in $\mathbb{E}^3$, which preserves the inner metric of the subspace, i.e. the length of curves on the surface.

this volume, p.88) of the "Disquisitiones generales" in 1827. After an explanation of the results (14), (15), (16), (17), Gauss continues with (♦) :

(18) These theorems lead to the consideration of the theory of curved surfaces from a new point of view, where a wide and still wholly uncultivated field is open to investigation. If we consider surfaces not as boundaries of bodies, but as bodies of which one dimension vanishes, and if at the same time we conceive them as flexible but not extensible, we see that two essentially different relations must be distinguished, namely, on the one hand, those that presuppose a definite form of the surface in space; on the other hand, those that are independent of the various forms which the surface may assume. This discussion is concerned with the latter. In accordance with what has been said, the measure of curvature belongs to this case. But it is easily seen that the consideration of figures constructed upon the surface, their angles, their areas and their integral curvatures, the joining of the points by means of shortest lines, and the like, also belong to this case. All such investigations must start from this, that the very nature of the curved surface is given by means of the expression of any linear element in the form $\sqrt{(E\,dp^2 + 2\,F\,dp\,.\,dq + G\,dq^2)}$.

This extract from Gauss' summary (this volume, p.88) essentially presents a shortened free translation of Article 13 of the "Disquisitiones generales", which is moreover complete except for one sentence (this volume, p.40, lines 8 ff.). We add this sentence, because it illustrates well the basic idea of the programme of (18) :

(19) "From this point of view, a plane surface and a surface developable upon a plane, for example a cylindrical surface, a conical surface, etc., are considered to be essentially the same."

Article 13 closes with the announcement that, in order to further illustrate the programme (18), the following articles will first derive the fundamentals of the theory of shortest paths on curved spaces.

(vii) In Articles 14, 15, 16, basic properties of certain families of normal [6] geodesics of a curved surface M in $\mathbb{E}^3$ are deduced. These are known today as the fundamental properties of the exponential map on M (partly under the name "Gauss Lemma"). More precisely :

In Article 14, the ordinary differential equations for normal geodesics of a given curved surface M in $\mathbb{E}^3$ are established, and in fact are expressed by means of the outer (= extrinsic) condition discovered already by Euler in 1744 (see [7]) : the acceleration vector of a normal geodesic of the surface M, considered as a curve in $\mathbb{E}^3$ (i.e., at noncritical points of this space curve,

♦) See German text p. 88

6) "normal geodesic" = geodesic parametrized by arc length.

the principal normal vector of the curve), is always orthogonal to the surface M. In Articles 15 and 16, Gauss proves the following theorem, which has important geometrical consequences.

Theorem (called today the *Gauss-Lemma*) .- Let M be a surface in $\mathbb{E}^3$. Let I, J be intervals of $\mathbb{R}$ containing 0 [7], and let $f : I \times J \to M$ be a differentiable mapping from the rectangle $I \times J$ of the (u,v)-plane $\mathbb{R}^2$ into M, such that for all $v \in J$,

(22) $$f(.,v) : I \to M \; (u \to v)) \text{ is a normal [6] geodesic in } M, \qquad (*)$$

and

$$\langle f_u(0,v), f_v(0,v) \rangle = 0 \quad . \qquad (**)$$

Then $\langle f_u(u,v), f_v(u,v) \rangle = 0$ for all $(u,v) \in I \times J$. (***)

In more geometric language, if the u-parameter curves of f are all normal geodesics in M (see (*)), which are orthogonal to at least one v-parameter curve of f (see (**)), then the u-parameter and v-parameter curves of f form an orthogonal net of curves in M (see (***)).

In the two special cases where $f(0,.) : J \to M$ is constant or is an injective immersion, one obtains (respectively) from (22) the following results of inner differential geometry :

(23) the end points of geodesic rays of constant length ε emanating from a fixed point A of the surface M (in $\mathbb{E}^3$) lie on a curve which is orthogonal to these rays (and which for small values of $\varepsilon > 0$ is just the so-called spherical shell of radius ε in M around A)

and

(24) the endpoints of the geodesic perpendiculars of constant length ε erected from the points of a regular curve c of the surface M (in $\mathbb{E}^3$) lie on a curve orthogonal to these perpendiculars (and which for small $\varepsilon > 0$ is just the so-called parallel curve to c at distance ε in M).

Remark .- a) In Article 15 (this volume, p.44) Gauss explicitly proves only

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7) In Gauss' Disquisitiones the interval I has always the special form $[0,\varepsilon]$ with $\varepsilon > 0$, but this makes no difference whatever.

statement (23), but he expressly states in Article 16 (this volume, p.46, line 11 from below) that his proof of (23) in Article 15 immediately yields the general results (22) and (24) without any analytical modification.

b) In modern terminology, the hypothesis (22) (*) on f is just :

$$f(u,v) = \exp^M_{f(0,v)}(u.f_u(0,v)) \quad \text{and} \quad \|f_u(0,v)\| = 1 \quad \text{for all } (u,v) \in I \times J \quad ,$$

so that the conclusion (22) (***) is a statement about the differential of the exponential map $\exp^M$ of M. In particular, (23) yields the well-known property of $\exp^M_A|_*$ which is quoted today (unfortunately alone) as the "Gauss-Lemma", and (24) gives the familiar property of the geodesic tubular neighborhood map (induced by $\exp^M$) for the curve c in M.

(viii) In Articles 17 and 18 Gauss again considers a surface M in $\mathbb{E}^3$ in the representation (4) : $f : U \to \mathbb{E}^3$. The oriented angle between a unit tangent vector $\underline{a}$ on M at the point f(u,v) and the tangent vector $f_u(u,v)$ of the u-parameter curve through f(u,v) is then given (this volume, p. 48, line 8 from below) by the unique number $\theta(\underline{a}) \in \,]-\pi,\pi]$ such that

$$(25) \qquad \cos(\theta(\underline{a})) = \frac{\langle f_u, \underline{a}\rangle}{\sqrt{E}} \quad \text{and} \quad \sin(\theta(\underline{a})) = \frac{\langle Ef_v - Ff_u, \underline{a}\rangle}{\sqrt{E}\,\sqrt{EG - F^2}} \quad .$$

Gauss then introduces (this volume, p.50, line 14) a differential 1-form, the so-called

<u>angular variation</u> Θ of f [8], defined as (see (13))

$$(26) \qquad \Theta = \frac{1}{2\sqrt{EG - F^2}} \left(\frac{F}{E}\, dE + E_v\, du - G_u\, dv - 2F_u\, du \right)$$

and proves that if $c : [0,\sigma] \to M$ is a normal geodesic with $\theta(\dot{c}(s)) \neq \pi$ for all $s \in [0,\sigma]$ [9], then (this volume, p. 50, line 14)

$$(27) \qquad (\theta(\dot{c}))' = \Theta(\dot{c}) \quad , \quad \text{i.e.} \quad \int_c \Theta = \theta(\dot{c}(\sigma)) - \theta(\dot{c}(0)) \quad .$$

---

8) Gauss denotes the form Θ by $d\theta$, which is very suggestive in view of (27). However we avoid this notation, since in general Θ is not exact (see (32) below). It would be more correct to write $f^*\Theta$ instead of Θ in (26).

9) This hypothesis, in general necessary for the validity of equation (27) (which would otherwise only hold modulo 2π) does not appear explicitly in Gauss, but is fulfilled in all the applications of (27) he gives.

That is, the integral (27) measures the variation of the oriented angle between the velocity vector of the geodesic and the positive directions of the u-parameters curves of f, over the curve c.

Note .- The preceding statement can also be expressed as follows : the integral (27) measures the variation over c of the angle between the positive tangent vector field f_u of the u-parameter lines and the direction of the geodesic c. Hence if we use the notion of Levi-Civita parallel transport (which of course Gauss did not possess), and observe that $\dot{c}$ is parallel along c in the sense of Levi-Civita, then the preceding formulation yields the following interesting interpretation of (27).

$(\widetilde{27})$ The integral (27) measures the angle by which the positive tangent vector field to the u-parameter curves f_u along c turns away from the Levi-Civita parallel direction along c.

(ix) Gauss introduces in Article 19 (this volume, p.52, line 4 ff.) a class of special charts of a surface M in $\mathbb{E}^3$, which are particularly well suited for trigonometric investigations on small geodesic triangles of M (to which the remaining articles of the "Disquisitiones generales" are devoted). Such a chart we shall call it a

(28) "geodesic-abscissa orthogonal chart" [10] which is an immersion $f : U \to M$ of an open rectangle U parallel to the axes of the (u,v)-plane $\mathbb{R}^2$, in which all u-parameter curves of f

a) are normal geodesics in M,

b) meet all v-parameter curves of f orthogonally.

For such a chart (28) it follows therefore that (see (13)) :

$$E \equiv 1 \quad , \quad F \equiv 0 \quad , \quad EG - F^2 = G \quad , \tag{29}$$

(however (29) does not conversely imply (28) a) !).

Author's note .- A significant example of (28) is a chart for a surface of revolution M in $\mathbb{E}^3$, in which the u-lines are meridians (parametrized by arc length !) and the v-lines are parallels.

Two special cases of such charts (28), which exist locally for every sur-

[10] This term, suggested by Gauss' explanations, is not in his work.

face M in $\mathbb{E}^3$, are explicitely mentioned by Gauss. We give them in modern notation (their geometrical significance is made clear respectively by (23) and (24)). Let A be a point of the surface M, and $(\underline{a}_1, \underline{a}_2)$ an orthonormal pair of tangent vectors of M at A. Then there is an $\varepsilon \in \mathbb{R}_+$ with the following property :

the mapping $f :]0,\varepsilon[\times \mathbb{R} \to M$ (see (30)) deduced from the so-called "geodesic polar coordinates at A (w.r.t. $(\underline{a}_1, \underline{a}_2)$)" [10], is defined by

$$(30) \qquad f(r,\varphi) := \exp_A^M(r \cdot (\cos\varphi \ \underline{a}_1 + \sin\varphi \ \underline{a}_2)) \quad \text{for} \quad (r,\varphi) \in]0,\varepsilon[\times \mathbb{R} ,$$

and provides a special example of a chart of type (28) according to corollary (23) of the Gauss-Lemma (22). Similarly, due to corollary (24), for sufficiently small $\varepsilon \in \mathbb{R}_+$, the mapping $f :]-\varepsilon,\varepsilon[^2 \to M$ (see (31)) deduced from the so-called "geodesic parallel coordinates (along c w.r.t. $\underline{a}_1$)" [10),11)] is another important special case of a chart of type (28) : it is defined by

$$(31) \qquad f(u,v) := \exp_{c(v)}^M(u \cdot \underline{n}(v)) \quad \text{for} \quad (u,v) \in]-\varepsilon,\varepsilon[^2 ,$$

where $c :]-\varepsilon,\varepsilon[\to M$ is the normal geodesic of M with $c(0) = A$ and $\dot{c}(0) = \underline{a}_1$ and $\underline{n} :]-\varepsilon,\varepsilon[\to TM$ denotes the continuous unit normal field of the geodesic c in M with $\underline{n}(0) = \underline{a}_2$.

For a general geodesic-abscissa orthogonal chart f of the surface M (see (28)), Gauss immediately deduces from (14) and (22) respectively, using (29) (this volume, p. 52, line 13 from below), the following formulas for the Gaussian curvature K and the differential form Θ giving the angular variation of f (see (26), (27)) :

$$(32) \qquad K = -\frac{1}{\sqrt{G}} (\sqrt{G})_{uu} \quad \text{and} \quad \Theta = -(\sqrt{G})_u \, dv .$$

In the special case of geodesic polar coordinates (see (30), where $u = r$, $v = \varphi$), Gauss completes (32) as follows (this volume, p.52, line 4 from below [12]) : the functions $\sqrt{G}$, $(\sqrt{G})_r$, $(\sqrt{G})_{rr}$, in case (30), together with f, can be continuously extended to $[0,\varepsilon[\times \mathbb{R}$ (i.e. to $r = 0$) in such a way that

---

11) The corresponding analogues of the charts (31) (with Levi-Civita parallel normal fields $\underline{n}$) along a normal geodesic c of a Riemannian manifold are today called "Fermi coordinates (along c)".

12) Gauss does not explicitly mention the third equation in (33). However, it is a trivial consequence of the first two equations in (33) and of (32).

(33) $\sqrt{G}(0,\varphi) = 0 \quad , \quad (\sqrt{G})_r(0,\varphi) = 1 \quad , \text{ and } \quad (\sqrt{G})_{rr}(0,\varphi) = 0$

for all $\varphi \in \mathbb{R}$.

Note -- Expressed in terms of E. Cartan's calculus of exterior differential forms, (32) reads, in view of $d\sigma = \sqrt{G}\, du \wedge dv$ (see (29)) :

$$(\widetilde{32}) \qquad K d\sigma = -(\sqrt{G})_{uu}\, du \wedge dv = \Theta \ .$$

This transcription of (32) brings to mind, in view of Stokes' formula, that (32) will be the essential analytic basis for Gauss' proof of the following theorem (34) (this volume, p.54, line 6 from below). This is the theorem known today as the

Gauss-Bonnet theorem for ("small") geodesic triangles Δ of a surface M in $\mathbb{E}^3$ with angles α, β, γ :

$$(34) \qquad \int_\Delta K d\sigma = (\alpha + \beta + \gamma) - \pi \ .$$

In his summary (this volume, p. 90 , line 9 from below), Gauss states this as :

"The excess of the angles of a triangle formed by shortest paths over two right angles is equal to the total curvature of the triangle".

He further remarks (this volume, p. 56, line 9 and also p. 92, line 2) that the following generalization is obtained by "dissection" (= discerptio) into triangles (he was quite familiar with the method of triangulation because of his practical experience in surveying, this volume, p. 92, lines 2 and ff.) :

$(\widetilde{34})$ The excess of the angles of an n-sided polygon, whose sides are shortest paths, over $(n-2)\pi$, is equal to the total curvature of the polygon.

Remark -- (34) does not hold in general for arbitrary geodesic triangles. Gauss' proof of (34), however, assumes implicitly that the geodesic triangle Δ with vertices A, B, C is "small" in the following sense : if $\alpha \in]0,\pi[$ is the angles at vertex A, then there exists an orthonormal 2-frame $(\underline{a}_1, \underline{a}_2)$ in $T_A M$ and a continuously differentiable function $\rho : [0,\alpha] \to \mathbb{R}_+$ such that the "sector"

$$\{r(\cos\varphi\ \underline{a}_1 + \sin\varphi\ \underline{a}_2) \mid \varphi \in [0,\alpha] \quad \text{and} \quad r \in [0,\rho(\varphi)]\}$$

in $T_A M$ is mapped diffeomorphically by $\exp_A^M$ onto Δ, the paths

$$\left.\begin{array}{lll} c : r \longmapsto \exp_A^M(r\underline{a}_1) , & & r \in [0,\rho(0)] \\ b : r \longmapsto \exp_A^M(r(\cos\varphi\ \underline{a}_1 + \sin\varphi\ \underline{a}_2)) , & & r \in [0,\rho(\alpha)] \\ a : \varphi \longmapsto \exp_1^M(\rho(\varphi)(\cos\varphi\ \underline{a}_1 + \sin\varphi\ \underline{a}_2)) , & & \varphi \in [0,\alpha] \end{array}\right\} \qquad (*)$$

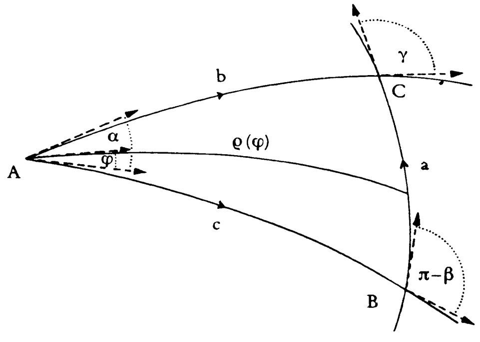
being the geodesic sides of the triangle Δ. Note that the side a of the triangle Δ (see (*)) never intersects the r-parameter lines of the mapping

$$(r,\varphi) \longmapsto \exp_A^M(r(\cos\varphi\ \underline{a}_1 + \sin\varphi\ \underline{a}_2))$$

in an angle equal to π, so that the result (27) can be applied to the integral $\int_a \Theta$. [We also observe that if, for example, the triangle Δ lies completely in a "geodesic-convex" neighborhood of the vertex A, which is always the case for sufficiently small Δ, then we are finally certain to be in the situation assumed in Gauss'proof of (34) and described above.]

Now, Gauss' proof for (34) consists of the following two-line argument (after the preparations (27), (32), (33), and in view of (*) and (30), according to which the sides b, c of the triangle are γ-parameter curves of the geodesic polar coordinates in A) (see also Figure 1) :

Figure 1



$$\int_\Delta K d\sigma \underset{(\widetilde{32})}{=} -\int_0^\alpha \int_0^{\rho(\varphi)} (\sqrt{G})_{rr}(r,\varphi)\,dr\,d\varphi = -\int_0^\alpha \Big((\sqrt{G})_r(\rho(\varphi),\varphi) - (\sqrt{G})_r(0,\varphi)\Big)\,d\varphi$$

$$\underset{(33)}{=} \alpha - \int_0^\alpha (\sqrt{G})_r(\rho(\varphi),\varphi)\,d\varphi \underset{(32),(*)}{=} \alpha + \int_a \Theta \underset{(27),(26)}{=} \alpha + \gamma - (\pi - \beta) .$$

(x) In the form (34) given by Gauss, the "Gauss-Bonnet theorem" can be interpreted as a "local comparison theorem", which compares the sum of the angles of a ("small") geodesic triangle on a curved surface with the sum π of the angles of a straight-sided triangle in the Euclidean plane, and measures the difference by means of the curvature of the surface. It is less well known that Gauss devotes the last nine articles (Articles 21 to 29, approximately one-third of the text !) of the "Disquisitiones generales" almost exclusively to proofs of theorems which compare individual angles (and not only the angle sums !) and the surface area of geodesic triangles in curved surfaces on the one hand with straight-sided triangles with the same side lengths in the Euclidean plane on the other hand [13].

In contrast to the local result (34), these last-mentioned comparison theorems are however only infinitesimal, i.e. under the hypothesis of real analyticity of the curved surface, the deviation of the curved situation from the Euclidean situation is expressed as a power series in the (common) side lengths a, b, c, and the coefficients of this power series are calculated up to and including terms of third order. The two most significant results of the Disquisitiones generales on this theme (see (35), (36) below) are contained in the

Theorem -- Let Δ be a "small" geodesic triangle on a curved surface M in $\mathbb{E}^3$ with vertices A, B, C, angles α, β, γ, and (opposite) side lengths a, b, c. Denote by σ the surface area of Δ in M and by $K(A)$, $K(B)$, $K(C)$ the values of the Gaussian curvature at the vertices of Δ. It follows from the minimizing property of the sides of Δ that $a \leq b + c$, so that there exists a straight-sided triangle Δ^* in the Euclidean plane with the same side lengths as Δ. Let α^*, β^*, γ^* be the respective angles of Δ^*, and let σ^* be the (Euclidean) surface area of Δ^*. Then the following series expansions in a, b, c are valid :

Infinitesimal Angle Comparison Theorem (this volume, p. 78, line 8 from below) :

$$(35) \qquad \alpha = \alpha^* + \frac{\sigma}{12}(2K(A)+K(B)+K(C)) + (\text{terms of } 4^{\text{th}} \text{ order and higher in } a,b,c)$$

---

[13] The lack of renown of these comparison theorems is even more remarkable if one takes into account that Gauss (this volume, p. 74, line 10 from below) announces them with distinct emphasis : "Magnam utilitatem affert consideratio trianguli plani rectilinei, cuius latera aequalia sunt ipsis a,b,c."

and

Infinitesimal Surface Area Comparison Theorem (this volume, p. 80, line 9 from below) :

(36) $$\sigma = \sigma^*\Big[1 + \frac{1}{120}\big(K(A)(s - a^2) + K(B)(s - b^2) + K(C)(s - c^2)\big) + + (\text{terms of } 4^{\text{th}} \text{ order and higher in } a, b, c)\Big]$$

(37) where $s = 2(a^2 + b^2 + c^2)$.

Remarks -- a) Of course, formulas analogous to (35) for β and γ are valid.

b) Gauss even calculates in (35) (this volume, p.76, formula 11) the terms of fourth order, though he does not give them in a completely explicit manner, and therefore we have chosen not to describe them here. However, if M is of constant curvature K_o (for example, $K_o = R^{-2}$, if M is a sphere of radius R in $\mathbb{E}^3$) , then these terms of fourth order can be easily calculated. This leads to the formula (this volume, p.78, line 10) :

(38) $$\alpha = \alpha^* + \frac{\sigma}{3} K_o + \frac{\sigma}{180} K_o^2(b^2 + c^2 - 2a^2) + (\text{terms of } 5^{\text{th}} \text{ order and higher}).$$

Notice that in the case of a sphere of radius R ($K_o = R^{-2}$), Legendre ([17], p. 426) already proved in 1787 that

(39) $$\alpha = \alpha^* + \frac{\sigma}{3} K_o + (\text{terms of } 4^{\text{th}} \text{ order and higher in } a, b, c) .$$

Gauss' generalization (35) of Legendre's result (39) for spherical geodesic triangles to geodesic triangles on arbitrary curved surfaces was developed with practical surveying strictly in mind. If one considers the earth's surface first as a sphere, and then as a spheroid (which is less curved towards the poles), then (neglecting terms of 4^{th} order) in the first case the angular correction of Legendre is the same for all three angles of the triangle, while in the second case, according to Gauss, the vertices nearer to the poles, where the Gaussian curvature is smaller, get a smaller angular correction. Gauss gives these different correction values according to (35) for one of the largest terrestial triangles measured by him, namely with "vertices" at Brocken, Hohehagen [14] and Inselsberg (ordered with increasing distances to

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[14] This less well-known mountain lies between Göttingen and (Hann.) Münden.

the north pole)[15], in Article 28 of the "Disquisitiones generales" (this volume, p. 80) as the following (in seconds) respectively :

$$4.95104'' \quad , \quad 4.95113'' \quad , \quad 4.95131'' \quad .$$

However the spherical Legendre correction, which is the same for all vertices, is 4.95116" in this case. The following interesting remark of Gauss concerning the theoretical and numerical comparison of his result (35) with Legendre's result (39), for the purpose of surveying, is contained in a letter dated 1 March 1827, to his friend Olbers[16] (see G.W. 9, p. 378) :

> "In practice this (i.e. the difference of the correction values for the different angles of terrestial geodesic triangles, (the author)) is of course not at all important, because it is negligeable for the largest triangles on earth that can be measured ; however, the dignity of science requires that we understand clearly the nature of this inequality."

c) We explain more precisely which meaning should be given to the word "small", in the hypotheses necessary for formulas (35) and (36) to hold. First Gauss chooses geodesic parallel coordinates $f :]-\varepsilon,\varepsilon[^2 \to M$ as in (31), so that the "line element" takes the form

$$(40) \qquad du^2 + G(u,v)dv^2 \quad ^{17)} \quad ,$$

and he assumes that ε is sufficiently small so that :

1) $f(]-\varepsilon,\varepsilon[^2)$ -in modern terminology- is contained in the image $\exp^M_{f(0,0)}(B)$, where B is a ball centred at the origin of $T_{f(0,0)}M$ on which the exponential map is a diffeomorphism (Gauss formulates this by means of the geodesic polar coordinates at f(0,0), see (30)).

2) The function $\sqrt{G(u,v)}$ (see (41) and note the analyticity hypothesis

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[15] The sides of this triangle are approximately 69, 85, 107 kilometers long.

[16] Heinrich Wilhelm Olbers was a general practitioner in Brennen, who operated privately a small observatory.

[17] To compare this formula and the following ones with the corresponding formulae in the "Disquisitiones generales", note that the parameters of the geodesic parallel coordinates (31), which we have denoted here (and throughout this lecture, for coherence) by u and v, are called q and p respectively by Gauss.

concerning M made above) can be developped in a power series in u and v, which converges for all $u,v \in]-\varepsilon,\varepsilon[$ and whose values lie in $]0,2[$ (this last guarantees that $\sqrt{G(u,v)}^{-1}$ can also be expressed as a power series which converges on $]-\varepsilon,\varepsilon[^2$).

Gauss now considers, for $u,u',v \in]-\varepsilon,\varepsilon[$ with $u' < u$ and $v > 0$, the geodesic triangle with vertices

$$(41) \qquad A = f(0,0) \quad , \quad B = f(u,v) \quad , \quad C = f(u',v)$$

and with the following normal geodesics $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ as sides :

$$(42) \qquad \begin{aligned} \tilde{a}(t) &:= f(t,v) && \text{for } t \in [u', u] \quad (f \text{ as in } (31)) \quad , \\ \tilde{b}(t) &:= \exp_A^M(t\underline{a}') && \text{for } t \in [0, r(u', v)] \quad , \\ \tilde{c}(t) &:= \exp_A^M(t\underline{a}) && \text{for } t \in [0, r(u,v)] \quad , \end{aligned}$$

where $\underline{a}$ and $\underline{a}'$ are suitable unit vectors in $T_A M$. Then the lengths a, b, c of the sides of the triangle are given by

$$(43) \qquad a = u - u' \quad , \quad b = r(u,v) \quad , \quad c = r(u',v) \quad .$$

It is precisely for these "small" triangles that Gauss proves (35) and (36).

In order to sketch the calculations in the "Disquisitiones generales" leading to the comparison theorems (35) and (36), we mention also Gauss' notation [17] for the following oriented angles with values in the interval $]-\pi,\pi]$ (see (31), (42)) :

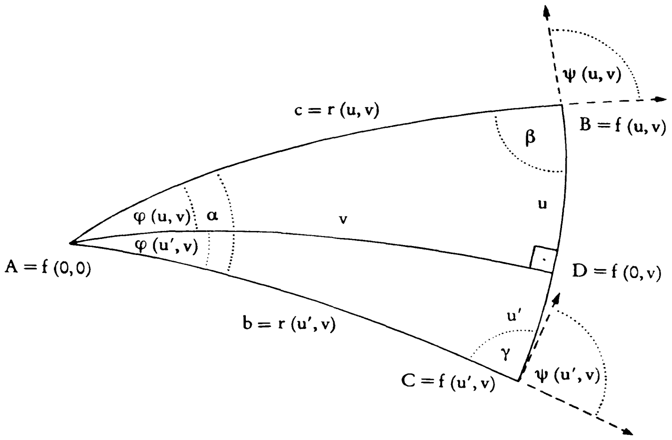
$$(44) \qquad \begin{aligned} \varphi(u,v) &:= \measuredangle(\underline{a}_1, \underline{a}) \quad , & \varphi(u',v) &:= \measuredangle(\underline{a}_1, \underline{a}') \\ \psi(u,v) &:= \measuredangle\big(\dot{\tilde{c}}(r(u,v)), f_u(u,v)\big) \quad , & \psi(u',v) &:= \measuredangle\big(\dot{\tilde{b}}(r(u',v)), f_u(u',v)\big) . \end{aligned}$$

This yields moreover for the angles α, β, γ of the triangle :

$$(44') \qquad \alpha = \varphi(u,v) - \varphi(u',v) \quad , \quad \beta = \psi(u,v) \quad , \quad \gamma = \pi - \psi(u',v) \quad .$$

The geometrical meaning of these quantities is elucidated by Figure 2.

Figure 2



The starting point of Gauss' proof of the series formulae (35) and (36) is the following. Since by (31) the particular v-parameter curve $w : v \to f(0,v)$ is a normal geodesic which cuts the u-parameter curves at right angles (hence at a constant angle !) it follows from (27) that $\Theta(\dot{w}(v)) = 0$ for all $v \in]-\varepsilon,\varepsilon[$, i.e. by (32) (this volume, p.64, line 5 from below)

(45)

$$(\sqrt{G})_u(0,v) = 0 \quad ,$$

and furthermore

$$\sqrt{G}(0,v) = 1 \quad \text{for all} \quad v \in]-\varepsilon,\varepsilon[\quad ,$$

since $\sqrt{G}(0,v) = \langle\dot{w}(v),\dot{w}(v)\rangle$ and w is a normal geodesic. Because of (45), the series expansion of $\sqrt{G}(u,v)$ for $u,v \in]-\varepsilon,\varepsilon[$ has the following form (this volume, p. 64, line 2 from below) :

(46)

$$\sqrt{G}(u,v) = 1 + f(v)u^2 + g(v)u^3 + h(v)u^4 + \dots \quad ,$$

where f(v), g(v), h(v) ..., are power series in v whose coefficients are denoted by $f^o, f', f'', \dots, g^o, g', g'', \dots, h^o, h', h'', \dots$, respectively.

Thus the coefficients of these power series f,g,h,... determine completely the inner metric (40) of M. Using (46) and the equation $\sqrt{G} \cdot K = -(\sqrt{G})_{uu}$, valid for geodesic parallel coordinates (see (32)), Gauss obtains the following series expansion for the Gaussian curvature (expressed now as a function of u and v) (this volume, p.72, line 6 from below) :

(47) $$K(u,v) = -2f(v) - 6g(v)u - (12h(v) - 2f^2(v))u^2 - \dots \quad ,$$

which thus yields an expression for K(u,v) in terms of the coefficients of the coefficients of the metric (see (40), (46)). Conversely, the coefficients f^o, f', f'', g^o, g', h^o of the metric (see (46)) can be expressed by virtue of (47) and Taylor's theorem in terms of the derivatives of the Gaussian curvature at $\underline{0} = (0,0)$:

(48) $$\begin{array}{lll} f^o = -K(\underline{0}) \quad , & f' = -\frac{1}{2}K_v(\underline{0}) \quad , & f'' = -\frac{1}{4}K_{vv}(\underline{0}) \quad , \\ g^o = -\frac{1}{6}K_u(\underline{0}) \quad , & g' = -\frac{1}{6}K_{uv}(\underline{0}) \quad , & h^o = \frac{1}{24}(K(\underline{0})^2 + K_{uu}(\underline{0})) \quad . \end{array}$$

(The formulae (48) are given by Gauss explicitly only for the special case K = constant, this volume, p. 78, line 2).

Using the transformation equations connecting two different parametrization of M developed in Article 21 (applied here in particular to the transition from geodesic parallel coordinates (31) to geodesic polar coordinates (30)), Gauss obtains the following partial differential equations for the functions (see (42), (44) above) r, φ and ψ in the variables u and v (this volume, p. 66, lines 10-11) [17] :

(49) $$\begin{gathered} 4r^2G = G\left((r^2)_u\right)^2 + \left((r^2)_v\right)^2 \quad , \\ 2\sqrt{G}\; r \sin\psi = (r^2)_v \quad , \quad 2r\cos\psi = (r^2)_u \quad , \end{gathered}$$

(50) $$G(r^2)_u\, \varphi_v + (r^2)_v\, \varphi_u = 0 \quad .$$

Then Gauss obtains a power series expansion for r, r sin ψ, and r cos ψ in the variables u and v from (49) and (46) (this volume, p. 66, formulas [1], [2] and p. 68, formula [3], and from this and (50) he gets an expansion for r cos φ and r sin φ (this volume, p. 68, formulas [4], [5]). For the oriented surface area S(u,v) of the geodesic triangle A,B,D with D = f(0,v) (see Figure 2), for which

(51) $$S(u,v) \ge 0 \text{ if } u \ge 0 \text{ and } S(u,v) < 0 \text{ if } u < 0 \quad ,$$

Gauss states, appealing to simple "geometric observations", the following partial differential equation (this volume, p. 70) :

(52) $$((r\sin\psi)S_v + \sqrt{G}(r\cos\psi)S_u)(u,v) = (r\sin\psi)(u,v)\left(\int_0^u \sqrt{G(\tau,v)}\,d\tau\right) \quad .$$

Then he obtains a power series expansion for $S(u,v)$ in the variables u and v from (46), from the power series expansion for $r \cos \psi$ and $r \sin \psi$ and from (52) (this volume, p.70, formula [7]). Therefore he also gets a power series expansion for the surface area σ of the geodesic triangle A,B,C (see (41), (51) and Figure 2) in the (geodesic parallel) coordinates u, u' and v of the vertices B and C, because

$$\sigma(u,u',v) = S(u,v) - S(u',v) \quad .$$

Comparison of the power series in u, u', v thus obtained with (43), (44) and (47) gives (after ingenious manipulations of power series) first the result (36), and then also (35).

This finishes our report on the contents of the "Disquisitiones generales".

* * * *

ON THE PRESENTATION OF THE "DISQUISITIONES GENERALES"

(i) The preceding report should indicate the high density and the deeply and carefully thought out composition of the definitions and theorems of this essay. Indeed, as we know from letters and unpublished notes, this presentation of the "Disquisitiones generales" is the ripe fruit of 15 years of repeated consideration and intellectual efforts concerning this subject and the result of two years hard work immediately preceding publication (in particular, the search for the "optimal analytic argument", i.e. the Gauss equation (14), for the proof of the "theorema egregium", i.e. the invariance under isometries of the Gaussian curvature). Interestingly, the most well-known statements of Gauss concerning his own style of work and presentation also come from this period of differential geometric activity. Concerning the latter phase of concrete preparation for the "Disquisitiones generales", we quote from a letter of Gauss to his friend Olbers [16] on 30 October, 1825 (see G.W. 8, p. 399, line 6 from below) :

> "Although the mathematical aspect of an investigation is usually the most interesting for me, I cannot deny on the other hand that in order to be pleased with a long-lasting investigation like this one, I finally have to see the emergence of a beautifully organized entirety, immaned by an unorderly appearance",

and further from a letter to his friend Schumacher [18] on 21 November, 1825 (see G.W. 8, p. 400, line 12 from below) :

> "My investigations are certainly made extremely difficult for me by the desire, which I have always possessed, of giving them

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[18] Heinrich Christian Schumacher (1780-1850), Doctor of Laws and from 1810 on Professor of Astronomy in Copenhagen (who lived in Altona close to Hambourg, which belonged at that time to Denmark), was a friend of Gauss from 1808 until his death.

such a degree of perfection, ut nihil amplius desiderari possit [19]."

Moreover, these and similar statements have been misinterpreted even by close friends of Gauss. Schumacher [18] and Bessel thus urged him repeatedly (particularly later, in view of his increasing age) to preserve his numerous ideas for future generations by a more rapid publication, and to leave the "polishing" of these ideas to others. The following sentence, from a letter of Gauss written on 5 February, 1850 (at the age of 73, almost exactly five years before his death ; see G.W. 10, 2, P. Stäckel : "Gauss als Geometer", p. 10, line 3), reveals most clearly how deeply Gauss felt that he was misunderstood by such demands and once again he pointed out precisely that component of his endeavour toward "perfection in presentation" which costed him so much time :

> "You are completely wrong if you think that I mean by this only the last layer of polish concerning the language and the elegance of presentation. Comparatively, these cause only an insignificant loss of time. What I really mean is the inner perfection. Points which have cost me years of thought exist in many of my texts and afterwards no one notices the difficulty which has (had) to be overcome, because of their concentrated and short presentation."

To this one can only add that this attitude was seen by Gauss exclusively as a personal demand on himself, and that he did not claim it as the only and compulsory standard for working in mathematics, as is shown by the tolerance of the following excerpt from a letter to Encke of 18 August, 1832 (see G.W. 11, 1, p. 84) :

> "This way of working can sometimes have as a consequence, and this has happened to me several times, that things that I have known for years are later discovered and published first by others ; it can also perhaps have as a consequence that some things disappear with me completely, and I know that some of my friends would like for me to work less in this spirit. However, this will never happen ; I can find no pleasure in fragmentary results, and a work in which I find no pleasure is a torment to me. Let each person work in that spirit which best suits him."

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19) i.e."that nothing more comprehensive could be desired". According to P. Stäckel (see "Gauss als Geometer", p. 7, G.W. 10, 2), this phrase is already found in Euler's work.

(ii) With respect to the type of presentation, Gauss clearly gives preference in the "Disquisitiones generales" to the analytical method. If they occur at all, geometric arguments are very brief, for example in the comparison theorems (35) and (36) (see above), where Gauss indicates that the partial equations necessary for the power series calculations "follow from elementary geometric considerations " without further comments.

Unfortunately illustrative figures which would be most helpful in increasing the legibility of several passages of the "Disquisitiones generales" (above all this concerns the last articles, which contain more than their share of mathematical symbols for various angles, distances, etc.), are completely missing.

Gauss saw clearly the ambivalence of the use of analytical calculations in geometrical problems (i.e. their effectiveness on the one hand, and on the other hand their inherent tendency to weaken the force of geometric intuition), as is shown by the following excerpts from his review of the "Géométrie descriptive" by G. Monge (see G.W. 4, p. 359 ff.) :

> "It is not to be denied that the advantages of an analytical treatment over a geometrical treatment, its conciseness, simplicity, uniformity, and especially its generality, usually become more and more decisive as the investigations become more difficult and more complicated. However, it is always very important to continue to cultivate the geometric method. ... In particular we must praise the work under consideration (i.e. the "Géométrie descriptive", the author) for its great clarity ..., and therefore recommend its study as nourishing intellectual substance, by which undoutedly much can be contributed to the revival and conservation of the genuine geometric spirit, sometimes missing in the mathematics of these times."

The latter recommendation is supplemented and rounded off in the review by the remark (which is also interesting didactically) that the geometric method will

> "remain indispensable in the early study of young people, to prevent one-sidedness ... and to give to the understanding a lineliness and directness, which are much less developped and -occasionally- rather jeopardized by the analytical method."

Finally we quote in this context two passages from letters of Gauss, which shed in a clear light his opinion of new mathematical calculi (to whose developments he himself also contributed) (see G.W. 8, p. 298) :

> "After all the situation concerning new calculi is such that one cannot accomplish anything with them which could not be accomplished without them ; the advantage, however, is that when such a calculus corresponds to the innermost nature of frequent and repeated needs, then anyone has mastered them (even without the almost unconscious inspiration of a genius, which nobody can gain by force, to solve problems belonging to this calculus), can solve them (even in such complicated cases that the genius would be powerless without such help). This has been the case concerning the invention of the differential calculus ; it is also the case (though only in more special cases) of Lagrange's calculus of variations, of my calculus of congruences and of Möbius calculus. Countless problems which otherwise would stand isolated and would demand in each special case new efforts (large or small) of ingenuity, are comprised by such concepts to become an organic realm."

However, Gauss criticized a purely mechanical use of calculi, without keeping in mind their origin in conceptual or geometric intuition, as a contribution to the detriment of the "solidity" of mathematical working (see G. W. 10, 1, p. 434) :

> "It is in the nature of mathematics of modern times (in contrast to that of antiquity), that we possess a lever in the form of our symbolic language and nomenclature, whereby most complicated reasoning are reduced to a certain mechanism. In this way Science has gained infinitely in richness, but has lost equivalently ... in beauty and character. How often is this lever applied purely mechanically ... . I demand that one should whenever using the calculus, ... remain aware of the original stipulations ..."

(iii) Finally it remains to be mentioned that Gauss does not indicate, in the "Disquisitiones generales", the geometrical intuition which has led him to the discovery of the Theorema egregium. Its presentation as a corollary to the Gauss equation (14), proved by (five pages of) pure calculation, conveys the reader in the smoothest possible manner to be sure, but nevertheless -due to its lack of geometric motivation- it hits the reader like an analytical "deus ex machina".

This effect is still experienced and confirmed by students in differential geometry today, when they are taught the invariance of the Gaussian (or Riemannian) curvature under isometries as a consequence of the Gauss equation, although the derivation of the latter equation nowadays (by means of Levi-Civita's covariant differentiation) has merely the character of a simple analytical exercise.

Having regard to this deficiency of any information about the leading, basic motivation, one is reminded of a statement of Gauss reported by Sartorius von Waltershausen in his article "Gauss zum Gedächtnis" (see [24], p. 6, line 5 from below) which appeared in 1856 : "After a building ... has been completed, one should no longer see the scaffold". In view of the above mentioned quote of Gauss concerning his endeavour towards the perfections of the contents of this results, this quotation is not only to be taken aesthetically. Indeed, the following interpretation seems to be much more probable (in particular if one takes into account the behaviour of Gauss during the completion of the final version of the "Disquisitiones generales", to be reported below) : according to Gauss' rich experience with that topic, the analytical "Gauss equation" (14) was expressing for him the content (and not merely the beauty !) of his key result (namely that the Gaussian curvature is determined solely by the inner metric of the surface) so perfectly, that the geometric ideas, which had led him first to the discovery of the invariance of the Gaussian curvature under isometries, were considered by him to be pushed into the background or even to be a rather disturbing framework. Moreover, the history of differential geometry has provided a belated justification of this analytical accentuation through Riemannian geometry, in so far as the importance and the central position of the Gauss equation are yet more evident here (than in the differential geometry of surfaces in $\mathbb{E}^3$). Still the question of geometric origin of the Theorema egregium remains challenging !

Fortunately, the history of the origin of the "Disquisitiones generales" (as it can be reconstructed from Gauss' papers which were made public only much later) sheds some light on this open question. This is what we shall now describe.

* * *

ON THE HISTORY OF THE ORIGIN OF THE "DISQUISITIONES GENERALES" AND ON THE HISTORY OF THE IDEA OF THE THEOREMA EGREGIUM

(i) The starting point for Gauss' work on differential geometry is found on the one hand in his lasting interest since his early years (at least since 1794 !) in the foundations of geometry (his interest there being focused almost exclusively in "finding the truth" about the independence of Euclid's axiom of parallels), and on the other hand in his occupation with surveying (which was required professionally of him as Director of the Göttingen Astronomical Observatory). One common ground of both of these main streams of his geometric interest was undoubtedly trigonometry (Euclidean, spherical, hyperbolic, and that of curved surfaces), and here in particular the theorem on the sum of angles of a triangle or a polygon in these geometries. Gauss dates his first fundamental insights about this as occuring already at the age of 17, as we learn from his letter of 10 October, 1846, to Gerling (see G.W. 8, p.26 p. 266) :

> "The theorem which Mr. Schweikart mentioned to you, that in any geometry the sum of all outer angles of a polygon differs from 360° by a quantity, ..., which is proportional to the surface area, is the first theorem lying almost on the threshold of that theory, a theorem whose necessity I already recognized in 1794."

In the case of spherical trigonometry, this theorem of angular excess (see (34) above, the "Gauss-Bonnet" theorem) was certainly already widely known at that time. Here Gauss is talking about the hyperbolic case, as is shown by the reference of Schweikart (who had investigated a "hyperbolic" geometry).

His work on surveying led him (between 1812 and 1816) to the study of geodesics on ellipsoids of revolution and to the question of determining "all" conformal charts for general curved spaces. He estimated these charts to be the most important ones in matters of surveying (see [24], p. 91, line 7) :

"You are quite right, Gauss writes to Hansen on 11 December, 1825, that the essential condition in every map-projection is the infinitesimal similarity, a condition which should be neglected only in very special cases of need."

In a letter to Schumacher of 5 July, 1816, Gauss states (see G.W. 8, p. 370) : "I have also conferred with Lindenau about a competition question(♦) which was to be posed in the new journal. I had thought of an interesting problem, namely :

(53) in the general case, to project (map) a given surface onto another (given) one in such a way, that the image and the original become infinitesimally similar.

A special case arises when the first surface is a sphere and the second a plane. Here the stereographic and Mercator's projections are particular solutions. However, one wants the general solution for all types of surfaces, containing all these particular ones."

This competition question, suggested by Gauss to a new journal for astronomy, was however not chosen by the editors [21]. His former student and friend Schumacher [18], whom he had told about this problem, hence used the first opportunity he had and induced the Copenhagen Scientific Society to pose the competition question (53) (see above) in 1821. Since no solution had been submitted in 1821, the problem (53) was set once again in 1822. When Schumacher reported this to Gauss on 4 June, 1822, Gauss answered on 10 June, 1822 : "I am sorry that I have only now learned of the renewal of your competition question ..." On 25 November, 1822, he asked Schumacher when the deadline for submission of solutions was set, and after Schumacher replied that it was the end of the year, Gauss submitted his treatment on 11 December, 1822 (see [24], p. 90). Thus the first important contribution of Gauss to the theory of surfaces was an answer (found under the pressure of a deadline) to a challenge which he himself had posed.

This competition essay was published only in 1825 in the "Astronomischen Abhandlungen" under the following title :

"A general solution to the problem of mapping the parts of a given surface onto another surface such that the image and the mapped part are similar in the smallest parts"

(♦) "Preisfage".

21) Presumably Gauss already possessed around this time (1816 ?) the main idea how to solve this problem (53) (see G.W. 8, p. 371, p. 372, line 2 from below).

and Gauss added the following latin sentence to the title, like a motto :

"Ab his via sternitur ad maiora",

a self-confident announcement [22] of a successful breakthrough in this work of his. The "way paved herewith to greater things" is certainly the way towards "Inner Differential Geometry". Precisely two days after he sent off his solution to the competition question, Gauss wrote up some private notes on 12 December, 1822, with the title "The state of my investigations on the transformation of surfaces"(see G.W. 8, p. p. 374-384), in which he emphasizes one single result (see G.W. 8, p. 381, Formula 25). Namely, if the line element of a curved surface with respect to some conformal chart is given as $\sqrt{m^2(du^2+dv^2)}$, then its Gaussian curvature K can be calculated to be

$$K = -\frac{1}{m^2}\left(\frac{\partial^2 \log m}{\partial u^2} + \frac{\partial^2 \log m}{\partial v^2}\right) \quad . \tag{54}$$

This is the Gauss equation of the "Disquisitiones generales" for the special case of a conformal chart (see (14) above), a useful formula which has been often used in differential geometry later but which unfortunately was not retained by Gauss for the "Disqnitiones generales". Gauss concludes from this : "... the curvature keeps the same value under all transformations of the surface which leave the line element $\sqrt{m^2(du^2+dv^2)}$ unchanged." [Since the fact of the invariance of K under isometries contained herein was known to Gauss already in 1816 by way of geometric arguments (see (56) below), the "breakthrough" experienced here by Gauss is no doubt to be attributed to obtaining the explicit analytical "binding" of K to the first fundamental form by the formula (54).]

We learn from Gauss' accompanying letter of submission of his competition essay that he was urged on to further investigations by this work (see G.W. 4, p. 191) : "The author of this treatise ... regrets that the latter situation (meant here is his late notification by Schmacher of the information concerning the renewal of the unsolved competition questior, the author) has obliged him to restrict himself ... to the bare essentials. If it were not for the deadline, the author would have liked ... to pursue the development of several secondary topics, which he must now reserve for another time and place."

22) I. Newton ([20], p. 244) added the words "Et his principiis via ad maiora sternitur" to his treatise "De quadratura curvarum", in which he published earlier investigations which had led him to the calculus of fluxions (i.e. to his version of the differential calculus).

(ii) From 1821 until August 1825, a considerable part of Gauss'time is taken up in laborious and time-consuming geodesic field measurements.

He writes to Olbers about this (see [23], p. 29) : "I cannot look back on my five years of measuring without discontent", and in another passage, concerning field measurements to be made in the summer of 1825 (see [23], p. 29) : "I would like very much to complete all works of that sort which yet remain at one swoop, in order to use the years of my life, which Heaven will still grant to me, working in my study, undisturbed."

Only afterwards is Gauss able to attack a new theory of the surfaces. He reports on this to Olbers on 9 October, 1825 (see G.W. 8, p. 397), and writes on 21 November, 1825, to Schumacher (see G.W. 8, p. 400) :

> "Recently I have taken up again a part of the general investigations on curved surfaces which are to form the basis of my projected essay on advanced geodesy. It is a subject which is as rich as it is difficult, and it takes me from accomplishing anything else. Unfortunately, I find that I have to go far back in the exposition because even what is known must be developed in another, different, form suitable to the new investigations. All roots of the tree must be followed to their ends, and some of these efforts cost me weeks of strenuous thought. Much of this belongs to the Geometria situs, a field almost completely uncultivated up to now."

The rough draft (which was already quite extensive) for this chapter on differential geometry of his planned work on advanced geodesy was also found in Gauss' manuscripts and carried the title

(55) "New general investigations on curved surfaces".

He had composed it during the last three months of 1825. From this draft and from further notes (collected painstakingly by P. Stäckel from the manuscripts of Gauss), we have the following :

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23) (Note for p. 130)
The name "Krümmungsmaß" (measure of curvature) for the number defined by (8) occurs for the first time only in private notes of Gauss of 12 December, 1822 (see G.W. 8, p. 381), whereas Gauss used the corresponding important geometric concept (introduced by definition (8)) already much earlier.

(iii) Dating of the discoveries of some of the new concepts and theorems in the (later) "Disquisitiones generales".

1) Concept of the Gauss map (see (7)) 2) Concept [23] of the Gaussian curvature (see (8)) 3) The result $K = \kappa_1 \cdot \kappa_2$ (see (13))	Between 1799 and 1813 (see G.W. 8, p. 367, p. 369).
4) Invariance under isometries of the total curvature (see (16))	Around 1816 (see G.W. 8, p. 372).

This relatively early discovery becomes the pacemaker and leitmotiv for the subsequent investigations of Gauss on the differential geometry of curved surfaces, and he himself designates it as the "beautiful theorem". We formulate it here once again explicitely, closely following Gauss, who did not yet use the words "total curvature" in 1816 (see G.W. 8, p. 372) :

(56) The *"beautiful theorem"* (Gauss, around 1816) -- If a curved surface on which a figure is fixed takes on different shapes in $\mathbb{E}^3$, then the surface area of the spherical image of the figure is always the same, for all possible shapes (of the surface).

Corollary -- The Theorema egregium (see (15)), i.e. the invariance under isometries of the pointwise defined Gaussian curvature, follows trivially from (56) by the limit procedure described in (8).

5) Derivation of the "Gauss equation" (i.e. calculation of the Curvature curvature K from the first fundamental form alone, the latter being given)
   a) in conformal coordinates (see (54)) : 1822 (see G.W. 8, p. 381),
   b) in geodesic polar coordinates (see (32)) : 1825 (see G.W. 8, p. 442).

6) Comparison theorems for the angles of geodesic triangles (see (35) and Legendre's result (39)) : 1825 (see G.W. 8, p. 399).

7) The sum of angles in small geodesic triangles (see (34) = "Gauss-Bonnet theorem") : 1825 (see G.W. 8, p. 435).

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[23] See p. 129.

Note .- Gauss gave no proof or indication of proof for the "beautiful theorem" in his notes of 1816 (see (56) above). In the fragment (55) of 1825, a proof of (56) (see (iv) below) can be found which makes use of the theorem (34) on the angular excess in small geodesic triangles. It is thus quite probable that Gauss already possessed in 1816 the theorem (34) on the angular excess, i.e. the "Gauss-Bonnet theorem", as a "forerunner" to (56). This presumption is supported by the state of knowledge of Gauss concerning properties of angles of triangles in hyperbolic geometry. In 1816, he already knew the theorem that triangles with equal angles are always congruent in hyperbolic geometry (and he added an interesting speculation on a possible universal unit for measuring distances, in case the universe is hyperbolic, see G.W. 8, p. 168), and in 1819 he stated for hyperbolic geometry (see G.W. 8, p. 182, line 18) : "The defect of the sum of the angles in a triangle from 180° is not only increasing as its surface area increases, but is exactly proportional to it."[24] In later years (1846), Gauss even back-dated his personal knowledge of this latter theorem, "almost lying on the threshold" (of hyperbolic differential geometry, the author) back to 1794 (see G.W. 8, p. 266) .

8) General "Gauss lemma" (see (22), (23), (24)) : 1825 (see G.W. 8, p. 439).

9) Derivation of the "Gauss equation" (see (14)) (i.e. the calculation of the Gaussian curvature K from the first fundamental form only if the latter is given in arbitrary coordinates) : 1825 (see [24], p. 97, line 5).

A glance at this chronological table shows that the ordering of the deductive presentation of the concepts and theorems in the "Disquisitiones generales" is at essential points inverse to the chronological order of their discoveries . This is not surprising, since Gauss certainly did not know the "Gauss equation" (14) in general coordinates, which is the cornerstone of his

24) The letter of 3 March, 1819 to Gerling, from which this quotation stems, is an important document about the state of knowledge of Gauss and about hyperbolic geometry around 1819. Passages from this letter (see in particular G.W. 8, p. 10 ff.) may have been the reason for the conjecture (which cannot be proved with certainty) that terrestial or astronomical angle measurements were made by Gauss as a "test" for the validity or invalidity of Euclidean geometry in the space which surrounds us.

presentation in the "Disquisitiones generales", before the end of 1825. For this reason, Gauss gives in the fragment (55) (found only in his manuscripts) a completely different proof.

(iv) Proof for the "Theorema egregium" (= invariance under isometries of the Gaussian curvature), which does not use the Gauss equation (14), but which for his part clears up the geometrical origin of this discovery. We now sketch this proof (see G.W. 8, pp. 435-436) :

The point of departure is essentially theorem (34) on the "angular excess of small geodesic triangles over two right angles", in which however the total curvature measuring the excess is not defined (as in (34)) by the integral over the Gaussian curvature, but directly as the "oriented" surface area of the spherical image (see (7)) of the triangle. We give here, in free translation, the formulation of this theorem by Gauss in the fragment (55) (see G.W. 8, p. 435) :

(57) The sum of the angles of a (small) geodesic triangle Δ in a curved surface in $\mathbb{E}^3$ is equal to the sum of π and the oriented surface area of the spherical image of Δ, where the oriented area is taken to be positive or negative according to whether the boundary of the spherical image of Δ winds around the image in the same direction or in the opposite direction as the boundary of Δ winds around Δ.

The result (57) was already widely known in the special case of a developable or a spherical surface. Gauss possessed the result analogous to (57) for hyperbolic geometry (and therefore essentially for surfaces of constant negative curvature, the author) already in 1794 (see G.W. 8, p. 266), and announced it to Gerling in a letter in 1819 (see G.W. 8, p. 182). It is therefore easy to imagine that Gauss was led to a geometrically intuitive "insight" concerning the validity of (57) for the general case of a surface of non-constant Gaussian curvature from the knowledge of this fundamental case and on the basis of his rich differential-geometric experience in geodesy (with geodesics and trigonometry spheroids , but also with questions of mapping and bending) obtained in the years 1812 to 1822.

In the fragment (55) Gauss sketches a proof of (57) (where he had to distinguish between different possible geometric cases). However, he did not seem to be too happy with it, as the following remark of his shows (see G.W. 8, p. 435) : "This proof will need explanation and some change in its form, if ..." (one of the different geometric cases occurs).

Although Gauss had studied intensively the concept of "oriented surface area" as used by him in (57) (see G.W. 8, p. 398, line 2 from below), he did not think that his corresponding studies had "matured" sufficiently. In any case, the simple analytic description which is common today is absent in Gauss :

(58) oriented surface area of the spherical image of $\Delta = \int_\Delta \zeta^* \sigma_2$,

where ζ is the Gauss map (7) of the curved surface under consideration and σ_2 is the surface area form of the sphere S^2 in $\mathbb{E}^3$.

The geometric (and historically the original !) proof of Gauss for the Theorema egregium (15) is found as follows in the fragment (55). From the previously proved theorem (57) on the excess of the sum of angles for geodesic triangles, Gauss next deduces the analogous theorem on the excess of the sum of angles for arbitrary geodesic polygons with $n \geq 3$ sides :

(59) The sum of all angles of a small geodesic polygon Π with n sides, lying in a curved surface M in $\mathbb{E}^3$, is equal to $(n-2)\pi$ plus the oriented surface area $\int K(\Pi)$ [25] of the spherical image of Π under the spherical map of M,

and then he argues (not verbally but essentially !) as follows (see G.W. 8, p. 435, line 3 from below, and p. 436) :

under an isometry ("development") $f : M \to M'$ of one curved surface M in $\mathbb{E}^3$ onto another one M', the lengths of curves on the surfaces remain invariant. Thus f maps geodesics of M onto geodesics of M'. For the same reason, every (closed) geodesic ε-neighbourhood D_ε of a point A in M is mapped onto the (closed) geodesic ε-neighbourhood of D'_ε of the point $A' = f(A)$, i.e. $D'_\varepsilon = f(D_\varepsilon)$, and hence the following equality of their surface areas follows from the property of f being isometric :

(60) $$\text{area}(D_\varepsilon) = \text{area}(D'_\varepsilon) \quad .$$

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25) Here we use "$\int K$" as a suggestive symbol (not to be divided into "$\int$" and "K", although suggestive of this) for the real valued set function which to each compact subset D of the surface M bounded by a curve, assigns the "oriented surface area $\int K(D)$" of the spherical image of D under the Gauss map ζ of M, where the sign is determined by comparing the winding directions of the boundaries ∂D and $\zeta(\partial D)$ respectively, analogous to the explanation of Gauss in (57).

On the other hand, angles between intersecting curves are also preserved by the isometry f. From this and (59), it follows immediately that :

(61) The oriented surface area $\int K(\Pi)$ of the spherical image of a geodesic polygon Π on M is equal to the oriented surface area $\int K(\Pi')$ of the spherical image of the geodesic polygon $\Pi' = f(\Pi)$ on M' corresponding to Π (under the isometry f) [25].

By approximating the closed ε-neighbourhood D_ε of A by (inscribed) geodesic polygons of the surface M, it follows by virtue of (61) that :

(62) The oriented surface area $\int K(D_\varepsilon)$ of the spherical image of D_ε under the Gauss map of M is equal to the oriented surface area $\int K'(D'_\varepsilon)$ of the spherical image of D'_ε under the Gauss map of M' [25].

Taking the limit as $\varepsilon \to 0$, it follows from (60) and (62) that :

$$K(A) \underset{(8)}{=} \lim_{\varepsilon\to 0} \frac{\int K(D_\varepsilon)}{\text{area}(D_\varepsilon)} \underset{(60),(62)}{=} \lim_{\varepsilon\to 0} \frac{\int K'(D'_\varepsilon)}{\text{area}(D'_\varepsilon)} \underset{(8)}{=} K'(A') \quad {}^{25)} .$$

That is, the Gaussian curvatures at points A and A' corresponding to each other under an isometry f of two isometric curved surfaces M and M' in $\mathbb{E}^3$ onto each other, are equal. This is the Theorema egregium !

(v) The proof (with its so impressive geometric arguments, which we have just retraced from the fragment (55)) was never published by Gauss ! The reason for this lies on the one hand certainly in his self-criticism of his own above mentioned sketch of a proof for (57), and particularly of his concept of the "oriented surface area of the spherical image on a curved surface" (involved in that proof) which he has not defined with that analytic rigour usually applied by him [26]. Over and above this, there surely is another eason. In the fragment (55), the proof of Gauss referred to above (see (iv)) for the Theorema egregium is followed by the "Gauss lemma" (see (23), (24)) and the "Gauss equation" specialized to geodesic polar coordinates (see (32)) ; then the fragment under the title (55) "new general investigations on

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26) Gauss judged non rigorous proofs very harskly. In his letter of July 1828 to Olbers, he says, concerning the question of the existence of straight line generators for developable surfaces in $\mathbb{E}^3$ (see G.W. 8, p. 44, line 2 from below) : "In all the so-called proofs that I know except mine, this existence of such straight lines is obtained quite surreptitiously ..."

curved spaces" breaks off abruptly [This can be dated as approximatively occuring at the end (December ?) of 1815.] Gauss must have recognized here that the Theorema egregium follows immediately from that last analytic result (32) obtained, if one knows in addition that (under an isometry of curved surfaces) geodesic polar coordinates of one surface always go over into geodesic polar coordinates of the other surface. This can indeed be immediately seen from the length- and angle-preserving properties of an isometry (and, moreover, much more easily than some of the geometric arguments in his proofs of (57) and (62) !). Thus it seems that Gauss considered his first proof, described in (iv) above, to be conclusively "dethroned" and outdated.

In view of Article 21 (this vol. p. 60, lines 7-5 from below) of the later "Disquisitiones generales" (for which there is no precursor in the fragment (35)!), we might even be led to conclude that at that time Gauss had perceived the possibility, in principle, of expressing the coefficients E, F, G, of the first fundamental form with respect to one local chart of the surface (see (13)) in terms of the analogous coefficients E', F', G', of another such chart. Hence it should be in principle possible to express the Gauss equation (32), obtained first by him in the special chart of geodesic polar coordinates, as an equation of the type (14). In this way Gauss could have found the explicit form (14) of the Gauss equation for general coordinates on the surface. It is almost certain that Gauss did not found this latter calculation till 1826, as well as the final proof of (14) appearing in the "Disquisitiones generales", which is "pushed through" by sheer computation in the given general coordinates. Thus Gauss, realizing the central position and the great effectiveness of this tool, put aside such already well-presented article as the fragment (55) and began developing a new presentation of these results, based entirely on the Gauss equation (14). One learns then from this letter of 20 Novembre, 1826, to Bessel (see G.W. 9, p. 392, line 19) that he has given up his original intention to include his investigation concerning the differential geometry of curved surfaces in a book on advanced geodesy, and instead will devote a separate essay to it. By this time, the part of the investigations concerning the Gaussian curvature seems to him to be so matured [27],

[27] Among Gauss' manuscript one can find (rather sketchy) notes, originating also from the time between 1822 and 1825, on the "Seitenkrümmung") (= geodesic curvature) of curves on a curved surface in $\mathbb{E}^3$ (see G.W. 8, p. 386-395). These notes had not yet reached a comparable state of working out. Consequently Gauss excluded completely the concept of "geodesic curvature" from the contents of the "Disquisitiones generales". Here F. Minding was prior to him with a publication (see [18], 1830) !

that he prepares it for publication without delay. Thus the "Disquisitiones generales circa superficies curvas" arises as a kind of concentrated research report on his investigations of the theory of surfaces, in which he not only bases the new presentation fully on the Gauss equation (14), which now comes at the beginning, but also omits all the elementary parts of the first draft with the title (55) (concerning curvature of curves, of surfaces, etc).

In this manner, the "Disquisitiones generales" exemplify most convincingly the motto which was engraved in the seal used by Gauss (see [24], p. 6, line 3 from below). This seal pictures a tree with a few fruits, and carries the words :

"Pauca, sed matura".

It is the culmination of more than fifteen years of thought and work on the geometry of surfaces, and at the same time an opening of new perspectives, almost unparallel in mathematical literature for its density and beauty of presentation, as well as for the originality of its contents and the stimulative force of its ideas [28].

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[28] It is above all this "evolutive" component of the "Disquisitiones generales" which makes the ("static"- or "finished"-sounding) title of "edifice", which P. Stäckel gave to this work of Gauss, seem so inadequate in our opinion. (See note p. 99).

SOME IMPORTANT THEMES, RESULTS AND DEVELOPMENTS IN DIFFERENTIAL GEOMETRY DURING THE LAST 150 YEARS

Even a cursary sketch of these is not possible without first recalling the following important basic concepts of differential geometry (which were introduced from 1854 on, essentially after Gauss' death in 1855) :

1) n-dimensional Riemannian manifold M (which we always assume here to be connected) with its inner metric (defined by the infimum of lengths over all continuously differentiable paths joining two points) and its sectional curvature (function) $\varkappa^M$, introduced by B. Riemann in 1854 using the (intrisic !) Gaussian curvature [29] (see [21], p. 272-287), as well as isometric immersions and isometries of Riemannian manifolds.

2) Lie groups and differentiable actions of these on manifolds [S. Lie and F. Klein (from 1869 on), E. Cartan].

3) Levi-Civita parallel transport L_c, for vectors along a differentiable path $c : [a,b] \to M$ in a Riemannian manifold M (T. Levi-Civita, 1917), which is a

(63) linear isometry $L_c : T_{c(a)}M \longrightarrow T_{c(b)}M$

between the tangent spaces at the initial and end points of c.

4) Topological concepts, such as for example, connectedness, compactness,

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29) If $\sigma \in G_2(T_pM)$, that is, σ is a two-dimensional vector subspace of the tangent space T_pM of M at the point $p \in M$, then Riemann defines the "curvature $\varkappa^M(\sigma)$ of σ (in M)" as the value of the Gaussian curvature at the point p of the two-dimensional surface in M which is spanned by the geodesics of M which pass through p and are tangent to σ. The Riemann manifold M is said to be "of constant sectional curvature C ($\in \mathbb{R}$)", if $\varkappa^M(\sigma) = C$ for all $p \in M$ and $\sigma \in G_2(T_pM)$.

orientability, simple connectedness, fundamental groups and coverings, homology and cohomology of manifolds (developed from around 1890 on, e.g. by H. Poincaré, H. Hopf, H. Whitney, S. Eilenberg, and many others).

5) Metrical completeness (M. Fréchet, F. Hausdorff, 1914) and geodesic completeness (H. Hopf and W.Rinow, 1931) of Riemannian manifolds.

Recommendations for literature : [1], [4], [9], [15], [22], [27]. Whenever no reference is given, consult the bibliographies in [15], [22], [27].

Using these basic concepts we now mention ("pars pro toto") the following themes of inner differential geometry.

(i) The converse to the "Theorema egregium", i.e. to the invariance under isometries of the Gaussian, or more generally of the Riemannian, curvature.

Because of its definition [29] and Gauss' Theorema egregium, the curvature of a Riemannian manifold is invariant under isometries. More precisely, if $f : M \to M'$ is a differentiable immersion of Riemannian manifolds of the same dimension ($\dim M \geq 2$), then in order for f to be an isometry we must have

$$\varkappa^{M'}(f_*(\sigma)) = \varkappa^{M}(\sigma) \text{ for all } p \in M \text{ and all } \sigma \in G_2(T_p(M)$$ [29] .

This result suggests at once the following "converse" :

Problem .- In which sense do "equality assumptions" on the curvature functions of equal-dimensional Riemannian manifolds imply that they are isometric ?

We mention three prominent results concerning this problem.

a) Any two n-dimensional ($n \geq 2$) Riemannian manifolds of equal constant sectional curvature are locally isometric (B. Riemann 1854, see [27] p. 59, corollary 2.3.8 and p. 69, 2.4.11).

b) An arbitrary simply connected complete n-dimensional ($n \geq 2$) Riemannian manifold of constant sectional curvature C ($\in \mathbb{R}$) is (globally !) isometric to :

1) the n-dimensional Euclidean space $\mathbb{E}^n$ (if $C = 0$),
2) the n-dimensional sphere of radius $\frac{1}{\sqrt{C}}$ in the (n+1)-dimensional Euclidean space (if $C > 0$),
3) the n-dimensional hyperbolic space in which the sum of angles of each geodesic triangle Δ is equal to $\pi + C \text{ area}(\Delta)$ (if $C < 0$).

c) The theorem of E. Cartan (1928) and W. Ambrose (1956) (see [27], p. 61 ; [4], p. 238 ; [1]).

Preliminary remark : Each r-sided (not necessarily closed !) geodesic polygone $c : [0,s] \to M$ in a Riemannian manifold M, parametrized by arc length and with "vertices" $0 = a_o < a_1 < \dots < a_r = S$, is uniquely determined by its side lengths $s_i = a_i - a_{i-1} \in \mathbb{R}_+$, $i = 1,\dots,r$, and by the initial velocity vectors of each of its r sides, transported by the Levi-Civita parallel translation along c back to the starting point $p = c(0) \in M$ to give vectors $v_1,\dots,v_r \in T_pM$. That is,

$$\dot{c}(a_{i-1^+}) = L_{c|[0,a_{i-1}]}(v_i) \quad \underset{\sim}{\text{for}} \quad i = 1,\dots,r \quad (\text{see } (63)) \quad .$$

Hence c can be identified with the (2r+1)-tuple $(p;v_1,\dots,v_t;s_1,\dots,s_r)$. Conversely, each such (2r+1)-tuple with $p \in M$, $v_1,\dots,v_r$ unit tangent vectors to M at p, and $s_1,\dots,s_r$ positive real numbers, yields a unique r-sided geodesic polygon starting at p, which we call

$$(64) \qquad c = (p;v_1,\dots,v_r;s_1,\dots,s_r) \quad .$$

Moreover, we obtain from each 2-dimensional subspace σ of the tangent space to M at the starting point p of the geodesic polygon c (see (64)) a 2-dimensional subspace $L_c(\sigma)$ of the tangent space to M at the end point of c by means of the Levi-Civita parallel translation along c (see (63)), and we denote its sectional curvature by

$$(65) \qquad \lambda_r^{(M,p)}(v_1,\dots,v_r;s_1,\dots,s_r;\sigma) = \kappa^M(L_c(\sigma)), \ \underset{\sim}{\text{where}}\ c\ \underset{\sim}{\text{is as in}}\ (64).$$

Now the theorem of Cartan and Ambrose says that <u>equality of the curvature</u> functions $\lambda_2^{(M,p)}$ (i.e. with $r = 2$) defined above, is not only necessary but also <u>sufficient for</u> global <u>isometry</u> of Riemann manifolds in the simply connected complete case. Thus one obtains in this case a fully satisfactory answer to the problem formulated above. More precisely, the theorem says :

> *Let M and M' be two n-dimensional ($n \geq 2$) simply-connected complete Riemannian manifolds, $p \in M$ and $p' \in M'$, and $\varphi : T_pM \to T_{p'}M'$ a linear isometry of the tangent spaces of M at p and of M' at p'. If for all pairs (v,w) of unit tangent vectors to M at p, for all $\varepsilon > 0$ and $\delta > 0$, and for all 2-dimensional subspaces σ of the tangent space to M at p (see (65)) we have*

$$\lambda_2^{(M,p)}(v,w;\varepsilon,\delta;\sigma) = \lambda_2^{(M',p')}(\varphi(v),\varphi(w);\varepsilon,\delta,\varphi(\sigma)) \quad ,$$

then there exists a (unique) isometry $f : M \to M'$ from M into M' (with $f(p) = p'$ and $f_*|T_pM = \varphi$).

In short, equality of the sectional curvature under parallel translation along two-sided geodesic polygons is a necessary and sufficient condition for isometry of simply connected complete n-dimensional Riemannian manifolds.

(ii) The Clifford-Klein space-form problem.

This problem treats the following question : to which extent are the global homeomorphism type or (affine) diffeomorphism type or isometry type of complete n-dimensional Riemannian manifolds of fixed constant sectional curvature C ($\in \mathbb{R}$) [which are of course all locally isometric to each other (see (i), a)) and are called "space-forms with curvature C"] restricted by the hypothesis on their curvature, as well as the larger problem of a complete classification of this class of Riemannian manifolds ?

Clifford found in 1873 a two-dimensional Riemannian manifold with constant zero curvature which was diffeomorphic to the two-dimensional torus, the so-called "flat Clifford torus" in S^3 ($\subset \mathbb{E}^4$). F. Klein then formulated the programme outlined above in 1890.

a) For $C = 0$, i.e. for the space-forms M locally isometric to the Euclidean space $\mathbb{E}^n$, the following results are known (see [22], p. 359 ff.) :

n = 2 : (W. Killing, 1891). If M is compact, then M is homeomorphic to the torus or the Klein bottle. If M is not compact, then M is homeomorphic to $\mathbb{R}^2$ or to a cylinder or to the Möbius strip.

n = 3 : (W. Hantzsche and H. Wendt, 1935). Among the compact three-dimensional Euclidean space forms, there are six orientable and four non-orientable homeomorphism types (see [27], 3.5.5 and 3.5.10). (W. Nowacki, 1935). Among the non-compact three-dimensional Euclidean space-forms there are four orientable and four non-orientable homeomorphism types (see [27], 3.5.5 and 3.5.10).

n = 4 : The classification problem is unsolved, but we have the following result of L. Bieberbach. There are only finitely many compact (1911) and non-compact (1929) homeomorphism types of n-dimensional Euclidean space forms.

b) For $C = 1$ (this is the general case $C > 0$ up to homothetic transformation), i.e. for the space-forms locally isometric to the unit sphere S^n in $\mathbb{E}^{n+1}$, the following is known :

n even : (H. Hopf, 1926). M is isometric to S^n or to $P^n(\mathbb{R})$ (obtained from S^n by identifying antipodal points).

$n = 3$: H. Seifert and W. Threlfall, 1930-32). There are only finitely many homeomorphism types of three-dimensional spherical space-forms.

$n \geq 4$: After G. Vincent (1947) cleared off the case $n \equiv 1 \mod 4$, J. Wolf (1967) solved the isometry classification problem for spherical space-forms of arbitrary dimension $n \geq 4$ (see [27], p. vii, line 6 from below and 7.4).

(iii) Riemannian manifolds with "rich" isometry groups.

Abundance of isometries of a Riemannian manifold onto itself (i.e. a high degree of "inner movability") must imply a certain regularity of the sectional curvature, because of its invariance under isometries and thus corresponding restrictions on the isometry types are to be expected. Results here are best approached from the following two theorems a) and b) (not given in the chronological order of their discovery) :

a) The isometry group of an n-dimensional Riemannian manifold is a Lie group which acts differentiably on M and has a dimension $\leq (n+1)n/2$ (see [15], vol. I, p. 239). [If equality holds in the dimension estimate, then M is of constant sectional curvature (see [15], vol. I, p. 238).]

b) If an n-dimensional Riemannian manifold M is homogeneous, i.e. if the isometry group G of M acts transitively on M, then the isotropy group H of a point $p \in M$ (i.e. the group of all isometries leaving p fixed) is a compact subgroup of the Lie group G (see a)), and M is diffeomorphic to the manifold G/H of the left cosets of H in G.

In particular it follows from this that the diffeomorphism types of all homogeneous Riemannian manifolds whose isometry groups are isomorphic to a fixed Lie group G can be enumerated by the manifolds G/H, where H runs through a system of representatives of the conjugacy classes of all compact subgroups of G.

A type of converse is given by :

c) Let G be a connected Lie group acting differentiably and transitively on a differentiably manifold M, such that the isotropy group H of a point p ∈ M is compact. Then there always exists a Riemannian metric for M such that G acts on M by isometries of the Riemannian manifold (M,g) (see [15], vol. I, p. 154).

Corollary .- If G possesses in addition a bi-invariant Riemannian metric $\tilde{g}$, i.e. one with respect to which the inner products of left and right invariant vector fields on G are constant (this is always the case if G is compact !), then the metric g on M in c) may be canonically obtained from this bi-invariant metric $\tilde{g}$ on G and the Riemannian sectional curvature of M is always non-negative (and can be explicitly calculated from g using algebraic operations of the Lie algebra of G alone, see [15], vol. II, p. 203). The best-known Riemannian manifolds fall into this category, e.g. projective spaces and Grassmann manifolds over $\mathbb{R}$ and $\mathbb{C}$.

d) A special class of homogeneous Riemannian manifolds which are particularly well understood (and fully classified up to isometry !) are the globally symmetric spaces of E. Cartan (see [28]). These are Riemannian manifolds M for which at each point p ∈ M there exists an involutive isometry of M onto itself having p as an isolated fixed point.

Among these globally symmetric spaces, the following examples have been important in the historical development :

e) Helmholtz-Lie space forms are n-dimensional Riemannian manifolds M on which the isometry group acts transitively on the set of all orthonormal n-frames (of M).

Since the "position" of a rigid body in space can be uniquely described by giving one of its points and the position of an orthonormal n-frame fixed to the body at this point, the Helmholtz-Lie space forms are, in a more physical language, those Riemannian manifolds in which a rigid body can be moved from one position to any other position by a global isometry (of the manifold onto itself). Then (S. Lie, H. Weyl) :

An n-dimensional Helmholtz-Lie space form M ($n \geq 2$) is obviously of constant sectional curvature $C \in \mathbb{R}$ (because its isometry group also acts transitively on the bundle $G_2(TM)$ of all two-dimensional vector subspaces of tangent spaces to M), and is -more precisely- either isometric to one of the simply connected, complete Riemannian manifolds of constant curvature (see (i), b) above), or if $C > 0$, possibly isometric to the n-dimensional real projective space obtained from the n-dimensional sphere of radius $\sqrt{C^{-1}}$ in $\mathbb{E}^{n+1}$ by identification of antipodal points.

(iv) Effects of the values of the sectional curvature of a complete Riemannian manifold on its global topology or its differentiable structure.

From around 1900 on, significant results were obtained concerning the question as to how far the global topology or differentiable structure of a complete Riemannian manifold is determined by the sign (or by the existence of bounds on the values) of its sectional curvature function. The following are a representative selection of such results.

a) J. Hadamard (1898) for $n = 2$, E. Cartan (1928) for $n \geq 3$: If M is an n-dimensional ($n \geq 2$) simply connected complete Riemannian manifold of non-positive sectional curvature, then M is diffeomorphic to $\mathbb{R}^n$.

More precisely : for each point $p \in M$ the exponential map of M is a diffeomorphism from the tangent space of M at p onto M. In particular, each pair of points of M can be connected by exactly one geodesic, and this is the shortest path between them.

b) S.B. Myers (1935) : If M is an n-dimensional ($n \geq 2$) complete Riemannian manifold whose sectional curvature is at least as large as a fixed positive number $\varepsilon > 0$, then the diameter of M is at most $\pi \sqrt{\varepsilon^{-1}}$.

In particular, M is compact and the fundamental group of M is finite, as the covering group of the (analogously) compact universal covering space of M.

c) J.L. Synge (1936) : If M is an even-dimensional compact Riemannian manifold with strictly positive sectional curvature and orientable (resp. non-orientable), then M is simply connected (resp. possesses a two-sheeted universal covering space).

Since the diffeomorphism types of two-dimensional compact differentiable manifolds are completely known, it follows that :
Each compact two-dimensional Riemannian manifold of strictly positive sectional curvature is diffeomorphic either to the two-dimensional sphere S^2 or to the two-dimensional real projective plane $P^2(\mathbb{R})$.

d) Topological and differentiable sphere theorems :
Let M be a compact simply connected Riemannian manifold with strictly positive sectional curvature. Consider the pinching number of M :

$$\delta(M) = \frac{\min \varkappa^M(G_2(TM))}{\max \varkappa^M(G_2(TM))} \in \left]0,1\right] \quad , \tag{66}$$

i.e. the quotient of the minimal and maximal values of the sectional curvature function on M.

Then $\delta(M) = 1$ if and only if (see (i), b)) M is isometric to an n-dimensional Euclidean sphere in (n+1)-dimensional Euclidean space. In the following "sphere theorems" , the question under investigation is that of how much $\delta(M)$ can deviate from 1 with M still remaining homeomorphic or diffeomorphic to such a sphere ?

M. Berger (1960, [2'], for n even), W. Klingenberg (1961, for n odd) [after important first results of H. E. Rauch (1951) and preliminary results of M. Berger (1958) and W. Klingenberg (1958-1959)] : If, under the hypotheses of d) one has $\delta(M) > \frac{1}{4}$ (see (66)), then M is homeomorphic to the unit sphere S^n in $\mathbb{E}^{n+1}$.

Supplement : For the complex projective space $P^k(\mathbb{C})$ with its Fubini-Study metric, we have $\delta(P^k(\mathbb{C})) = \frac{1}{4}$, but $P^k(\mathbb{C})$ is not homeomorphic to S^{2k} for $k > 1$. Thus the above lower bound is sharp at least for n even (for more refined results, see M. Berger [2']).

E. Ruh (1973) [after important preliminary results of D. Gromoll (1965), E. Calabi (1966), Shiohama, Tsugimoto and H. Karcher (1971)] : If, under the hypotheses of (66), $\delta(M) > \frac{4}{5}$, then M is diffeomorphic to the unit sphere S^n in $\mathbb{E}^{n+1}$.

[The bound $\frac{4}{5}$ in the preceding results is presumably not sharp.]

e) The diffeomorphism types of complete, non-compact Riemannian manifolds of non-negative sectional curvature.

St. Cohn-Vossen (1935/36) : If M is a two-dimensional complete non-compact Riemannian manifold of non-negative sectional curvature, then either $\varkappa^M \equiv 0$ (and hence M is isometric to $\mathbb{E}^2$, to a cylinder, or to a Möbius band) or M is diffeomorphic to $\mathbb{R}^2$ (if $\varkappa^M \not\equiv 0$).

The following generalization is valid for dim $M > 2$:
D. Gromoll and W. Meyer (1969) : If M is an n-dimensional ($n \geq 2$) complete, non-compact Riemannian manifold with strictly positive sectional curvature, then M is diffeomorphic to $\mathbb{R}^n$ (see [10]).

The zero value is allowed for the curvature, then M need no longer to be diffeomorphic to $\mathbb{R}^n$. However, one has :
J. Cheeger and D. Gromoll (1972) : If M is an n-dimensional ($n \geq 2$) complete, non-compact Riemannian manifold with non-negative sectional curvature, then M is diffeomorphic to the total space of the normal bundle of a compact (totally geodesic, totally convex) submanifold S with boundary of M (see [5], theorem 2.2).

(v) The Gauss-Bonnet theorem and characteristic differential forms on Riemannian and Kähler manifolds, according to S.S. Chern.

Shortly after its appearance, the "Gauss-Bonnet theorem" of the "Disquisitiones generales" (see (34) above) for small geodesic triangles of a surface underwent a curious, "extrinsinc" generalization :

a) C.G. Jacobi (1837, see [13]) : Consider a "small" triangle in Euclidean space $\mathbb{E}^3$ formed by three curves without points of inflection, such that the principal normal vector fields of each pair of sides agree in their common vertex. Then the image of the principal normal fields of the three sides bounds a compact subset of S^2 whose oriented surface area [30] is equal to the sum of the angles of the triangle minus π.

b) O. Bonnet discovered the following integral theorem in 1848 (see [3]) : Let N be a two-dimensional, compact, simply connected submanifold with boundary of a two-dimensional surface M in $\mathbb{E}^3$. (M may actually be any two-dimensional Riemannian manifold.) Then the sum of the integral of the Gauss curvature K over N and of the integral of the geodesic curvature $\varkappa_g$ over the boundary ∂N of N is equal to 2π :

$$\int_N K d\sigma + \int_{\partial N} \varkappa_g \, ds = 2\pi \quad . \tag{67}$$

This theorem yields (by "rounding offcorners") the following more general result :

[30] The sign of the surface area is determined by a convention analogous to the one used in (57).

c) Let N be a simply connected open subset of a two-dimensional Riemannian manifold M, bounded by a "polygon" made of r smooth curves C_i (indices $i \in \mathbb{Z}$ mod r), such that the end point of C_i is equal to the initial point of C_{i+1} and such that the inner angle at this vertex is $\alpha_i \in [0,\pi]$. Then, with notations analogous to (67), we have

$$(68) \qquad \int_N K d\sigma + \sum_{i=1}^{r} \int_{C_i} \varkappa_g \, ds = \sum_{i=1}^{r} \alpha_i + (2-r)\pi \quad .$$

Gauss' formula (34) follows from (68) with $r = 3$, if C_1, C_2, C_3 are geodesic curves on M. On the other hand, by suitable dissection into simply connected subregions using regular curves and by balancing sums of angles, it follows that :

d) If M is a compact oriented two-dimensional surface (without boundary) in $\mathbb{E}^3$, with genus p, then

$$(69) \qquad \frac{1}{2\pi} \int_M K d\sigma = 2(1-p) = \chi(M) \left(= \text{Euler-Poincaré characteristic of } M \right) \quad .$$

Hence the total curvature of M introduced by Gauss is not only an isometry invariant, fact that Gauss knew from 1816 on and proved in the "Disquisitiones generales", but even a topological invariant of the compact orientable surface M, and in particular independent of the special differentiable imbedding of M in $\mathbb{E}^3$!

In 1925, H. Hopf (see [11]) interpreted the integral in (69) as :

$$\frac{1}{4\pi} \int_M K d\sigma = \text{Brouwer degree of the spherical Gauss map of } M$$

(see (7)) and calculated this degree for arbitrary compact oriented hypersurfaces M without boundary in $\mathbb{E}^{n+1}$ using only topological methods (and Poincaré's theorem, according to which the sum of indices of a tangential vector field on M with only finitely many zeroes is equal to $\chi(M)$), and obtained the value $\frac{1}{2}\chi(M)$. Thus (69) was proved, in a way which was to become a model for further development.

e) In 1940, C.B. Allendoerfer and W. Fenchel independently discovered an analog of (69) for an arbitrary n-dimensional ($n \geq 2$) compact, oriented Riemannian manifold M which permits an isometric embedding in an Euclidean space $\mathbb{E}^{n+k}$ (with arbitrary codimension k). The latter condition is, as we know today, according to the embedding theorem of J. Nash (1956, see [19]) not a restriction on M, but the proofs given for this theorem, which can be formulated in terms of the inner differential geometry of M alone, used extrinsic components, i.e. the explicit embedding of M in $\mathbb{E}^{n+k}$. The first proof completely within the domain of "inner differential geometry" was given by :

S.S. Chern (1944, [6]) : If M is an n-dimensional compact oriented Riemannian manifold without boundary, then there is a universal differential form Ω of n^{th} degree on M, the so-called Euler form, which can be calculated from the Riemannian curvature tensor of M alone, and for which the general "Gauss-Bonnet theorem" holds :

$$\int_M \Omega = \chi(M) \quad . \tag{70}$$

Remark .- S.S. Chern uses a differentiable unit vector field E on $M \setminus \{p\}$ in his proof. That is, the vector field E has only one singularity $p \in M$, and it defines in a canonical way a continuous map f from the (n-1)-dimensional sphere of all unit vectors on M at p onto itself, whose degree on the one hand according to Poincaré-Hopf is equal to the Euler characteristic $\chi(M)$. Chern cannot show (because the "extrinsic" Gauss map is not available !) that this degree is equal on the other hand to the integral on the left hand side in (70) as Hopf did, but instead he proves this by a completely new geometric integration (using Kronecker's integral formula).

In this connection it must be emphasized that the "intrisic" proof of (70) given by S.S. Chern in [6] opened a new horizon for "inner differential geometry". Up until then, "inner differential geometry" was understood to be the geometry of the Riemannian manifold M which only operated "in M". Here this is no longer the case in a strict sense. The proof of S.S. Chern involves in addition to M other manifolds and differential forms on them, as well as differentiable maps from M (or embeddings of $M \setminus \{p\}$ with $p \in M$) into them. However, these manifolds "grow naturally" from M; for example, the unit tangent bundle of M or the bundle of orthonormal n-frames and the corresponding differential forms arise in a natural way from the Riemannian manifold M alone (as H. Weyl paraphrased this "like a snail builds its own house by itself"). These higher dimensional intrinsically constructed bundle manifolds "over M" play the role of a "substitute" for the missing Euclidean space in S.S. Chern's proof (as can be seen by comparing it with H. Hopf's proof for (69) quoted above), an idea which has contributed considerably to the development of the methods and reasoning of recent "inner differential geometry".

f) Because of (70) and (69), a Riemannian metric on a compact oriented differentiable manifold cannot have a completely "arbitrary" sectional curvature function. For example, according to (69), each Riemannian metric on the two-dimensional sphere S^2 must have a Gaussian curvature function that takes on some positive values (so that the integral in (69) can attain the value $\chi(S^2) = 2 > 0$). The question as to whether or when a given function K on a two-dimensional differentiable manifold M can be realized as the Gaussian curvature of a Riemannian metric for M (and analogously for higher dimensions), has been recently treated with success, in particular by J. Kazdan and F. Warner (1973/1974), and H. Gluck (1971/1974). For this, see the survey article [8] with its extensive bibliography.

g) For a compact, oriented surface M in $\mathbb{E}^3$, the intrinsic Euler form Ω (whose integral over M gave us a topological invariant for M, see (70)), as comparison with (69) and (58) shows, was just (up to a universal constant factor) the pull back under the Gauss map $\zeta : M \to S^2$ (see (7)) of the surface area form σ_2 of S^2 which is a closed differential form of degree two.

An analogous extrinsic construction leads, in the case of higher dimensional Riemannian manifolds, also to new intrinsic forms, i.e. to the so-called "characteristic differential forms of M".

Let M be an n-dimensional compact oriented Riemannian manifold which has an isometric embedding $f : M \to \mathbb{E}^{n+k}$ (because of the well-known embedding theorem of J. Nash (1956, see [19]), such an embedding always exists for $k \geq \frac{1}{2}(n+1)\, n(3n+11)$). According to H. Whitney, we have the following generalization of the Gauss map, called the Whitney map W by parallel tangents, which assigns to each $p \in M$ the n-dimensional vector subspace W(p) of $\mathbb{E}^{n+k}$ obtained by parallel translation of the image (under the embedding f) of the tangent space to M at p to the origin of $\mathbb{E}^{n+k}$. Hence W is a differentiable map from M into the Grassmann manifold of all n-dimensional vector subspaces of $\mathbb{E}^{n+k}$, which is canonically a homogeneous Riemannian manifold ($= \mathbf{O}(n+k)/\mathbf{O}(n) \times \mathbf{O}(k)$) on which there exist certain standard closed differential 4ℓ-forms. If one pulls these forms back to M through W, then one obtains the so-called "characteristic differential forms of M" or Pontrjagin forms of M. Similarly these do not depend on the special isometric embedding f of M is a Euclidean, and can be calculated from the (intrinsic) Riemannian curvature tensor alone. Their de Rham cohomology classes (i.e. the so-called Pontrjagin classes of M) are even independent of the Riemannian metric of M and turn out to be invariants of the underlying differentiable manifold of M. For n-dimensional (complex) Kähler manifolds M, the analogous so-called "Chern forms", which are closed differential forms of degrees 2,4,...,2n, can be again intrinsically constructed using only the Riemannian curvature tensor of M. Their de Rham cohomology classes are integral valued (like the Pontryagin classes) and are invariants for the underlying holomorphic structure of M. These characteristic cohomology classes of Riemannian and Kähler manifolds, calculated by differential geometric methods, have become an important instrument linking topology (resp. differential topology) with differential geometry.

(vi) Angle comparison theorem.

Already in the "Disquisitiones generales", Gauss gave (see (35) above) an infinitesimal angle comparison theorem for small geodesic triangles on a surface in $\mathbb{E}^3$ ("compared" with a planar Euclidean triangle the corresponding sides of which are of the same length).

Global angle comparison theorems between geodesic triangles on Riemannian manifolds, whose sectional curvature values are bounded from above or below by fixed values, have proved to be fundamental and very powerful tools in global differential geometry. We quote here only a simple, but particularly impressive theorem from the many theorems of this type, developed principally by A.D. Alexandroff and W.A. Toponogoff (for more subtle versions, see e.g. [5], theorem 1.1 or [14]).

W.A. Toponogoff (1958) : Let M be an n-dimensional ($n \geq 2$) complete Riemannian manifold whose sectional curvature function κ^M takes on only values $\geq C$ ($\in \mathbb{R}$). Denote by M_C^2 "the" two-dimensional simply connected complete Riemannian manifold of constant sectional curvature C (see (i), b)), which will be used as the "level of comparison". Let Δ be a triangle whose sides are shortest paths in M, with side lengths a, b, c and (opposite) angles α, β, γ. Then there exists a geodesic triangle Δ_C in M_C^2 with the same side lengths as Δ, and whose angles α_C, β_C, γ_C permit the following "comparison" with those of Δ :

$$\alpha_C \leq \alpha \quad , \quad \beta_C \leq \beta \quad , \quad \gamma_C \leq \gamma \quad .$$

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There is no doubt that this survey is a very incomplete selection (and at times possibly appearing quite arbitrary) from the entire collection of results in differential geometry during the last 150 years (and in particular, it was not meant to contain any judgment concerning the relative importance of the results). Highly interesting questions on geodesics (their shortest path properties both locally and globally, conjugate points, cut points, ..., existence of closed geodesics), questions on minimal surfaces (the Plateau problem and regularity questions around it, Berstein's theorem and higher-dimensional analogues,...), more generally questions concerning the so-called "harmonic maps" of Riemannian manifolds, and finally all questions about the shape, rigidity or flexibility of submanifolds in Riemannian manifolds of constant curvature, all of these questions had to be left aside (in the lecture) because of lack of time, although the results in these areas are not only numerous but they are even, in their contents, extremely impressive !

However, the goal of the selection was to present fundamental "classical", and at the same time, "live" themes of differential geometry, in order to show (even to a non-specialist in the field , if possible) on the one hand how deeply differential geometry still depends on the concepts and results, respectively is still involved with themes, first presented by Gauss in his "Disquisitiones generales", and on the other hand how far the growth and the development of these ideas (which partly were only intimated by Gauss) has been carried. When the full breadth and the lasting influence of the "Disquisitiones generales" on the growth of differential geometry became clear to me (for the first time on the occasion of the preparation of this lecture !), I was reminded of the short address delivered by Heinz Hopf, at the International Congress of Mathematicians of 1958 in Edinburgh, as President of the Fields Medal Committee at that time. Among other things, he said (see [12], p. liii, line 2 from below) :

"The great variety within mathematics is due not only to the multiplicity of the branches of mathematics, but also to the diversity of the general tasks that face a mathematician in any branch. A task which is particularly fundamental is : to solve old problems ; and another, no less fundamental, is :

to open the way to new developments."

Few mathematical papers have fulfilled this latter task to the extent that the "Disquitiones generales circa superficies curvas" by Carl Friedrich Gauss have, and it is above all for this reason, in my opinion, that this work shall continue to be a "showpiece" of mathematical literature !

* * *

LITERATURE

In this manuscript, the several-volume collection

"Carl Friedrich Gauss, Werke",

edited by the "Gesellschaft der Wissenschaften zu Göttingen" and published by B.G. Teubner (Leipzig) and J. Springer (Berlin), is abbreviated to "G.W." (followed by the corresponding volume number).

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