

# *Astérisque*

DAVID W. BOYD

**Pisot sequences, Pisot numbers and Salem numbers**

*Astérisque*, tome 61 (1979), p. 35-42

[http://www.numdam.org/item?id=AST\\_1979\\_\\_61\\_\\_35\\_0](http://www.numdam.org/item?id=AST_1979__61__35_0)

© Société mathématique de France, 1979, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

PISOT SEQUENCES, PISOT NUMBERS AND SALEM NUMBERS

by

DAVID W. BOYD

1. The sets S and H: The well known set  $S$  of Pisot (or Pisot-Vijayaraghavan) numbers is the set of algebraic integers  $\theta > 1$  all of whose other conjugates lie strictly within the unit circle. The initial interest in  $S$  stems from the fact that if  $\lambda \in \mathbf{Z}(\theta)$ , then  $\|\lambda\theta^n\| = \text{dist}(\lambda\theta^n, \mathbf{Z}) \rightarrow 0$  as  $n \rightarrow \infty$ . Let us denote by  $H$  the set of real  $\theta > 1$  for which there is a  $\lambda > 0$  such that  $\|\lambda\theta^n\| \rightarrow 0$ . A still unanswered question is whether  $S = H$ . This was considered by Thue [16] and Hardy [17], who showed that if  $\|\lambda\theta^n\| = O(b^n)$  with  $b < 1$ , then  $\theta \in S$ . Hardy also pointed out that the only algebraic elements in  $H$  are the elements of  $S$ . Generalizations of this result were given by Vijayaraghavan in [17].

Until recently Pisot's result [13], that  $\sum \|\lambda\theta^n\|^2 < \infty$  implies  $\theta \in S$  was essentially the closest approach to a proof that  $S = H$ , but Cantor [7] has recently given a substantial improvement of this which is somewhat technical to describe here. Salem [14] used Pisot's result to prove that the set  $S$  is closed and hence is nowhere dense in  $[1, \infty)$ .

An interesting fact about  $H$  is that it is a countable set. Thus, if  $H$  contains any transcendental numbers then it does not do so for trivial reasons. We will see Pisot's [13] proof that  $H$  is countable in what follows. It should be mentioned that Vijayaraghavan [18] proved that the set of  $\theta$  for which  $\|\theta^n\| \rightarrow 0$  is countable by a somewhat different method.

2. E-sequences: Pisot's method of proof is to examine a certain interesting class of sequences of integers, now called E-sequences or Pisot sequences. To see

how these arise, suppose that  $a_n = \lambda\theta^n + \epsilon_n$ , where  $\lambda > 0$ ,  $\theta > 1$ ,  $a_n \in \mathbf{Z}$  and  $\epsilon_n$  is bounded. We observe that

$$a_{n+1}a_{n-1} - a_n^2 = \lambda\theta^{n-1}(\theta^2\epsilon_{n-1} - 2\theta\epsilon_n + \epsilon_{n+1}) + (\epsilon_{n+1}\epsilon_{n-1} - \epsilon_n^2),$$

so that

$$\limsup |a_{n+1} - a_n^2/a_{n-1}| = \limsup |\theta^2\epsilon_{n-1} - 2\theta\epsilon_n + \epsilon_{n+1}| = \delta, \text{ say.}$$

If  $\delta < 1/2$ , then eventually  $a_{n+1}$  is determined uniquely by  $a_n$  and  $a_{n-1}$ .

By deleting some initial terms, we have that

$$(1) \quad a_{n+1} = N(a_n^2/a_{n-1}), \quad n = 0, 1, \dots,$$

where  $N(x) = [x + 1/2]$  = "the nearest integer to  $x$ ". The formula (1) defines the E-sequence  $E(a_0, a_1)$  for arbitrary integers  $0 < a_0 < a_1$ . Pisot showed that the limit  $a_{n+1}/a_n \rightarrow \theta$  always exists, and this defines a certain set  $E$ . Clearly  $E$  is countable and contains  $H$ , ( $\delta = 0$ ), so  $H$  is countable. On the other hand  $E$  is dense in  $[1, \infty)$  so  $E \neq S$ , since  $S$  is nowhere dense by Salem's result.

One can show that  $\lambda = \lim a_n/\theta^n$  exists if  $\theta > 1$ , and if one defines  $\epsilon_n = a_n - \lambda\theta^n$ , then the above discussion shows that  $E$  is essentially characterized by the inequality

$$(2) \quad \limsup |\theta^2\epsilon_{n-1} - 2\theta\epsilon_n + \epsilon_{n+1}| \leq 1/2,$$

in the sense that (2) is necessary for  $a_n$  to be an E-sequence, while (2) with strict inequality is sufficient for  $\{a_{n-n_0}\}$ ,  $n \geq n_0$  to be an E-sequence for some  $n_0$ .

In addition to the set  $S$ ,  $E$  also contains the set  $T$  of Salem numbers which are real algebraic integers  $\theta > 1$  such that all other conjugates lie within the unit circle, with at least one conjugate on the circle. This in fact implies that  $\theta$  satisfies a reciprocal equation, so its conjugates are  $\theta^{-1}$  and a certain set of numbers of modulus one [15]. To see that  $E \supset T$ , just choose  $\lambda \in \mathbf{Z}(\theta)$  so that the other conjugates of  $\lambda$  are small enough so that (2) holds.

3. Recurrent E-sequences: The interesting question now is whether  $E = S \cup T$ , since this would tell us that  $T$  is dense in  $[1, \infty)$  and hence that  $\inf T = 1$ , settling Lehmer's conjecture [12]. It would also imply that  $H = S$ , settling Pisot's conjecture.

One notes that the proof that  $E \supset S \cup T$  shows somewhat more, namely that the corresponding E-sequence satisfies a linear recurrence relation, or equivalently that the generating function of the sequence is rational, so

$$(3) \quad \sum_{n=0}^{\infty} a_n z^n = A(z)/Q(z) \quad ,$$

where  $A$  and  $Q$  are polynomials with integer coefficients, and  $Q(0) = 1$ . In [9], Flor shows that if  $E(a_0, a_1)$  satisfies (3), then  $\theta$  is in  $S$  or in  $T$ . We shall refer to these two possibilities as S-recurrence and T-recurrence.

In fact, in [13], Pisot already showed that  $E(2, a_1)$  and  $E(3, a_1)$  are S-recurrent with  $\deg(Q) \leq a_0$ . For example  $E(3, 5) = 3, 5, 8, 13, 21, \dots$  has degree 2. His proof distinguishes  $E(a_0, a_1)$  according to the congruence class  $a_1 \pmod{a_0^2}$ . Cantor [6] has given the explanation of why this is natural, and has studied the families  $E(a_0, ma_0^2 + b)$ , giving conditions on  $a_0$  and  $b$  in order that this sequence is S-recurrent for all  $m \geq m_0$ . The corresponding generating function is of the form  $A(z)/(Q(z) - mzA(z))$ .

However, Cantor and his student Galyean [5], by use of a computer algorithm designed for testing for linear recurrences showed that if  $E(4, 13)$  is recurrent, then  $\deg(Q) \geq 100$ , suggesting strongly that no such recurrence exists. In his thesis [10], Galyean found many examples of  $E(a_0, a_1)$  satisfying no recurrence of degree  $\leq 20$ , when  $4 \leq a_0 \leq 10$ .

4. Non-recurrent E-sequences: I was aware only of the example  $E(4, 13)$  when I proved [1] that indeed there are E-sequences which are non-recurrent, and in fact that the set of  $\theta$  produced from such sequences is dense in  $[(\sqrt{5} + 1)/2, \infty)$ .

The proof is rather amusing since it concentrates its attention on T-recurrence,

which one might expect to be the difficult case. The point is that, although we have very little quantitative information about T itself, T-recurrent sequences are so distinctive that non-T-recurrence is rather easily detected. In principle, S-recurrence causes no difficulty since one can work in intervals disjoint from S. However, as we shall see later, for specific E-sequences, S-recurrence is more difficult to handle because the intervals in the complement of S are extremely short for even moderately large  $\theta$ .

To see how T-recurrence is dealt with, suppose then that  $E(a_0, a_1)$  is T-recurrent, then, taking into account the structure of the conjugates of  $\theta$ ,

$$(4) \quad a_n = \lambda\theta^n + \mu\theta^{-n} + \delta_n, \quad n \geq n_0,$$

where  $\delta_n$  is a linear combination of powers of numbers of modulus 1 and hence is almost periodic. Using (2) and the almost periodicity of  $\delta_n$ , we find that, for all  $n$ , including negative  $n$ ,

$$(5) \quad |\theta^2\delta_{n-1} - 2\theta\delta_n + \delta_{n+1}| \leq 1/2.$$

Furthermore, (4) can be used to define  $a_n$  for  $n < n_0$ , and since  $Q$  is reciprocal or antireciprocal, one finds that  $a_n$  is an integer for all  $n$ . Combining these two facts one then obtains a constructive estimate for  $n_0$  (and this is where the condition  $\theta > (\sqrt{5} + 1)/2$  seems unavoidable). For example, if  $\theta > 2$  then  $n_0 = 0$ . Assuming then that  $n_0 = 0$  (by shifting the sequence if necessary), the conditions that  $a_n$  be an integer for  $n < 0$ , combined with (5), produce various inequalities which must be satisfied by T-recurrent sequences. As a simple example, the condition that  $a_{-1}$  is an integer implies that

$$\|a_0^2/a_1\| \leq (1 + 2\theta)/(2\theta^2) + 1/a_1.$$

This is an extremely restrictive condition for large  $\theta$ , and shows that non-recurrent E-sequences produce a set of  $\theta$  dense in  $[1 + \sqrt{2}, \infty)$ .

Applying this inequality to a family  $E(a_0, ma_0^2 + b)$  with  $b > 0$ , we find that  $\|a_0^2/a_1\| = a_0^2/a_1$  if  $m \geq 2$ , while on the other hand  $(1+2\theta)/(2\theta^2) + 1/a_1$

is approximately  $(a_0 + 1)/a_1$ . Thus, within such a family, T-recurrence can only occur if  $m = 0$  or  $1$ . As another example consider  $E(2m, 7m)$  with  $m \equiv 1 \pmod{7}$ . Then  $\|a_0^2/a_1\| = 3/7$  while  $(1 + 2\theta)/(2\theta^2) \rightarrow 16/49 < 3/7$  as  $m \rightarrow \infty$ . Thus, for sufficiently large  $m$ , none of these sequences is T-recurrent. Since  $\theta \rightarrow 7/2 \notin S$  as  $m \rightarrow \infty$ , and since  $S$  is closed, it follows that  $\theta \notin S$  for sufficiently large  $m$ , so  $E(a_0, a_1)$  is not S-recurrent either.

5. Specific cases of non-recurrence: In spite of the ease of producing infinitely many non-recurrent E-sequences, one would still like to be able to answer the question of whether any specific  $E(a_0, a_1)$  is recurrent or not. In his thesis [10], Galyean conjectured that if  $E(a_0, a_1)$  is recurrent, then the degree of the recurrence is at most  $a_0$ . A proof of this would certainly provide the desired criterion. A result of this type seems reasonable when one considers that, in an E-sequence  $\lambda \approx a_0$ , and in order to make  $\epsilon_n$  small enough for (2) to hold, it seems necessary to have the other conjugates of  $\lambda$  small. This in turn forces  $\lambda$  to be fairly large since the product of these numbers is at least as large as  $1/\text{disc}(\theta)$ .

However, lacking such a quantitative result, we have based our proofs of non-recurrence for specific E-sequences on a different method. Proofs of non-T-recurrence are based on refinements of the ideas discussed above. It seems likely that the infinite set of necessary conditions for T-recurrence so obtained are also sufficient; this has certainly proved to be the case in practice. To prove non-S-recurrence we simply have to show that  $\theta \notin S$ , a constructively feasible procedure since  $S$  is closed and since we can generate arbitrarily good approximations to  $\theta$ . The practical difficulties grow with  $\theta$  so our success with this method is confined to  $\theta < 2.5$ . The main tool is a computer algorithm based on ideas of Dufresnoy and Pisot [8] and described in more detail in [3]. It is capable of finding all the elements in  $S \cap (\alpha, \beta)$ , provided this number is finite. The idea is that, if  $P$  is the minimal polynomial of  $\theta$ , and  $Q(z) = z^{\text{deg}(P)} P(z^{-1})$ , then

$$(6) \quad f(z) = (\text{sgn } P(0))P(z)/Q(z) = u_0 + u_1z + \dots ,$$

where the  $u_n$  are integers and where  $|f(z)| = 1$  on  $|z| = 1$ . The  $u_n$  are characterized by inequalities obtained from Schur's algorithm:

$$(7) \quad w_n(u_0, \dots, u_{n-1}) \leq u_n \leq w_n^*(u_0, \dots, u_{n-1}) .$$

If in addition  $\alpha \leq \theta \leq \beta$ , then there are additional inequalities

$$(8) \quad v_n(u_0, \dots, u_{n-1}; \alpha) \leq u_n \leq v_n^*(u_0, \dots, u_{n-1}; \beta) .$$

These lead to the search of a finite tree if  $S \cap (\alpha, \beta)$  is a finite set.

An instructive example is the sequence  $E(10,22)$ , with  $\theta = 2.190327956\dots$ . The criteria for T-recurrence are easily shown to be violated. A search of a small interval containing  $\theta$  shows that  $\text{dist}(\theta, S) = .905 \times 10^{-8}$ , the closest point of  $S$  being a root of the following 32nd degree polynomial:

$P = 1 - 2 0 0 - 1 - 2 0 0 - 2 0 0 0 - 1 2 0 0 1 2 0 0 2 0 0 0 1 - 2 0 0 - 1 - 1 0 0 - 1$   
 (notation:  $a b c \dots$  means  $ax^k + bx^{k-1} + \dots$ ). Thus  $E(10,22)$  is non-recurrent. From Galyean's thesis, we find that  $E(10,22)$  is predicted to  $a_{21}$  by the generating function  $(10 + 2z + 4z^2 + 9z^3)/(1 - 2z - 2z^4)$ . However the polynomial  $z^4 - 2z^3 - 2$ , in addition to a root  $\phi = 2.190327947$ , has roots  $\gamma, \bar{\gamma}$  with  $|\gamma| \approx 1.0157$ . Hence this is not the generating function of an E-sequence. The fact that  $|\gamma|^{44} < 2$  makes it clear how this sequence can masquerade as an E-sequence for many terms. Intuitively, it appears that  $E(10,22)$  is diverted away from nearby S-numbers of small degree by the presence of this "pseudo"-S-number of degree 4. Since  $a_0 = 10$  is apparently too small to allow  $E(10,22)$  to satisfy a recurrence of high degree, the sequence is unable to satisfy any recurrence whatsoever.

An extremely interesting example of this type, mentioned in [5], is  $E(6,16)$  which is connected with the polynomial  $P(z) = z^5 - 3z^4 + z^3 - z - 1$ , which has roots at  $\phi = 2.699\dots$  and  $\gamma, \bar{\gamma}$  with  $|\gamma| \approx 1.007$ . This polynomial turns out to be a limit point of polynomials with the same properties. Since  $\text{dist}(\phi, S) < 10^{-46}$ , we have as yet been unable to show  $E(6,16)$  is not S-recurrent.

## PISOT NUMBERS

There are in addition many other examples of non-recurrence which are not explainable by this mechanism. For example, the non-recurrence of  $E(7,15)$  seems to be explained by our arbitrary choice of "rounding up" in the definition of  $N(x)$ . For details of this and other examples, the reader may consult [3].

6. Concluding Remarks: Space has not permitted a discussion of the new characterization of  $T$  given in [2], nor the application of the above-mentioned computer algorithm to questions concerning the distribution of  $T$  in the real line, but this is adequately described in [3].

As far as applications of E-sequences to finding T-numbers, as suggested in [5], it seems that a more fruitful type of sequence to use is given by the following non-linear recurrence:

$$a_{n+2} = N(a_{n+1}(a_{n+1} + a_{n-1})/a_n - a_n) \quad , \quad n = 1, 2, \dots$$

If one takes  $a_0 = 0$ ,  $a_1 > 0$  and  $a_2 \geq 2a_1 + 1$ , then one obtains all Salem numbers as limits of the ratios  $a_{n+1}/a_n$ . The criterion for T-recurrence is now valid for all  $\theta > 1$ , because the inequality (5) is replaced by a more amenable form. Some details concerning these sequences are to be found in [4].

## REFERENCES

1. D.W. Boyd, Pisot sequences which satisfy no linear recurrence, *Acta Arith.* 32(1977), pp.89-98.
2. \_\_\_\_\_, Small Salem numbers, *Duke Math. Jour.* 44(1977), pp.315-328.
3. \_\_\_\_\_, Pisot and Salem numbers in intervals of the real line, *Math.of Comp.* (to appear in 1978).
4. \_\_\_\_\_, Some integer sequences related to Pisot sequences, *Acta Arith.* (to appear).
5. D.G. Cantor, Investigation of T-numbers and E-sequences, in *Computers in Number Theory*, ed A.O.L. Atkins and B.J. Birch, Academic Press, N.Y. 1971.



6. \_\_\_\_\_, On families of Pisot E-sequences, Ann.Sci.Éc.Norm.Sup. 4<sup>e</sup> Série, 9(1976),pp.283-308.
7. \_\_\_\_\_, On power series with only finitely many coefficients (mod 1): solution of a problem of Pisot and Salem, Acta Arith. 34(1977),pp.43-55.
8. J. Dufresnoy and Ch. Pisot, Étude de certaines fonctions méromorphes bornées sur le cercle unité, application à un ensemble fermé d'entiers algébriques, Ann.Sci.Éc.Norm.Sup. 3<sup>e</sup> Série, 72(1955),pp.69-92.
9. P. Flor, Über eine Klasse von Folgen natürlicher Zahlen, Math. Annalen 140 (1960),pp.299-307.
10. P. Galyean, On linear recurrence relations for E-sequences, Thesis, University of California Los Angeles, 1971.
11. G.H. Hardy, A problem of diophantine approximation, Jour.Ind.Math.Soc. 11 (1919),162-166; Collected works I, pp.124-129.
12. D.H. Lehmer, Factorization of certain cyclotomic functions, Ann.Math. 34(1933), pp.461-479.
13. Ch. Pisot, La repartition modulo 1 et les nombres algébriques, Ann. Scuola Norm.Sup. Pisa 7(1938),205-248.
14. R. Salem, A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan, Duke Math. Jour. 11(1944),pp.103-107.
15. \_\_\_\_\_, Power series with integral coefficients, Duke Math. Jour. 12(1945), pp.153-171.
16. A. Thue, Über eine Eigenschaft die keine transzendente Größe haben kann, Skrifter Vidensk.I. Kristiania 2(1912), No.20, pp.1-15.
17. T. Vijayaraghavan, On the fractional parts of the powers of a number (II), Proc. Camb. Phil. Soc. 37(1941),pp.349-357.
18. \_\_\_\_\_, On the fractional parts of the powers of a number (III), Jour. Lond. Math. Soc. 17(1942), pp.137-138.

David W. Boyd  
Department of Mathematics  
University of British Columbia  
Vancouver, B.C., Canada  
V6T 1W5