WILLIAM A. VEECH

Ergodic theory and uniform distribution

Astérisque, tome 61 (1979), p. 223-234
<http://www.numdam.org/item?id=AST_1979__61__223_0>

Société Mathématique de France
Astérisque 61 (1979) p. 223-234

Ergodic Theory and Uniform Distribution

by
William A. Veech*

1. Introduction. We shall discuss the applications of ergodic theory to two problems in the theory of uniform distribution. The first problem concerns uniform distribution in a general compact group, the second uniform distribution modulo 1.

If K is a compact (Hausdorff, topological) group, a sequence $S = \{s_n\}$ in K is a K-sequence if S generates a dense subgroup of K. S is a K_σ-sequence if it has the additional properties that (i) for every $n > 0$ $(s_1, \ldots, s_n) = (s_{k+1}, \ldots, s_{k+n})$ for infinitely many k, and (ii) $S^{-1}S = \{s_i^{-1}s_j\}$ generates a dense subgroup of K. Any K-sequence may be used to construct a K_σ-sequence.

We recall that a sequence $R = \{r_n\}$ is called a uniformly (resp. well) distributed sequence generator, u.d.s.g. (resp. w.d.s.g.), if for every compact group K and every K-sequence $S \subseteq K$, the sequence $T(R,S) = \{t_n\}$, where

$$t_n = \prod_{j=1}^{n} s_{r_j} \tag{1.1}$$

is uniformly (resp. well) distributed in K ([13], [15], [17]).

Examples of u.d.s.g.'s are given in [13], [15]. One such is $r_1 = 9$, $r_2 = 2$, and in general r_n = the length of the gap between the n^{th} and $(n+1)^{st}$ '1' in the sequence 123456789101112... .

At the present time one knows no example of a w.d.s.g. . However, Losert and Rindler [8] have proved there exist sequences $R \subseteq \mathbb{Z}$ which satisfy a similar condition which we shall not describe

*Research supported by NSF - MCS 78-01858

here . Any Losert-Rindler sequence serves as a "program" (like (1.1)) for writing down a well distributed sequence in terms of a given K-sequence. This is the purpose for which the notion of a w.d.s.g. was introduced, and the Losert-Rindler result suffers only an aesthetic defect of being nonexplicit.

In preparation of the statement of the first theorm, let $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ be a sequence of integers such that $\lambda_n \geq 2$. Also, set $\lambda_0 = 1$. For every $k \in \mathbf{Z}$ such that $k \neq -1$ there is a unique integer $\tau = \tau(k) \geq 0$ such that

$$(1.2) \qquad k+1 = \lambda_0 \lambda_1 \cdots \lambda_\tau (a\lambda_{\tau+1} + b)$$

with $a \in \mathbf{Z}$ and $0 < b < \lambda_{\tau+1}$.

Notice in the theorem to follow that the K_σ-sequence begins at 0 (the definition is analogous).

1.3 Theorem. With notations as above, assume the sequence λ is bounded, and define $R = \{\tau(1), \tau(2), \ldots\}$. If K is a compact group, and if $S = \{s_0, s_1, \ldots\}$ is a K_σ-sequence in K, then T(R,S) (see (1.1)) is well distributed in K.

Next, let $X = \mathbb{R}/\mathbf{Z}$, and let $\theta \in X$ be an irrational. Given an "interval" $I \subseteq X$. whose length is denoted $|I|$, define $S_n(x) = S_n(x, \theta, I)$, $x \in X$, $n > 0$, to be the number of j such that $0 \leq j < n$ and $x + j\theta \in I$.

A theorem of Kesten [7] asserts that there exists $x \in X$ such that $S_n(x) - n|I|$ is bounded (in n) only if $|I| \in \mathbf{Z}\theta$ modulo 1. (The converse is easy and classical.) A simple proof of Kesten's theorem is given by Furstenberg-Keynes -Shapiro [6] (see also [17]). The following is a sharpening of Kesten's theorem:

1.4 Theorem. With notations as above, suppose there exist $x \in X$ and $M < \infty$ such that

$$E_M(x) = \{n \mid |S_n(x) - n|I|| \le M\} \tag{1.5}$$

has positive upper density. Then modulo 1, $|I| \in \mathbb{Z}\theta$.

2. Monothetic groups. In this section X denotes an infinite compact monothetic group and $\theta \in X$ an element which generates a dense subgroup. X will be written additively. Let μ be normalized Haar measure on X.

Fix a finite set $E \subseteq X$ such that E contains a coset of no subgroup of X other than $\{0\}$. Let K be a compact group, and let there be given a continuous map $\varphi: E^c \to K$ such that φ does not extend to be continuous on X.

Define $X' = E + \mathbb{Z}\theta$, and define a map $X' \to K^{\mathbb{Z}}$ by $m_x(n) = \varphi(x+n\theta)$, $x \in X'$, $n \in \mathbb{Z}$. The closure, M, of the image of X' is invariant under the left shift, $\sigma(\sigma m(n) = m(n+1))$. In addition one has from [16], Section 2, that (a) (σ, M) is minimal (every σ-orbit in M is dense in M), (b) (σ, M) is uniquely ergodic (there is a unique normalized σ-invariant Borel measure on M), and (c) the map $\pi m_x = x$, $x \in X'$, is well defined and extends to a continuous map $M \xrightarrow{\pi} X$ such that $\pi\sigma m = \pi m + \theta$, $m \in M$; moreover, π is one-to-one on $\pi^{-1}X'$. Because of (b) and (c), we shall write μ also for the normalized invariant measure on M.

Next, let $N = M \times K$, and define $T: N \to N$ by

$$T(m,k) = (\sigma m, m(0)k) \quad . \tag{2.1}$$

Let ν be normalized Haar measure on K, and set $\omega = \mu \times \nu$. Clearly, ω is T-invariant.

If (T,N) is uniquely ergodic, a theorem of Oxtoby [9] implies that for each $z \in N$ the sequence $\{T^n z, n \geq 1\}$ is ω-well distributed in N. In particular, the sequence of "second coordinates" is well distributed in K. When $z = (m_x, e)$, $x \in X'$, the second coordinate of $T^n z$, $n > 0$, is

$$(2.2) \qquad \varphi^{(n)}(x) = \varphi(x+(n-1)\theta)\varphi(x+(n-2)\theta)\cdots\varphi(x) \ .$$

It is Furstenberg's observation that (T,N) is uniquely ergodic if ω is ergodic for T (if $A \subseteq N$ is measurable, and if $T^{-1}A = A$, then $\omega(A) = 0$ or $\omega(A^c) = 0$)([5]). The necessary and sufficient condition that ω fail to be ergodic is that there exist a nontrivial continuous irreducible unitary representation $\rho: K \to U(d)$ and a nonconstant measurable function $F: X \to \mathbb{C}^d$ such that

$$(2.3) \qquad F(x+\theta) = \rho(\varphi(x))F(x) \quad (\text{a.e. } \mu) \ .$$

(See [5], [14].)

3. Proof of Theorem 1.3. Let λ be as in the introduction, and define $\Lambda_0 = 0$ and $\Lambda_n = \lambda_1\lambda_2\cdots\lambda_n$, $n > 0$. We set $X = \varprojlim_n \mathbb{Z}/\Lambda_n\mathbb{Z}$ and view X as the set of sequences, $x = (x_1, x_2, \ldots)$, such that $0 \leq x_n = x_n(x) < \Lambda_n$ and $x_{n+1} - x_n \in \Lambda_n\mathbb{Z}$ for all $n > 0$. Letting $\theta = (1,1,\ldots)$, the subgroup $\mathbb{Z}\theta$ is dense in X. μ denotes normalized Haar measure on X.

Let $E = \{-\theta\}$. If $x \notin E$, define $\tau(x) = \ell - 1$, where ℓ is the least integer such that $x_\ell \neq \Lambda_\ell - 1$. $\tau(\cdot)$ is continuous on E^c, and $\lim_{x \to -\theta} \tau(x) = \infty$. In terms of the function $\tau(k)$, $k \neq -1$, defined in (1.2), one has (a) $\tau(k\theta) = \tau(k)$, $k \neq -1$, and (b) $\tau(x) = \tau(x_n(x))$ for any n such that $x_n(x) \neq \Lambda_n - 1$.

Define partitions $\mathcal{P}_n = \{P_{nk} | 0 \le k < \Lambda_n\}$ by setting $P_{nk} = \{x | x_n(x) = k\}$. The function $T_n(x) = \Lambda_n - 1 - x_n(x)$ assumes the constant value $\Lambda_n - 1 - k$ on P_{nk} for each k. Remark (b) of the preceding paragraph implies $\tau(x+j\theta)$ is constant on P_{nk} if $j \neq \Lambda_n - 1 - k$. As for the exceptional value of j, define $P_{nk}^{\ell} = \{x \in P_{nk} | \tau(x+(\Lambda_n - 1 - k)\theta) = n+\ell\}$, $\ell \ge 0$. An easy counting argument shows $\mu(P_{nk}^{\ell}) = (\lambda_{n+\ell} - 1)\frac{\Lambda_{n-1}}{\Lambda_{n+\ell}} \mu(P_{nk})$ holds for $\ell \ge 0$. If in particular λ is bounded (by Q), the last inequality implies

$$(3.1) \qquad \mu(P_{nk}^{\ell}) \ge Q^{-(\ell+1)} \mu(P_{nk}) \quad .$$

If $x \in X$, write $P_n = P_n(x)$ for the element of $\mathcal{P}$ which contains x. Given an $L^1(\mu)$ function $F: X \to \mathbb{C}^d$, the martingale theorem, together with a standard argument, shows

$$(3.2) \qquad \lim_{n\to\infty} \frac{1}{\mu(P_n)} \int_{P_n} |F(y) - F(x)| \mu(dy) = 0$$

Next, suppose $K \neq \{e\}$ is a compact group, and let $S = \{\psi(0), \psi(1), \dots\}$ be a K_σ-sequence in K. Using τ and S, we define $\varphi(x) = \psi(\tau(x))$, $x \in E^c$. The facts $K \neq \{e\}$ and S is a K_σ-sequence easily imply φ has no limit at $-\theta$. We shall be interested in $\varphi^{(\Lambda_n)}$ which we denote by φ_n. Our earlier discussion implies there exist $A_{nk}, B_{nk} \in K$, $0 \le k \le \Lambda_n$, such that

$$(3.3) \qquad \varphi_n(x) = A_{nk} \psi(n+\ell) B_{nk} \qquad (x \in P_{nk}^{\ell}) \ .$$

Indeed, of the Λ_n factors determining φ_n, all but one are constant on P_{nk}, and that factor is constantly $\psi(n+\ell) = \varphi(\tau(x+T_n(x)\theta))$ on P_{nk}^{ℓ}.

Suppose now that ρ is a nontrivial continuous irreducible unitary representation of K on $\mathbb{C}^d$, and suppose also that (2.3) has a nontrivial measurable solution. We replace K by $\rho(K) \neq \{e\}$, and reletter, so that (2.3) becomes

(2.3′) $$F(x+\theta) = \varphi(x)F(x) \ .$$

Now $\varphi(x) \in U(d)$, and (2.3′) implies $|F(\cdot)|$ is invariant under translation by θ, hence constant a.e. As F is assumed to be nontrivial, we may and shall assume that $|F(x)| = 1$ a.e. This will lead us to a contradiction, assumming λ is bounded (by Q).

Iterating (2.3′), one finds $F(x+m\theta) = \varphi^{(m)}(x)F(x)$, and this, plus the continuity of translation in $L^1(\mu)$, implies

(3.4) $$\lim_{m\theta\to 0} \|\varphi^{(m)}F-F\|_1 = 0 \ .$$

3.5 Lemma. With notations as above, there exists for every pair $\epsilon, q > 0$ a vector $v = v(\epsilon,q)$, $|v| = 1$, such that $|\psi(i)v-\psi(j)v| < 2\epsilon$, $0 \le i, j \le q$.

Proof: S is a K_σ-sequence, and therefore there exists an infinite set Γ such that $\psi(n+j) = \psi(j)$, $0 \le j \le q$, $n \in \Gamma$. Apply (3.4) ($m = \Lambda_n, n\in\Gamma$), and (3.2) to conclude that if $n \in \Gamma$ is large there exist $P_{nk} \in \mathcal{P}_n$, such that $(P^\epsilon_{nk})^c = \{y \in P_{nk} \mid |\varphi_n(y)F(x)-F(x)| \ge \epsilon\}$ has measure less than $Q^{-(q+1)}\mu(P_{nk})$. From (3.1) one concludes $P^\epsilon_{nk} \cap P^\ell_{nk} \ne \emptyset$, $0 \le \ell \le q$. Finally, (3.3), the definition of P^ϵ_{nk}, and the facts $n \in \Gamma$ and $A_{nk}, B_{nk} \in U(d)$ imply that if $v = B_{nk}F(x)$, then $|v| = 1$ and $|\psi(i)v-\psi(j)v| < 2\epsilon$, $0 \le i, j \le q$. The lemma is proved.

Notice in the above that also $|\psi(i)^{-1}\psi(j)v-v| < 2\epsilon$, $0 \le i, j \le q$, $v = v(\epsilon,q)$. If we let $\epsilon \to 0$, $q \to \infty$ in such a way that $v(\epsilon,q) \to v_0$, then $|v_0| = 1$, and $\psi(i)^{-1}\psi(j)v_0 = v_0$, $i,j \ge 0$. As S is a K_σ-sequence $kv_0 = v_0$, $k \in K$. Irreducibility then implies $d = 1$, $K = \{e\}$, a contradiction. We conclude that (2.3) cannot have a nontrivial measurable solution. The discussion of Section 2 now implies Theorem 1.3. (The second coordinate of $T^n(\theta,\epsilon)$ is $\varphi^{(n)}(\theta) = \psi(\tau(n))\psi(\tau(n-1))\dots\psi(\tau(1))$, where $\tau(k)$ is defined by (1.2)).

Remark on the case d = 1. Let λ be as in Section 1, possibly unbounded, and let $S = \{\Psi(n)\}_{n\geq 0}$ be a sequence of complex numbers of absolute value 1. Define K to be the closed subgroup of U(1) generated by the terms of S. Form $X = X(\lambda)$, and set $\varphi(x) = \psi(\tau(x))$, $x \neq -\theta$. We wish to allow for the possibility that φ has a limit at $-\theta$; this means that $M = M(\lambda,\Psi)$, rather than having $X(\lambda)$ for a "factor," may in fact itself be a "factor" of $X(\lambda)$ (more precisely, the quotient of $X(\lambda)$ by the periods of the extended function φ). Let $N = N(\lambda,\Psi) = M\times K$ and $T = T(\lambda,\Psi)$ be as in Section 2. Also, set $\omega = \omega(\lambda,\Psi) = \mu\times\nu$, as in Section 2. Using the above, one may prove

3.6 Theorem. With notations as above, suppose $\sum_{n=0}^{\infty} |\psi(n+1)-\psi(n)| = \infty$. Then (T,N) is uniquely ergodic. Moreover, the point spectrum of T, relative to ω, is contained in $\Gamma(\lambda) = \{\chi(\theta) \mid \chi$ a continuous character on $X(\lambda)\}$.

If $\tilde{\lambda}$ is a second sequence, we write $\tilde{\lambda} \perp \tilde{\lambda}$ if $(\Lambda_n, \tilde{\Lambda}_n) = 1$ for all n. When $\tilde{\lambda} \perp \lambda$, the Chinese Remainder Theorem implies $\mathbb{Z}(\theta,\tilde{\theta})$ is dense in $X(\lambda) \times X(\tilde{\lambda})$, and this in turn implies $\sigma\times\tilde{\sigma}$ is uniquely ergodic on $M\times M$ for any given $\tilde{\Psi}$. Suppose now that both ψ and $\tilde{\psi}$ satisfy the hypothesis of Theorem 3.6. As $\Gamma(\lambda) \cap \Gamma(\tilde{\lambda}) = \{1\}$, the point spectra of $T,\tilde{T}$, relative to $\omega,\tilde{\omega}$, have trivial intersection ($\{1\}$), and so by a well known result in ergodic theory, $T\times\tilde{T}$ is ergodic relative to $\omega\times\tilde{\omega}$. But $\omega\times\tilde{\omega}$ may be viewed as $(\mu\times\tilde{\mu})\times(\nu\times\tilde{\nu})$, $\nu\times\tilde{\nu}$ = Haar measure on $K\times\tilde{K}$, and so Furstenberg's princple (Section 2), plus the unique ergodicity of $\sigma\times\tilde{\sigma}$, implies $T\times\tilde{T}$ in uniquely ergodic.

The sequences $\varphi^{(n)}(0)$, $\tilde{\varphi}^{(n)}(0)$ are "q-muliplicative sequences" (see [3] for definition and references). An immediate consequence of the above is that when $\lambda \perp \tilde{\lambda}$ and $\psi, \tilde{\psi}$ satisfy the hypothesis of Theorem 3.6, one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi^{(n)}(0)\tilde{\varphi}^{(n)}(0) = 0 \quad .$$

It would be interesting to know whether other known (and unknown) properties of q-multiplicative sequences can be obtained from such considerations.

4. Irregularities of distribution modulo 1. In this section we suppose $X = \mathbb{R}/\mathbb{Z}$, and we fix $\theta \in X$ irrational. If $I \subseteq X$ is an interval and $\alpha, \beta \in \mathbb{R}$, define $\varphi = (\alpha-\beta)\chi_I - \beta\chi_{I^c}$. We regard φ as having values in $K = K(\alpha,\beta)$, the closed subgroup of X generated by α and β (modulo 1). We note that $\varphi^{(n)}(x) = S_n(x)\alpha - n\beta$, where $S_n(x) = S_n(x,\theta,I)$ is defined in Section 1.

Let $\{\frac{p_n}{q_n}\}$ be the sequence of convergents to θ, and define $\Gamma^0(\theta) \subseteq X$ to be the set of t which admit a representation $t = \sum_{n=1}^{\infty} b_n q_n \theta$ (in X) such that $b_n \in \mathbb{Z}$ and $\lim_n b_n q_n \|q_n\theta\| = 0$. (Any two such representations agree for large n [16].) If $\alpha \in \mathbb{R}$, we also define $\Gamma^0_\alpha(\theta) = \{t \in \Gamma^0(\theta) \mid \lim_n b_n\alpha = 0 \text{ in } X\}$. As noted in [16], [17] we have (i) if θ has bounded partial quotients, then $\Gamma^0(\theta) = \mathbb{Z}\theta$, and (ii) if $t \notin \mathbb{Z}\theta$, then for almost all α, $t \notin \Gamma^0_\alpha(\theta)$.

The theorem below is proved in [16] for $\alpha = \frac{1}{2}$. Extension to the general case is sketched in [18], [17] and the details are carried out by Stewart in [12].

4.1 Theorem. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \notin \mathbb{Z}$. If for every k such that $k\alpha \neq 0$ (in X) $|I| \notin \Gamma^0_{k\alpha}(\theta)$ modulo 1, then (T,N) (Section 2) is uniquely ergodic.

4.2 Corollary. ([17],[12]) *If* $|I| \notin \mathbb{Z}\theta$ *modulo* 1, *then for almost all* $\alpha \in \mathbb{R}$ *the sequence* $\{S_n(x)\alpha - n\beta\}$ *is well distributed modulo* 1 *for any choice of* $x \in X$ *and* $\beta \in \mathbb{R}$.

The corollary may be used to prove Theorem 1.4. To this end, suppose $|I| \notin \mathbb{Z}\theta$ modulo 1 but for some $x \in X$ and $M < \infty$ the set $E_M(x)$ (Section 1) has upper density $2\epsilon > 0$. Corollary 4.2 implies there exists α, $0 < \alpha < \frac{\epsilon}{2M}$ such that $\{S_n(x)-n\beta\}$ is well distributed modulo 1 for all β. Set $\beta = |I|\alpha$, and note for this choice that $\|S_n(x)\alpha - n|I|\alpha\| \le |S_n(x)\alpha - n|I|\alpha| < \frac{\epsilon}{2}$ if $n \in E_N(x)$. Well distribution implies the set of n such that $\|S_n(x)\alpha - n|I|\alpha\| < \frac{\epsilon}{2}$ has upper density $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon < 2\epsilon$, and we have a contradiction. That is, $E_M(x)$ has upper density 0, and the theorem is proved.

When $\varphi = (\alpha-\beta)\chi_I - \beta\chi_{I^c}$ is regarded as taking values in $\mathbb{R}$, it is natural to prevent "drift" by requiring φ to have integral 0. But for a change of scale, this is tantamount to requiring $\varphi = (1-|I|)\chi_I - |I|\chi_{I^c}$. In what follows, $G = G(I)$ is the closed subgroup of $\mathbb{R}$ generated by $|I|$ and $1-|I|$. We assume $0 < |I| < 1$.

Define $T: X\times G \to X\times G$ by $T(x,y) = (x+\theta, y+\varphi(x))$. T preserves Haar measure on $X\times G$, which of course is infinite. Using a topological analogue of K. Schmidt's notion of an "essential value" of a cocycle ([11]), it is not difficult to prove

4.3 Proposition. Assume $|I|$ is rational or else $1, \theta$, and $|I|$ are rationally independent. Then T has a residual set of points with dense orbits. In particular, for a residual set of $x \in X$ the sequence $S_n(x) - n|I|$ is dense in $G(I)$.

One conjectures the conclusion of the proposition holds with residual set of x replaced by 'measure 1 set of x.' (It does not hold for 'all x'. See Dupain [4].) One way to prove this is to prove T is <u>ergodic</u> (relative to Haar measure). This is so for $|I|=\frac{1}{2}$ (K. Schmidt [10]; Conze-Keane [2]) and also for almost all values of $|I|$ (Conze [1]). In [17] the question was raised whether $|I| \notin \Gamma^0(\theta)$ implies ergodicity. This is proved by M. Stewart [12] when θ has bounded partial quotients, and Stewart now claims a proof for general θ (oral communication). It is open whether <u>any</u> condition on $|I|$ is necessary for ergodicity (save $|I| \in \mathbb{Q}$ or $1, \theta, |I|$ rationally independent).

Stewart's work relies heavily on the work of Schmidt and Conze. The most important ingredients are Schmidt's notion of essential value, the Denjoy-Koksma lemma (used by Conze), and the following

4.4 Theorem (M. Stewart [12]). *Assume θ has bounded partial quotients. If $t \notin \mathbb{Z}\theta$ modulo 1, then*

$$\limsup_{n\to\infty} (\|q_n t\| - \tfrac{1}{2}q_n\|q_n\theta\|) > 0 \quad .$$

It would be of interest to have a formulation and proof of a nonabelian analogue of Theorem 4.1. At the present time one knows only that if θ has bounded partial quotients, if $|I| \notin \mathbb{Z}\theta$ modulo 1, and if K is a <u>finite</u> group with generators α,β, the homeomorphism (T,N) corresponding to $\varphi(x) = \alpha,\beta$ as $x \in I, I^c$ is uniquely ergodic [14].

References

[1] Conze, J. P. "Équirépartition et Ergodicité de Transformations Cylindriques," Preprint, Université de Rennes (1976).

[2] Conze, J. P. and Keane, M. "Ergodicité d'un flot cylindrique," C. R. Acad. Sci. Paris (to appear).

[3] Coquet, J., Kamae, T., and Mendes-France, M. "Sur la mesure spectrale de certaines suites arithmétiques," preprint, l'Universite de Bordeaux.

[4] Dupain, Y. "Intervalles à restes majorés pour la suite $\{n\alpha\}$," Acta. Math. Acad. Sci. Hung., 29(1977), 289-303.

[5] Furstenberg, H. "Strict ergodicity and transformations of the torus," Amer. J. Math. 88(1961), 573-601.

[6] Furstenberg, H., Keynes, H. and Shapiro, L., "Prime flows in topological dynamics," Israel J. Math. 14(1973), 26-38.

[7] Kesten, H. "On a conjecture of Erdös and Szüsz related to uniform distribution mod 1," Acta. Arith. 12(1966/67), 193-212.

[8] Losert, V. and Rindler, H. "Uniform distribution and the mean ergodic theorem," preprint.

[9] Oxtoby, J. C. "Ergodic sets," Bull. Amer. Math. Soc. 58(1952) 116-136.

[10] Schmidt, K. "A Cylinder Flow Arising From Irregularity of Distribution," preprint, University of Warwick, 1975.

[11] Schmidt, K. "Cohomology amd Skew Products of Ergodic Transformations," preprint, University of Warwick, 1974.

[12] Stewart, M. "Irregularities of uniform distribution," Ph.D. Thesis, Rice University, 1978.

[13] Veech, W. A. "Applications of ergodic theory to some problems of uniform distribution," Proc. Conf. on Ergodic Theory and Topological Dynamics (Univ. of Kentucky, 1971), Math Dept., Univ. of Kentucky, Lexington, Ky., 26-33.

[14] _____, "Finite group extension of irrational rotations, Israel J. Math. 21(1975) 240-259.

[15] _____, "Some questions of uniform distribution," Ann. of Math. (2)94(1971), 125-138.

[16] _____, "Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2," Trans. Amer. Math. Soc. 140(1969), 1-33.

[17] _____, "Topological dynamics," Bull. Amer. Math. Soc., 83 (1977), 775-830.

[18] _____, "Well distributed sequences of integers," Trans. Amer. Math. Soc. 161(1971), 63-70.

William A. VEECH
Department of Mathematics
Rice University HOUSTON
U.S.A.