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# A SURVEY OF SOME IMPORTANT PROBLEMS IN ADDITIVE NUMBER THEORY

by

R.C. VAUGHAN

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Since many aspects of additive number theory were covered by Halberstam's adress [27] to the recent meeting on Additive Number Theory in Bordeaux, I shall content myself by adumbrating just two of the principal areas which have interested me particularly. These are problems dealing with

(A) sums of kth powers,

(B) sums of primes.

One of the fascinating aspects of these problems is the interplay between them and other areas of analytic number theory.

A. - The typical problem involving sums of kth powers is

1. - <u>Waring's problem</u>, regarding which there is an excellent survey article by Ellison [24]. Let $g(k)$ denote the smallest $s$ such that for every $n \geq 1$ there exist $x_i \geq 0$ such that $n = x_1^k + \ldots + x_s^k$. The problem of evaluating $g$ has been essentially solved for all $k$ except $k = 4$. It is thought that

$$(1) \qquad g(k) = 2^k + [(\tfrac{3}{2})^k] - 2 .$$

It is classical that this holds whenever $k \neq 4, 5$ and

$$(2) \qquad \{(\tfrac{3}{2})^k\} < 1 - 2^{-k}[(\tfrac{3}{2})^k] .$$

Mahler [38] has shown that (2) has at most a finite number of exceptions, and Stemmler [50] has verified that (2) holds for $k \leq 200\,000$. Incidently, this has lead to interesting questions concerning distribution modulo 1, see Mahler [39].

More recently, Chen [3,4,5] has shown that (1) also holds when $k=5$. This leaves $k=4$. Here there has been considerable recent progress. The upper bound for $g(4)$ has been reduced first to 34, then to 30 and 23 and finally to 22 by Dress [22], Dress [23], Thomas [51] and Thomas [52] respectively. It is trivial that $g(4) \geq 19$.

2. - The more interesting and challenging problem is that of the estimation of $G(k)$, the smallest $s$ such that every sufficiently large integer is the sum of at most $s$ kth powers of positive integers. So far only $G(2)$ and $G(4)$ are known. If one defines $\Gamma(k)$ to be the least $s$ such that for every $q, n$ the congruence $x_1^k + \dots + x_s^k \equiv n \pmod q$ is soluble, then one has $G(k) \geq \max(k+1, \Gamma(k))$. One might guess that equality occurs. The current of play for small values of $k$ is as follows ;

$G(2) = 4$, Lagrange [34],
$G(3) \leq 7$, Linnik [35], Watson [66],
$G(4) = 16$, Davenport [12],
$G(5) \leq 23$, $G(6) \leq 36$, Davenport [13, 14],
$G(7) \leq 53$, Davenport's method (the claim $G(7) \leq 52$ of Sambasiva Rao [46] is fallacious),
$G(8) \leq 73$, Narasimhamurti [43].

For larger $k$, the principle results in the last thirty years have been

$G(k) < k(3\log k + 11)$, Vinogradov [63, 64],
$G(k) < k(3\log k + 9)$ $(k = 2^m)$, $G(k) < k(3\log k + 7)$ $(k \neq 2^m)$, Tong [53],
$G(k) < k(3\log k + 5.2)$, Chen [2],
$G(k) < k(2\log k + 4\log\log k + 2\log\log\log k + 13)$, Vinogradov [65].

This last result is superior to Chen's only when $k > 6\,103\,975\,350$. More recently, the method described in Vaughan [59] gives $G(9) \leq 91$, $G(10) \leq 107$, $G(11) \leq 122$, $G(12) \leq 137$, $G(13) \leq 153$, $G(14) \leq 168$, $G(15) \leq 184$, $G(16) \leq 200$, $G(17) \leq 216$ and $G(k) < k(3\log k + 4.2)$.

3. - <u>Homogeneous additive equations</u>. - Davenport and Lewis [15] have shown that there is an $s(k)$ such that if $s \geq s(k)$, then for every $c_1, \dots, c_s$ (with $c_1c_2 < 0$

if $k$ is even) the equation $c_1x_1^k + \dots + c_sx_s^k = 0$ has a non-trivial solution in integers $x_1, \dots, x_s$. They showed that it is possible to take $s(k) \leq k^2+1$ when $k \leq 6$ or $k \geq 18$ giving partial verification of Artin's conjecture that any form of odd degree represents 0 non-trivially whenever $s \geq k^2+1$.

Vaughan [59] has partly filled the gap by showing that $s(k) \leq k^2+1$ is permissible when $11 \leq k \leq 17$.

4. - Simultaneous homogeneous additive equations. - Davenport and Lewis [17] have treated the system

$$\begin{cases} c_{11}x_1^k + \dots + c_{1n}x_n^k = 0 , \\ \vdots \qquad\qquad \vdots \\ c_{r1}x_1^k + \dots + c_{rn}x_n^k = 0 . \end{cases}$$

There are many open questions in connection with this. Earlier [16], they had studied pairs of additive cubics

$$(2) \quad \begin{cases} c_1x_1^3 + \dots + c_nx_n^3 = 0 , \\ d_1x_1^3 + \dots + d_nx_n^3 = 0 . \end{cases}$$

They showed that if $n \geq 18$, then there is a non-trivial solution of (2), and that there exist $c_1, d_1, \dots, c_{15}, d_{15}$ such that (2) with $n = 15$ has only the trivial solution. Cook [10] has replaced the 18 by 17 and Vaughan [57] has reduced this to 16, the best possible.

For related matters see Davenport and Lewis [18].

5. - Vinogradov's mean value theorem. - Let $I(X, s, k)$ denote the number of solutions of

$$\begin{cases} x_1 + \dots + x_s = y_1 + \dots + y_s \\ x_1^2 + \dots + x_s^2 = y_1^2 + \dots + y_s^2 \\ \cdot \qquad \cdot \quad \cdot \qquad \cdot \\ x_1^k + \dots + x_s^k = y_1^k + \dots + y_s^k \end{cases}$$

with $0 < x_i, y_i \leq X$. Karatsuba and Korobov [31] have shown that

$$I(X, s, k) < C(k, \ell)\, X^{2s - \frac{1}{2}(k+1) + \delta}$$

with $\delta = \frac{1}{2}k(k+1)(1-\frac{1}{k})^{\ell}$ whenever $s \geq k^2+k\ell$. For an earlier account of this see Vinogradov's book [63, 64]. There are a number of important applications.

The value of $C(k, \ell)$ is not usually very important in additive number theory, but the contrary is true in the applications to multiplicative number theory.

Recently Bombieri has shown that it is possible to take $\delta = \frac{1}{2}k^2(1-\frac{1}{k})^{\ell}$ whenever $s \geq k\ell$.

B. - The archetypal problem concerning sums of primes is Goldbach's problem. This stems from two letters from Goldbach to Euler in 1742 in which he conjectures that every even natural number is the sum of two primes and that every integer greater than 2 is the sum of three primes. He included unity as a prime. There have been three lines of attack on these problems.

1. - Direct applications of sieve methods. - There are excellent surveys of earlier work in Halberstam and Roth [29] and Halberstam and Richert [28]. The most recent result is the celebrated theorem of Chen [6, 7] to the effect that for $n>n_o$ either $2n = p+p_1$ or $2n = p+p_1p_2$. There are shorter proofs by Ding, Pan and Wang [21], and Ross [44]. Ross [45] has also shown that the primes can be restricted in various ways. Graham [26] has made $n_o$ effectively computable.

2. - Indirect applications of sieve methods. - This stems from Shnirel'man [47, 48]. He showed that there exists a constant $C$ such that if $n>n_o$, then $n = p_1+\ldots+p_s$ with $s \leq C$. His $C$ is very large, and the method was later superceded by the more powerful Hardy-Littlewood-Vinogradov method (see below). However, alternative lines of approach are always of interest in connection with difficult problems. In recent times Chechuro and Kuzjashev [1], and Siebert [49] obtained $C = 10$ by this method. This is improved to $C = 6$ in Vaughan [56]. This last paper contains a brief survey of previous work via this method.

Perhaps more interesting is the fact that this method readily yields a $C_o$ such that every $n>1$ can be written as the sum of at most $C_o$ primes. The most recent work in this direction is

$C_o = 6 \times 10^9$, Klimov [32],
$C_o = 115$, Klimov, Pil'tai and Sheptitskaya [33],

$C_o = 75$, Deshouillers [19],
$C_o = 27$, Vaughan [58].

This last paper contains two different methods, in one of which the calculations are easier. However the more difficult method would permit a smaller $C_o$ provided certain calculations could be carried out. Deshouillers [20] has thereby obtained $C_o = 26$.

3. - The Hardy-Littlewood-Vinogradov method. - By obtaining non-trivial estimates for

$$(3) \qquad \sum_{p \leq N} e^{2\pi i \alpha p} \text{ when } |\alpha - \frac{a}{q}| \leq q^{-2}, \ (a,q) = 1, \ (\log N)^A < q \leq N (\log N)^{-A},$$

Vinogradov [62] gave an unconditional proof that every sufficiently large odd integer is the sum of at most three primes. Linnik [36, 37] (see also Chudakov [9]), Montgomery [40] and Vaughan [60] have given different ways of estimating (3).

Immediately following Vinogradov's work, Chudakov [8], van der Corput [11] and Estermann [25] all showed that if $E(x) = |\{n \leq x : 2n \nmid p + p'\}|$, then $E(x) = O_A(x \log^{-A} x)$. Thix was later improved to $O(x \exp(-c\sqrt{\log x}))$ and $O(x^{1-\delta})$ by Vaughan [54] and Montgomery and Vaughan [42] respectively.

For another question connected with Goldbach's problem, see Montgomery and Vaughan [41], and Vaughan [55].

Let me conclude by emphasizing the interaction between this subject and others of analytic number theory. Recently the ideas contained in Vaughan [60] have been used

(a) to give (Vaughan [61]) a new and simple proof of Bombieri's prime number theorem,

(b) by Heath-Brown and Patterson [30] as an aid in their resolution of Kummer's problem concerning cubic Gaussian sums, to the effect that the arguments are uniformly distributed modulo $2\pi$.

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